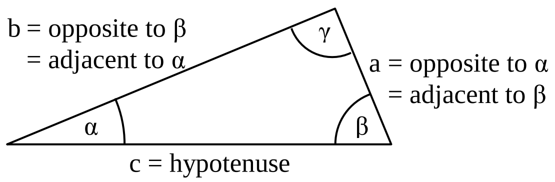
4.8. Trigonometric Functions

4.8.1. Basic definitions

In a plane the angles of a triangle add up to $\alpha + \beta + \gamma = 180°$, and in a **right-angled triangle** with $\gamma = 90°$ we have $\alpha + \beta = 90°$.
Two right-angled triangles are **similar** with equal side ratios if they have equal angles α or β. Consequently the following **side ratios** depend only on **one angle** α or β and are called **trigonometric functions**:



Sine: $\sin(\alpha) = \frac{a}{c} = \cos(\beta)$ **Cosine**: $\cos(\alpha) = \frac{b}{c} = \sin(\beta)$ **Tangent**: $\tan(\alpha) = \frac{a}{b} = \frac{1}{\tan(\beta)}$

The values of these ratios can be obtained from an electronic calculator and are used to calculate the missing data of a right-angle triangle if one side and either an angle or a second side are given.

Example 1
Given are a = 4 cm und $\alpha = 40°$. Calculate b, c and β.

Solution
Angle sum: $\underline{\beta} = 90° - \alpha = \underline{50°}$

Sine: $\sin(\alpha) = \frac{a}{c} \Leftrightarrow \underline{c} = \frac{a}{\sin(\alpha)} = \frac{4\text{ cm}}{\sin(40°)} \approx \underline{6{,}22\text{ cm}}$

Pythagoras: $a^2 + b^2 = c^2 \Rightarrow \underline{b} = \sqrt{c^2 - a^2} \approx \underline{5{,}22\text{ cm}}$

Example 2
Given are a = 5 cm und c = 8 cm. Calculate b, α and β.

Solution
Sine: $\sin(\alpha) = \frac{a}{c} \Leftrightarrow \underline{\alpha} = \sin^{-1}\left(\frac{a}{c}\right) = \sin^{-1}\left(\frac{5}{8}\right) \approx \underline{38{,}7°}$

Angle sum: $\underline{\beta} = 90° - \alpha = \underline{51{,}3°}$

Pythagoras: $a^2 + b^2 = c^2 \Rightarrow \underline{b} = \sqrt{c^2 - a^2} = \underline{\sqrt{39}\text{ cm}}$

Exercises on trigonometric functions No. 1

4.8.2. Basic relations

Theorem
In a right-angled triangle ($0 \le \alpha, \beta \le 90°$) we have the following obvious identities:

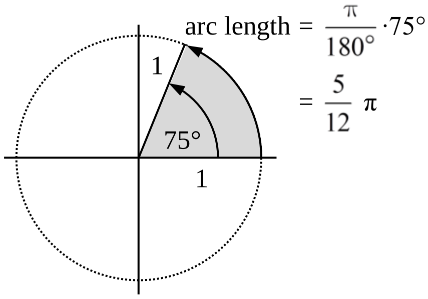
1. $\sin(\alpha) = \cos(\beta) = \cos(90° - \alpha)$
2. $\tan(\alpha) = \frac{a}{b} = \frac{c}{c}\frac{\sin(\alpha)}{\cos(\alpha)} = \frac{\sin(\alpha)}{\cos(\alpha)}$
3. Pythagoras: $a^2 + b^2 = c^2 \Leftrightarrow \left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1 \Leftrightarrow [\sin(\alpha)]^2 + [\cos(\alpha)]^2 = 1$

Exercises on trigonometric functions No. 2 - 5

4.8.3. Measuring angles in radians

Definition
The subdivision of the full cycle into 360 degrees is highly arbitrary and does not allow the approximation of the trigonometric functions in terms of simple polynomials. However the trigonometric functions become compatible with polynomials when the angle is expressed in radians instead of degrees.
The **radian** of an angle α is defined as the **arc length of the corresponding sector in the unit circle** with radius r = 1. Since a full cycle of 360° corresponds to the circumference 2πr = 2π of the unit circle and likewise a half cycle of 180° has an arc length of π, we can convert degrees into radians and vice versa with the formula:

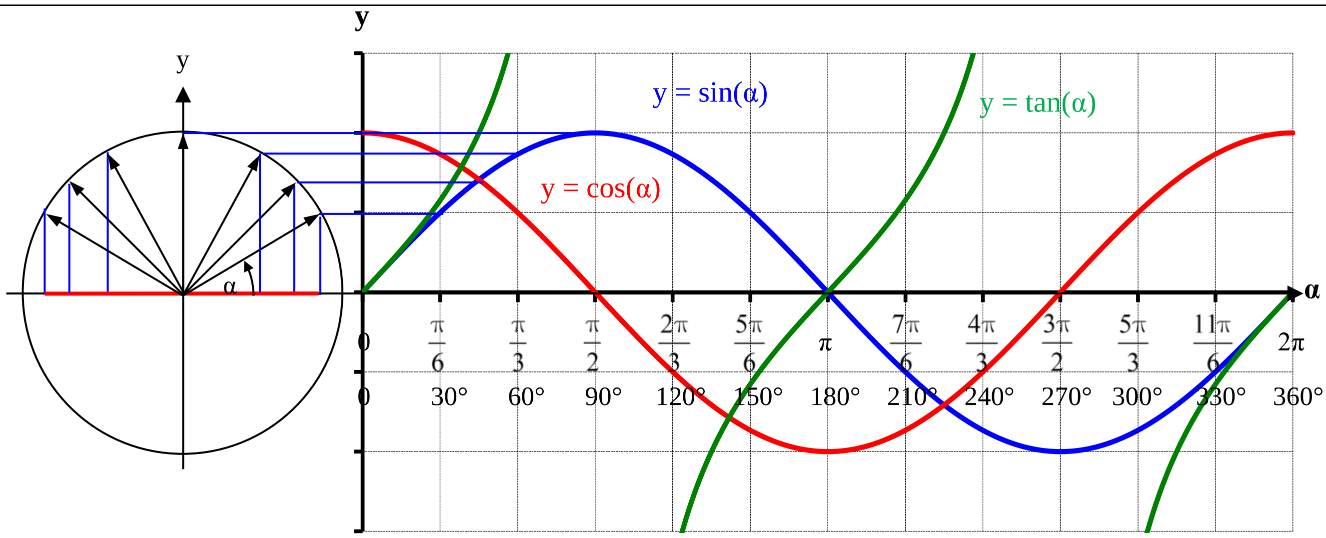
$$\alpha \text{ in radian} = \frac{\pi}{180} \cdot \alpha \text{ in degree.}$$



Exercises on trigonometric functions No. 6

4.8.4. Graphs of the trigonometric functions

Definition
Let P(x|y) the endpoint of a **pointer** in the **unit circle** (i.e. radius r = 1, which rotates to an angle α ∈ ℝ **counterclockwise** (α > 0) resp. **clockwise** (α < 0) starting from the horizontal position. Then the right-angled triangle under the pointer has the **hypotenuse 1** so that the **trigonometric functions** become sin(α) = y, cos(α) = x und tan(α) = $\frac{y}{x}$

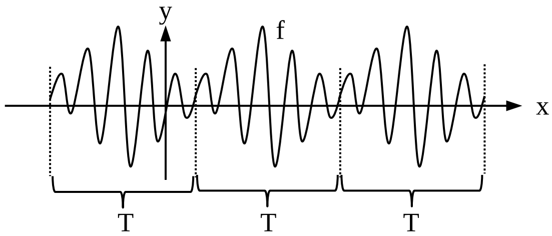


Example: Graphs of the trigonometric functions No. 1

4.8.5. Properties of the trigonometric functions

Definition: A function f is periodic with **period T**, if its course is repeated after each period T, i.e.

$$f(x) = f(x + T) \text{ for all } x \in D.$$



Example: Graphs of the trigonometric functions No. 2

Function **f(α) =**	Symmetry **f(-α) =**	Neighbor angle **f(π - α) =**	Period **T =**	Domain **D =**	Range **W =**
sin(α)	-f(α) (uneven)	f(α)	2π	$\mathbb{R}$	$[-1;1]$
cos(α)	f(α) (even)	-f(α)	2π	$\mathbb{R}$	$[-1;1]$
tan(α)	-f(α) (uneven)	-f(α)	π	$\mathbb{R}\backslash\{\frac{\pi}{2} + z\cdot\pi\colon z \in \mathbb{Z}\}$	$\mathbb{R}$

4.8.6. Angular velocity and amplitude

Examples: Exercises on trigonometric functions No. 7a

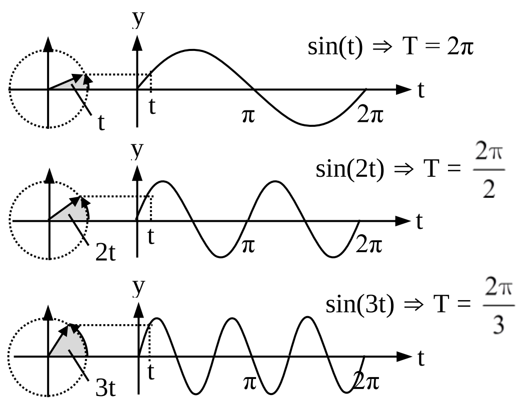
Angular velocity and period

The pointer moves with constant **angular velocity** $\omega = \frac{\alpha}{t}$.

At a given time t the angle is $\alpha = \omega\cdot t$ and the vertical height is $y(t) = \sin(\alpha) = \sin(\omega\cdot t)$.
The pointer needs exactly one **period** T for a complete cycle with angle $\alpha = 2\pi$.

Therefore the angular velocity is $\omega = \frac{2\pi}{T}$.

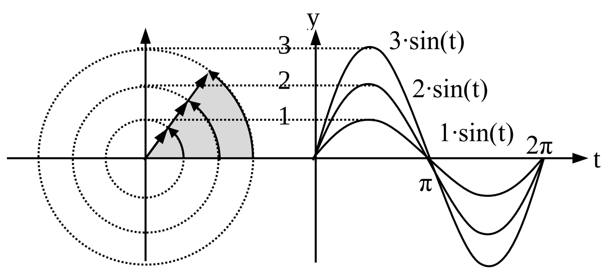
When the pointer moves faster the angular velocity ω increases but the period $T = \frac{2\pi}{\omega}$ decreases.



Examples: Exercises on trigonometric Functions No. 7b

The amplitude
The length of the pointer defines the range of the sine function and is called **amplitude A**.
At a given time t the vertical height of a pointer with length A moving with angular velocity ω is

$$y(t) = A\cdot\sin(\omega\cdot t)$$

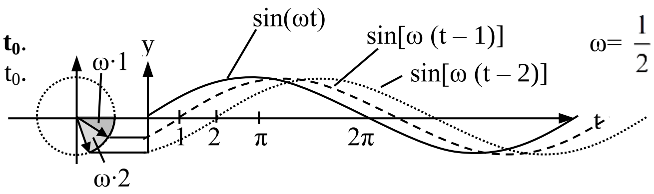


4.8.7. Vertical shift and phase

Examples: Exercises on trigonometric functions No. 7c

The phase
If the pointer starts late, his delay is called the **phase** $\mathbf{t_0}$. Accordingly the graph starts with a delay or **horizontal shift** of t_0.
The formula is

$$y(t) = A\cdot\sin[\omega(t - t_0)]$$

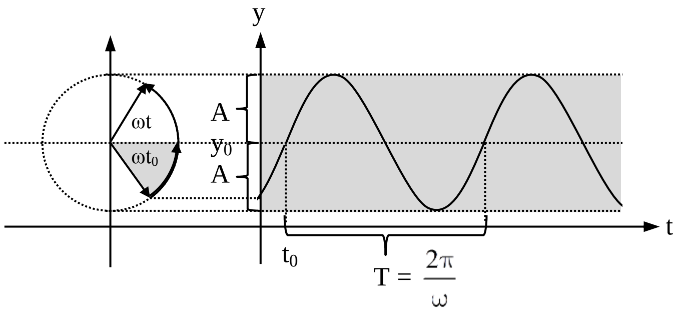


Examples: Exercises on trigonometric functions No. 7d

The vertical shift
If the pointer starts its movement at the height y_0 the graph follows and is shifted **vertically** Its formula is

$$y(t) = A \cdot \sin[\omega(t - t_0)] + y_0$$

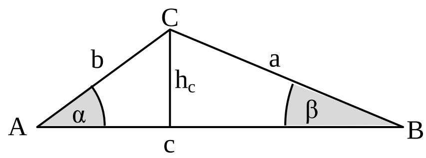
with
A = amplitude = length of pointer
ω = angular velocity of pointer
T = period = duration of one complete turn
t_0 = phase = delay of pointer = horizontal shift
y_0 = vertical shift



Exercises on trigonometric functions No. 8

4.8.8. The sine rule

Theorem (sine rule):
In any triangle we have $\dfrac{\sin(\alpha)}{a} = \dfrac{\sin(\beta)}{b} = \dfrac{\sin(\gamma)}{c}$.
"The larger the angle the longer the opposite side"



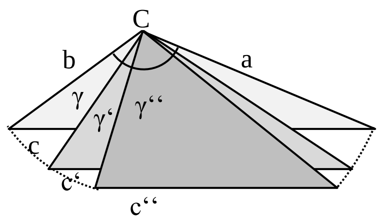
Proof for acute angles (for obtuse angles use supplementary angle):
$h_C = b \cdot \sin(\alpha) = a \cdot \sin(\beta) \Leftrightarrow b \cdot \sin(\alpha) = a \cdot \sin(\beta) \Leftrightarrow \dfrac{\sin(\alpha)}{a} = \dfrac{\sin(\beta)}{b}$.

Exercises on Trigonometric Functions No. 9 and 10

4.8.9. The cosine rule

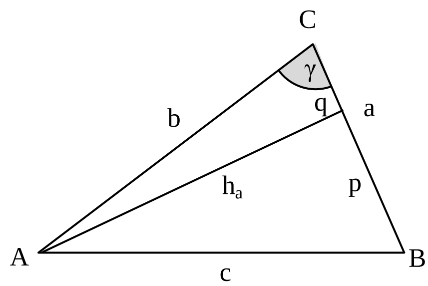
Theorem (cosine rule)
In any triangle we have $c^2 = a^2 + b^2 - 2 \cdot a \cdot b \cdot \cos(\gamma)$

commentary:
The cosine rule ist he **generalization of Pythagoras' theorem** for angles $\gamma \neq 90°$:
a) For **acute** angles $\gamma < 90°$ we have $\cos(\gamma) > 0$
   $\Rightarrow$ additional term $-2ab \cdot \cos(\gamma) < 0$,
   $\Rightarrow$ c is **shorter** than with Pythagoras. **(light trangle)**
b) For **right** angles $\gamma = 90°$ we have $\cos(\gamma) = 0$
   $\Rightarrow$ additional term $-2ab \cdot \cos(\gamma) = 0$,
   Pythagoras: $c^2 = a^2 + b^2$. **(middle trangle)**
c) For **obtuse** angles $\gamma > 90°$ we have $\cos(\gamma) < 0$
   $\Rightarrow$ additional term $-2ab \cdot \cos(\gamma) > 0$,
   $\Rightarrow$ c is **longer** than with Pythagoras **(dark triangle)**



Proof for acute angles (for obtuse angles use supplementary angle):

$c^2 = h_a^2 + p^2$	$\mid h_a = b \cdot \sin(\gamma)$ bzw. $p = a - q$
$= (b \cdot \sin(\gamma))^2 + (a - q)^2$	| expand
$= (b \cdot \sin(\gamma))^2 + a^2 - 2aq + q^2$	$\mid q = b \cdot \cos(\gamma)$
$= b^2 \cdot (\sin(\gamma))^2 + a^2 - 2 \cdot a \cdot b \cdot \cos(\gamma) + b^2 \cdot (\cos(\gamma))^2$	| common factor b^2
$= b^2 \cdot [(\sin(\gamma))^2 + (\cos(\gamma))^2] + a^2 - 2 \cdot a \cdot b \cdot \cos(\gamma)$	| Pythagoras
$= a^2 + b^2 - 2 \cdot a \cdot b \cdot \cos(\gamma)$	



Exercises on trigonometric functions No. 11 and 12