5.3. Prüfungsaufgaben zur Kurvenuntersuchung ganzrationaler Funktionen

Problem 1a (10)

Find intercepts, extrema and inflexion points of $f(x) = \frac{1}{2}x^3 - \frac{3}{2}x^2$ and draw its graph including all significant points.

Solution

$f(x) = \frac{1}{2}x^2(x-3) \Rightarrow$ Intersects $S_{x1}(0|0)$ (double $\Rightarrow$ Max/Min) and $S_{x2}(3|0)$ (1)

$f'(x) = \frac{3}{2}x^2 - 3x = \frac{3}{2}x(x-2)$ (1)

$f''(x) = 3x - 3$ (1)

$f'''(x) = 1$ (0)

Relative maximum: ($f'(x) = 0$ and $f''(x) < 0$): Max(0|0) (1)

Relative minimum: ($f'(x) = 0$ and $f''(x) > 0$): Min(2|–2) (2)

Inflexion point: ($f''(x) = 0$ and $f'''(x) \neq 0$): Inf(1|−1) (2)

Graph (2)

Problem 1b (10)

Find intercepts, extrema and inflexion points of $f(x) = \frac{1}{2}x^3 - 3x^2 + \frac{9}{2}x$ and draw its graph including all significant points.

Solution

$f(x) = \frac{1}{2}x(x-3)^2 \Rightarrow$ Intersects $S_{x1}(3|0)$ (double $\Rightarrow$ Max/Min) and $S_{x2}(0|0)$ (1)

$f'(x) = \frac{3}{2}x^2 - 6x + \frac{9}{2} = \frac{3}{2}(x^2 - 4x + 3) = \frac{3}{2}(x-1)(x-3)$ (1)

$f''(x) = 3x - 6$ (1)

$f'''(x) = 1$ (0)

Relative maximum: ($f'(x) = 0$ and $f''(x) < 0$): Max(1|2) (1)

Relative minimum: ($f'(x) = 0$ and $f''(x) > 0$): Min(3|0) (2)

Inflexion point: ($f''(x) = 0$ and $f'''(x) \neq 0$): Inf(2|1) (2)

Graph (2)

Problem 1c (10)

Find intercepts, extrema and inflexion points of $f(x) = \frac{1}{6}x^3 + x^2 + \frac{3}{2}x$ and draw its graph including all significant points.

Solution

Achsenschnittpunkte: $f(x) = \frac{1}{6}x^3 + x^2 + \frac{3}{2}x = \frac{1}{6}x(x+3)^2 \Rightarrow S_{x1}(0|0)$ und $S_{x2}(-3|0)$ (doppelt) (2)

Ableitungen: $f'(x) = \frac{1}{2}x^2 + 2x + \frac{3}{2} = \frac{1}{2}(x+3)(x+1)$, $f''(x) = x + 2$ und $f'''(x) = 1$ (2)

Extrempunkte ($f'(x) = 0$ und $f''(x)$ </> 0): $H(-3|0)$ und $T(-1|-\frac{2}{3})$ (2)

Wendepunkte ($f''(x) = 0$ mit VZW): $W(-2|-\frac{1}{3})$ (2)

Graph (2)

Problem 1d (10)

Find intercepts, extrema and inflexion points of f(x) = $\frac{1}{12}x^3 - \frac{1}{4}x^2 - \frac{3}{4}x$ and draw ist graph including all significant points.

Solution

Intercepts: $f(x) = \frac{1}{12}x^3 - \frac{1}{4}x^2 - \frac{3}{4}x = \frac{1}{12}x(x^2 - 3x - 9) \Rightarrow S_{x1}(0|0)$ und $S_{x2/3}(\frac{3}{2} \pm \frac{3}{2}\sqrt{5}\,|0)$ (2)

Derivatives: $f'(x) = \frac{1}{4}x^2 - \frac{1}{2}x - \frac{3}{4} = \frac{1}{4}(x-3)(x+1)$, $f''(x) = \frac{1}{4}(x-2)$ und $f'''(x) = \frac{1}{4}$ (2)

Extrema (f'(x) = 0 and f''(x) </> 0): Max$(-1|\frac{5}{12})$ and Min$(3|-\frac{9}{4})$ (2)

Inflexion point (f''(x) = 0 and f'''(x) ≠ 0): Inf$(2|-\frac{11}{6})$ (2)

Graph (2)

Problem 1e (10)

Find intercepts, extrema and inflexion points of f(x) = $-\frac{1}{10}x^3 - \frac{6}{5}x^2 - \frac{18}{5}x$ and draw its graph including all significant points.

Solution

Intercepts: $f(x) = -\frac{1}{10}x^3 - \frac{6}{5}x^2 - \frac{18}{5}x = -\frac{1}{10}x(x^2 + 12x + 36) \Rightarrow S_{x1}(0|0)$ und $S_{x2}(-6|0)$ (double) (2)

Derivatives: $f'(x) = -\frac{3}{10}x^2 - \frac{12}{5}x - \frac{18}{5} = -\frac{3}{10}(x+6)(x+2)$, $f''(x) = -\frac{3}{5}(x+4)$ und $f'''(x) = -\frac{3}{5}$ (2)

Extrema (f'(x) = 0 and f''(x) </> 0): Min$(-6|0)$ and Max$(-2|\frac{16}{5})$ (2)

Inflexion point (f''(x) = 0 and f'''(x) ≠ 0): Inf$(-4|\frac{8}{5})$ (2)

Graph (2)

Problem 1f (10)

Find intercepts, extrema and inflexion points of f(x) = $\frac{1}{10}x^3 - \frac{6}{5}x^2 + \frac{18}{5}x$ and draw its graph including all significant points.

Solution

Intercepts: $f(x) = \frac{1}{10}x^3 - \frac{6}{5}x^2 + \frac{18}{5}x = \frac{1}{10}x(x^2 - 12x + 36) \Rightarrow S_{x1}(0|0)$ und $S_{x2}(6|0)$ (double) (2)

Derivatives: $f'(x) = \frac{3}{10}x^2 - \frac{12}{5}x + \frac{18}{5} = \frac{3}{10}(x-6)(x-2)$, $f''(x) = \frac{3}{5}(x-4)$ und $f'''(x) = \frac{3}{5}$ (2)

Extrema (f'(x) = 0 and f''(x) </> 0): Max$(6|0)$ and Min$(2|-\frac{16}{5})$ (2)

Inflexion point (f''(x) = 0 and f'''(x) ≠ 0): Inf$(4|-\frac{8}{5})$ (2)

Graph (2)

Problem 1g (10)

Find intercepts, extrema and inflexion points of $f(x) = \frac{1}{30}x^3 - \frac{1}{5}x^2 - \frac{1}{2}x$ and draw ist graph including all significant points.

Solution

Intercepts: $f(x) = \frac{1}{30}x^3 - \frac{1}{5}x^2 - \frac{1}{2}x = \frac{1}{30}x(x^2 - 6x - 15) \Rightarrow S_{x1}(0|0)$ und $S_{x2/3}(3 \pm 2\sqrt{6}\,|0)$ (2)

Derivatives: $f'(x) = \frac{1}{10}x^2 - \frac{2}{5}x - \frac{1}{2} = \frac{1}{10}(x-5)(x+1)$, $f''(x) = \frac{1}{5}(x-2)$ und $f'''(x) = \frac{1}{5}$ (2)

Extrema ($f'(x) = 0$ and $f''(x)$ </> 0): $Max(-1|\frac{4}{15})$ and $Min(5|-\frac{10}{3})$ (2)

Inflexion point ($f''(x) = 0$ and $f'''(x) \neq 0$): $Inf(2|-\frac{23}{15})$ (2)

Graph (2)

Aufgabe 2a (10)

Untersuche das Schaubild von $f(x) = \frac{9}{32}x^4 - \frac{3}{4}x^3$ auf Achsenschnittpunkte, Extrem- und Wendepunkte. Skizziere seinen Verlauf.

Lösung

$f(x) = \frac{9}{32}x^4 - \frac{3}{4}x^3 = \frac{3}{4}x^3(\frac{3}{8}x - 1) \Rightarrow$ Achsenschnittpunkte $S_{x1}(0|0)$ (dreifach $\Rightarrow$ SP) und $S_{x2}(\frac{8}{3}|0)$ (2)

$f'(x) = \frac{9}{8}x^3 - \frac{9}{4}x^2 = \frac{9}{4}x^2(\frac{1}{2}x - 1)$ (1)

$f''(x) = \frac{27}{8}x^2 - \frac{9}{2}x = \frac{9}{2}x(\frac{3}{4}x - 1)$ (1)

$f'''(x) = \frac{27}{4}x - \frac{9}{2} = \frac{9}{2}(\frac{3}{2}x - 1)$ (0)

Tiefpunkt: ($f'(x) = 0$ und $f''(x) > 0$): $T(2|-\frac{3}{2})$ (2)

Wendepunkte: ($f''(x) = 0$ und $f'''(x) \neq 0$): $W_1(0|0)$ (Sattelpunkt) und $W_2(\frac{4}{3}|-\frac{8}{9})$ (4)

Aufgabe 2b (10)

Untersuche $f(x) = x^4 - 7x^2 + 12$ auf Achsenschnittpunkte sowie Extrem- und Wendepunkte. Skizziere seinen Verlauf.

Lösung

$f(x) = (x^2 - 3)(x^2 - 4) \Rightarrow$ Achsenschnittpunkte $S_{x1/2}(\pm\sqrt{3}\,|0)$ sowie $S_{x1/2}(\pm 2|0)$ und $S_y(0|12)$ (2)

$f'(x) = 4x^3 - 14x = 4x^2(x - \frac{7}{2})$ (1)

$f''(x) = 12x^2 - 14 = 12(x^2 - \frac{7}{6})$ (1)

$f'''(x) = 24x$ (0)

Tiefpunkte: ($f'(x) = 0$ mit VZW von – nach + oder $f''(x) > 0$): $T_{1/2}(\pm\sqrt{\frac{7}{2}}\,|-\frac{1}{2})$ (2)

Hochpunkt: ($f'(x) = 0$ mit VZW von + nach − oder $f''(x) < 0$): $H(0|12)$ (2)

Wendepunkt: ($f''(x) = 0$ mit VZW oder $f'''(x) \neq 0$): $W_{1/2}(\pm\sqrt{\frac{7}{6}}\,|\frac{187}{36})$ (4)

Aufgabe 2c (10)

Untersuche $f(x) = -\frac{1}{2}x^4 + 3x^2 - 4$ auf Achsenschnittpunkte sowie Extrem- und Wendepunkte. Skizziere seinen Verlauf.

Lösung

$f(x) = -\frac{1}{2}(x^2 - 2)(x^2 - 4) \Rightarrow$ Achsenschnittpunkte $S_{x1/2}(\pm\sqrt{2}|0)$ sowie $S_{x1/2}(\pm 2|0)$ und $S_y(0|12)$ (2)

$f'(x) = -2x^3 + 6x = -2x(x^2 - 3)$ (1)

$f''(x) = -6x + 6 = -6(x - 1)$ (1)

$f'''(x) = -6$ (0)

Hochpunkte: ($f'(x) = 0$ mit VZW von + nach − oder $f''(x) < 0$): $H_{1/2}(\pm\sqrt{3}|\frac{1}{2})$ (2)

Tiefpunkt: ($f'(x) = 0$ mit VZW von − nach + oder $f''(x) > 0$): $T(0|-4)$ (2)

Wendepunkt: ($f''(x) = 0$ mit VZW oder $f'''(x) \neq 0$): $W_{1/2}(\pm 1|-\frac{3}{2})$ (4)

Exercise 3 (10)

Find the intercepts, extrema and inflexion points of $f(x) = \frac{1}{5}x^5 - \frac{5}{3}x^3 + 4x$ and draw its graph.

Solution

$f(x) = \frac{1}{5}x(x^4 - \frac{25}{3}x^2 + 20) \Rightarrow$ Intercept $S_{x1}(0|0)$ (1)

$f'(x) = x^4 - 5x^2 + 4 = (x^2 - 4)(x^2 - 1)$ (1)

$f''(x) = 4x^3 - 10x = 4x(x^2 - \frac{5}{2})$ (1)

$f'''(x) = 12x - 10 = 12(x - \frac{5}{6})$ (0)

Relative minima: ($f'(x) = 0$ and $f''(x) > 0$): $Min_1(-1|-\frac{38}{15})$ and $Min_2(2|\frac{16}{15})$ (2)

Relative maxima: ($f'(x) = 0$ and $f''(x) < 0$): $Max_1(-2|-\frac{16}{15})$ and $Max_2(1|\frac{38}{15})$ (2)

Inflexion points: ($f''(x) = 0$ and $f'''(x) \neq 0$): $Inf_1(0|0)$ and $Inf_{2/3}(\pm\sqrt{\frac{5}{2}}|\pm\frac{13}{12}\sqrt{\frac{5}{2}})$ (3)

Graph (3)

Exercise 6 (10)

Find the intercepts, extrema and inflexion points of $f(x) = -\frac{1}{150}x^5 + \frac{1}{6}x^3$ and draw its graph.

Solution

$f(x) = -\frac{1}{150}x^5 + \frac{1}{6}x^3 = -\frac{1}{150}x^3(x^2 - 25)$

Symmetry: $f(x) = -f(-x) \Rightarrow$ Symmetry with respect to the origin (1)

Intercepts: $S_{x1}(0|0)$ (triple) and $S_{x2/3}(\pm 5|0)$

Derivatives: $f'(x) = \frac{1}{30}x^4 - \frac{1}{2}x^2 = \frac{1}{30}x^2(x^2 - 15)$, $f''(x) = \frac{2}{15}x^3 - x$ and $f'''(x) = \frac{2}{5}x^2 - 1$ (2)

Extrema: $f'(x) = 0$, $f''(x) <\!/\!> 0 \Rightarrow$ rel $Max(\sqrt{15}|\sqrt{15}) \approx Max(3{,}87|3{,}87)$ and rel $Min(-\sqrt{15}|-\sqrt{15}) \approx Min(-3{,}87|-3{,}87)$ (3)

Saddle point: $f'(0) = 0$, $f''(0) = 0$, $f'''(0) \neq 0 \Rightarrow$ saddle point $S(0|0)$ (1)

Inflexion point: $f''(x) = 0$, $f'''(x) \neq 0) \Rightarrow Inf_{1/2}(\pm\sqrt{\frac{15}{2}}|\mp\frac{21}{24}\sqrt{\frac{15}{2}}) \approx Inf_{1/2}(\pm 2{,}74|\mp 2{,}40)$ (2)

Graph (2)