Analysis

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Contents

1 Differentiation in $\mathbb{R}^n$

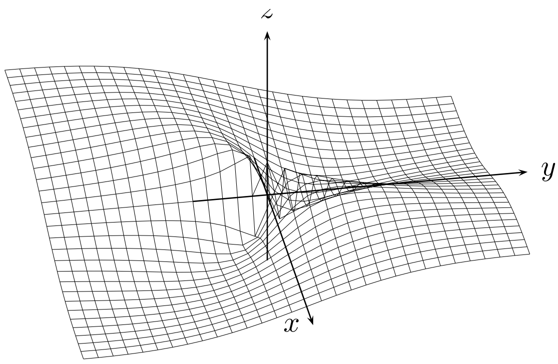
1.1 Shrinking lemma

For $B = B_1(\mathbf{0})$ and every **continuous** $\mathbf{g} : \overline{B} \to \mathbb{R}^n$ with $|\mathbf{g}(\mathbf{x}) - \mathbf{x}| < \epsilon$ for all $\mathbf{x} \in \delta B$ and $0 < \epsilon < 1$ we have $B_{1-\epsilon}(\mathbf{0}) \subset \mathbf{g}[B_1(\mathbf{0})]$.

Proof: Assuming there exists a $\mathbf{y} \in B_{1-\epsilon}(\mathbf{0}) \setminus \mathbf{g}[B]$ we also have $\mathbf{y} \notin \mathbf{g}[\delta B]$ and hence $\mathbf{y} \notin \mathbf{g}\left[\overline{B}\right]$ such that we can define a **continuous** $\mathbf{h} : \overline{B} \to \delta B$ by $\mathbf{h}(\mathbf{x}) = \frac{\mathbf{y}-\mathbf{g}(\mathbf{x})}{|\mathbf{y}-\mathbf{g}(\mathbf{x})|}$. For $\mathbf{x} \in \delta B$ we have $\mathbf{x} \cdot (\mathbf{y} - \mathbf{g}(\mathbf{x})) = \mathbf{x} \cdot \mathbf{y} + \mathbf{x} \cdot (\mathbf{x} - \mathbf{g}(\mathbf{x})) - \mathbf{x} \cdot \mathbf{x} < |\mathbf{y}| + \epsilon - 1 < 0$ and hence $\mathbf{x} \cdot \mathbf{h}(\mathbf{x}) < 0$, particularly $\mathbf{x} \neq \mathbf{h}(\mathbf{x})$ in contradiction to **Brouwer's fixed point theorem** [10, 22.11].

1.2 Differentiable functions

A function $\mathbf{f} : U \to \mathbb{R}^n$ on some **open** $U \subset \mathbb{R}^m$ is **differentiable** at $\mathbf{x} \in U$ iff there exists a **linear** $\frac{d\mathbf{f}}{d\boldsymbol{\xi}}(\mathbf{x}) : \mathbb{R}^m \to \mathbb{R}^n$ with $\lim_{|\mathbf{h}| \to 0} \frac{1}{|\mathbf{h}|} \left| \mathbf{f}(\mathbf{x} + \mathbf{h}) - \mathbf{f}(\mathbf{x}) - \frac{d\mathbf{f}}{d\boldsymbol{\xi}}(\mathbf{x})\,\mathbf{h} \right| = 0$ or equivalently $\mathbf{f}(\mathbf{x} + \mathbf{h}) = \mathbf{f}(\mathbf{x}) + \frac{d\mathbf{f}}{d\boldsymbol{\xi}}(\mathbf{x})\,\mathbf{h} + o(|\mathbf{h}|)$ using **Hardy's small order symbol** $o(x) = \varphi(x)$ for any otherwise unspecified **residue function** $\varphi : \mathbb{R} \to \mathbb{R}$ with $\lim_{x \to 0} \frac{\varphi(x)}{x} = 0$. In the same context we somtimes use **Hardy's large order symbol** $O(x) = x \cdot \varphi(x)$.



In the case of **(total) differentiability** the linear map $\frac{d\mathbf{f}}{d\boldsymbol{\xi}}(\mathbf{x})$ is represented by the **Jacobian matrix** whose elements $\left(\frac{d\mathbf{f}}{d\boldsymbol{\xi}}(\mathbf{x})\right)_{ij} = \frac{\partial f_i}{\partial \xi_j}(\mathbf{x}) = \lim_{h \to 0} \frac{f_i(\mathbf{x}+h\mathbf{e}_j) - f_i(\mathbf{x})}{h}$ are the **partial derivatives** of the **components** in $\mathbf{f} = \sum_{i=1}^{n} f_i \mathbf{e}_i$ according to definition [8, 11.1]. The existence of partial derivatives may not be sufficient for (total) differentiability: The function $f : \mathbb{R}^2 \to \mathbb{R}$ with $f(0;0) = 0$ and $f(x_1; x_2) = \frac{x_1 x_2}{x_1^2 + x_2^2}$ has partial derivatives $\frac{\partial}{\partial \xi_1}(\mathbf{x}) = \frac{x_2\left(x_2^2 - x_1^2\right)}{\left(x_2^2 + x_1^2\right)^2}$ resp. $\frac{\partial}{\partial \xi_2}(\mathbf{x}) = \frac{x_1\left(x_1^2 - x_2^2\right)}{\left(x_1^2 + x_2^2\right)^2}$ and $\frac{\partial}{\partial \xi_1}(\mathbf{0}) = \frac{\partial}{\partial \xi_2}(\mathbf{0}) = 0$ although it is not even continuous at $\mathbf{0}$. The problem is that the linear approximation is **restricted to the direction of the coordinate axes** with gradient zero wheras any deviation from these two direction results in a jump to a singularity.

1.3 Continuous differentiability

A function $\mathbf{f} : U \to \mathbb{R}^n$ on some **open** $U \subset \mathbb{R}^m$ is **continuously differentiable** at $\mathbf{x} \in U$ iff all **partial derivatives** $\frac{\partial f_i}{\partial \xi_j}(\mathbf{x})$ exist and are **continuous**.

Proof:

$\Rightarrow$: The partial derivatives $\frac{\partial f_i}{\partial \xi_j}(\mathbf{x}) = \lim_{h \to 0} \frac{f_i(\mathbf{x}+h\mathbf{e}_j) - f_i(\mathbf{x})}{h} = \lim_{h \to 0} \frac{\langle (\mathbf{f}(\mathbf{x}+|h|\mathbf{e}_j) - \mathbf{f}(\mathbf{x})), \mathbf{e}_i \rangle}{|h|} = \lim_{h \to 0} \frac{1}{|h|} \left\langle \frac{d\mathbf{f}}{d\boldsymbol{\xi}}(\mathbf{x})\,|h|\mathbf{e}_j, \mathbf{e}_i \right\rangle = \left(\frac{d\mathbf{f}}{d\boldsymbol{\xi}}(\mathbf{x})\right)_{ij}$ **exist** and they are **continuous** since $\frac{\partial f_i}{\partial \xi_j}(\mathbf{x}) - \frac{\partial f_i}{\partial \xi_j}(\mathbf{y}) = \left\langle \frac{d\mathbf{f}}{d\boldsymbol{\xi}}(\mathbf{x})\,\mathbf{e}_j, \mathbf{e}_i \right\rangle - \left\langle \frac{d\mathbf{f}}{d\boldsymbol{\xi}}(\mathbf{y})\,\mathbf{e}_j, \mathbf{e}_i \right\rangle = \left\langle \left(\frac{d\mathbf{f}}{d\boldsymbol{\xi}}(\mathbf{x}) - \frac{d\mathbf{f}}{d\boldsymbol{\xi}}(\mathbf{y})\right)\mathbf{e}_j, \mathbf{e}_i \right\rangle$ for $1 \leq j \leq m$ resp. $1 \leq i \leq n$.

$\Leftarrow$: According to the hypothesis for any $\epsilon > 0$ there is a $B_\delta(\mathbf{x}) \subset U$ with $\left|\frac{\partial f_i}{\partial \xi_j}(\mathbf{x}) - \frac{\partial f_i}{\partial \xi_j}(\mathbf{y})\right| < \frac{\epsilon}{m}$ for every $y \in B_\delta(\mathbf{x})$. For $\mathbf{h} = \sum_{j=1}^{m} h_j \mathbf{e}_j$ with $|\mathbf{h}| < \delta$ we define $\mathbf{h}_0 = \mathbf{0}$ and $\mathbf{h}_j = \sum_{k=1}^{j} h_k \mathbf{e}_k$ for $1 \leq j \leq m$ such that $\mathbf{f}(\mathbf{x}+\mathbf{h})-\mathbf{f}(\mathbf{x})=\sum_{j=1}^{m}(\mathbf{f}(\mathbf{x}+\mathbf{h}_j)-\mathbf{f}(\mathbf{x}+\mathbf{h}_{j-1}))$. Since $\mathbf{x}+\mathbf{h}_{j-1}\in B_\delta(\mathbf{x})$ and $B_\delta(\mathbf{x})$ is **convex** the segments $\{p(\mathbf{x}+\mathbf{h}_j)+(1-p)(\mathbf{x}+\mathbf{h}_j):0\le p\le 1\}\subset B_\delta(\mathbf{x})$ whence the **mean value theorem** [8, 12.11] gives a $0\le p_j\le 1$ such that $\mathbf{f}(\mathbf{x}+\mathbf{h}_j)-\mathbf{f}(\mathbf{x}+\mathbf{h}_{j-1})=h_j\cdot\frac{\partial\mathbf{f}}{\partial\xi_j}(\mathbf{x}+\mathbf{h}_{j-1}+p_jh_j\mathbf{e}_j)=h_j\cdot\frac{\partial\mathbf{f}}{\partial\xi_j}(\mathbf{x})+\vartheta_jh_j\cdot\frac{\epsilon}{m}$ for some $0\le\vartheta_j\le 1$ and every $1\le j\le m$. Note that we uase the derivative $\frac{\partial\mathbf{f}}{\partial\xi_j}=\sum_{i=1}^{n}\frac{\partial f_i}{\partial\xi_j}\cdot\mathbf{e}_i$ of the simple vector valued function $p_j\to\mathbf{f}(\mathbf{x}+\mathbf{h}_{j-1}+p_jh_j\mathbf{e}_j)$ whose components coincide with the partial derivatives $\frac{\partial f_i}{\partial\xi_j}$ of $\mathbf{f}$. Summing up we obtain $\left|\mathbf{f}(\mathbf{x}+\mathbf{h})-\mathbf{f}(\mathbf{x})-\sum_{j=1}^{m}h_j\cdot\frac{\partial\mathbf{f}}{\partial\xi_j}(\mathbf{x})\right|\le\frac{\epsilon}{m}\sum_{j=1}^{m}h_j\le|\mathbf{h}|\cdot\epsilon$. Hence f is totally differentiable at $\mathbf{x}$ and the **continuity of all components** $\frac{\partial f_i}{\partial\xi_j}(\mathbf{x})$ extends to the derivative $\frac{d\mathbf{f}}{d\boldsymbol{\xi}}$.

1.4 The product rule

For **differentiable** $\mathbf{f},\boldsymbol{g}:U\to\mathbb{R}^n$ on some **open** $U\subset\mathbb{R}^m$ with a **continuous bilinear** map $\cdot:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^p$ the **product** $\boldsymbol{f}\cdot\boldsymbol{g}$ is again differentiable with $\frac{d\mathbf{f}\cdot\boldsymbol{g}}{d\mathbf{y}}(\mathbf{x})=\frac{d\mathbf{f}}{d\mathbf{y}}(\mathbf{x})\cdot\boldsymbol{g}(\boldsymbol{x})+\boldsymbol{f}(\boldsymbol{x})\cdot\frac{d\mathbf{g}}{d\mathbf{y}}(\mathbf{x})$ meaning the **linear map** $\frac{d\mathbf{f}\cdot\boldsymbol{g}}{d\mathbf{y}}(\mathbf{x}):\mathbb{R}^m\to\mathbb{R}^n$ with $\frac{d\mathbf{f}\cdot\boldsymbol{g}}{d\mathbf{y}}(\mathbf{x})\boldsymbol{y}=\frac{d\mathbf{f}}{d\mathbf{y}}(\mathbf{x})\boldsymbol{y}\cdot\boldsymbol{g}(\boldsymbol{x})+\boldsymbol{f}(\boldsymbol{x})\cdot\frac{d\mathbf{g}}{d\mathbf{y}}(\mathbf{x})\boldsymbol{y}$.

Proof:

$\boldsymbol{f}(\boldsymbol{x}+\boldsymbol{h})\cdot\boldsymbol{g}(\boldsymbol{x}+\boldsymbol{h})-\boldsymbol{f}(\boldsymbol{x})\cdot\boldsymbol{g}(\boldsymbol{x})$

$=\boldsymbol{f}(\boldsymbol{x}+\boldsymbol{h})\cdot\boldsymbol{g}(\boldsymbol{x}+\boldsymbol{h})-\boldsymbol{f}(\boldsymbol{x})\cdot\boldsymbol{g}(\boldsymbol{x}+\boldsymbol{h})+\boldsymbol{f}(\boldsymbol{x})\cdot\boldsymbol{g}(\boldsymbol{x}+\boldsymbol{h})-\boldsymbol{f}(\boldsymbol{x})\cdot\boldsymbol{g}(\boldsymbol{x})$

$=[\boldsymbol{f}(\boldsymbol{x}+\boldsymbol{h})-\boldsymbol{f}(\boldsymbol{x})]\cdot\boldsymbol{g}(\boldsymbol{x}+\boldsymbol{h})+\boldsymbol{f}(\boldsymbol{x})\cdot[\boldsymbol{g}(\boldsymbol{x}+\boldsymbol{h})-\boldsymbol{g}(\boldsymbol{x})]$

$=\left[\frac{d\mathbf{f}}{d\mathbf{y}}(\mathbf{x})\boldsymbol{h}+|\boldsymbol{h}|\varphi_f(\boldsymbol{h})\right]\cdot\boldsymbol{g}(\boldsymbol{x}+\boldsymbol{h})+\boldsymbol{f}(\boldsymbol{x})\cdot\left[\frac{d\mathbf{g}}{d\mathbf{y}}(\mathbf{x})\boldsymbol{h}+|\boldsymbol{h}|\varphi_g(\boldsymbol{h})\right]$

$=\frac{d\mathbf{f}}{d\mathbf{y}}(\mathbf{x})\boldsymbol{h}\cdot\boldsymbol{g}(\boldsymbol{x})+\boldsymbol{f}(\boldsymbol{x})\cdot\frac{d\mathbf{g}}{d\mathbf{y}}(\mathbf{x})\boldsymbol{h}+|\boldsymbol{h}|\left[\frac{d\mathbf{f}}{d\mathbf{y}}(\mathbf{x})\frac{\boldsymbol{h}}{|\boldsymbol{h}|}\cdot[\boldsymbol{g}(\boldsymbol{x}+\boldsymbol{h})-\boldsymbol{g}(\boldsymbol{x})]+\varphi_f(\boldsymbol{h})\cdot\boldsymbol{g}(\boldsymbol{x}+\boldsymbol{h})+\boldsymbol{f}(\boldsymbol{x})\cdot\varphi_g(\boldsymbol{h})\right]$.

with $\varphi_f=\mathbf{f}(\mathbf{x}+\mathbf{h})-\mathbf{f}(\mathbf{x})-\frac{d\mathbf{f}}{d\mathbf{y}}(\mathbf{x})\mathbf{h}$ resp. $\varphi_g=\mathbf{g}(\mathbf{x}+\mathbf{h})-\mathbf{g}(\mathbf{x})-\frac{d\mathbf{g}}{d\mathbf{y}}(\mathbf{x})\mathbf{h}$.

Since $\boldsymbol{g}$ is **continuous** for $|\boldsymbol{h}|\to 0$ the last bracket vanishes and the formula is proven.

1.5 Integration by parts

For $f\in\mathcal{C}^1(U)$, $g\in\mathcal{C}_c^1(U)$ and $1\le i\le n$ we have $\int_U\frac{\partial f}{\partial x_i}(\boldsymbol{x})\cdot g(\boldsymbol{x})\,d\boldsymbol{x}=-\int_U f(\boldsymbol{x})\cdot\frac{\partial g}{\partial x_i}(\boldsymbol{x})\,d\boldsymbol{x}$.

Proof: Every $h\in\mathcal{C}_c^1(U)$ can be extended to $\mathbb{R}^n$ by $h(x)=0$ for every $x\in\mathbb{R}^n\setminus\operatorname{supp}(h)$. Due to the **Heine Borel theorem** [10, 9.10] there is an $R>0$ with $\operatorname{supp}(h)\subset[-R;R]^n$ so that the **fundamental theorem of calculus** [8, 12.10] yields $\int_{-R}^{R}\frac{\partial h}{\partial x_i}(\boldsymbol{x})\,dx_i=h(\boldsymbol{x}+\boldsymbol{e}_i(R-x_i))-h(\boldsymbol{x}-\boldsymbol{e}_i(R+x_i))=0-0=0$ and with **Fubini's theorem** [8, 8.5] we infer $\int_U\frac{\partial h}{\partial x_i}(\boldsymbol{x})\,d\boldsymbol{x}=\int_{[-R;R]}\frac{\partial h}{\partial x_i}(\boldsymbol{x})\,d\boldsymbol{x}=0$. The **product rule** 1.4 applied to $h=f\cdot g\in\mathcal{C}_c^1(U)$ provides $\frac{\partial h}{\partial x_i}(\boldsymbol{x})=\frac{\partial f}{\partial x_i}(\boldsymbol{x})\cdot g(\boldsymbol{x})+f(\boldsymbol{x})\cdot\frac{\partial g}{\partial x_i}(\boldsymbol{x})$ and hence the assertion.

1.6 The chain rule

For **differentiable** $\mathbf{g}:U\to V$ resp. $\mathbf{f}:V\to\mathbb{R}^p$ on some **open** $U\subset\mathbb{R}^m,V\subset\mathbb{R}^n$ the **composition** $\boldsymbol{f}\circ\boldsymbol{g}$ is again differentiable with $\frac{d\mathbf{f}\circ\boldsymbol{g}}{d\mathbf{y}}(\mathbf{x})=\frac{d\mathbf{f}}{d\mathbf{g}}(\boldsymbol{g}(\boldsymbol{x}))\circ\frac{d\mathbf{g}}{d\mathbf{y}}(\mathbf{x})$

Proof: $\boldsymbol{f}(\boldsymbol{g}(\boldsymbol{x}+\boldsymbol{h}))-\boldsymbol{f}(\boldsymbol{g}(\boldsymbol{x}))$

$= \frac{\mathbf{df}}{\mathbf{dg}}(\boldsymbol{g}(\boldsymbol{x}))\,\boldsymbol{k}(\boldsymbol{h})+|\boldsymbol{k}(\boldsymbol{h})|\,\varphi_f(\boldsymbol{k}(\boldsymbol{h}))$

$= \frac{\mathbf{df}}{\mathbf{dg}}(\boldsymbol{g}(\boldsymbol{x}))\circ\frac{\mathbf{dg}}{\mathbf{dy}}(\mathbf{x})\,\boldsymbol{h}+\frac{\mathbf{df}}{\mathbf{dg}}(\boldsymbol{g}(\boldsymbol{x}))\,|\boldsymbol{h}|\,\varphi_g(\boldsymbol{h})+|\boldsymbol{k}(\boldsymbol{h})|\,\varphi_f(\boldsymbol{k}(\boldsymbol{h}))$

$= \frac{\mathbf{df}}{\mathbf{dg}}(\boldsymbol{g}(\boldsymbol{x}))\circ\frac{\mathbf{dg}}{\mathbf{dy}}(\mathbf{x})\,\boldsymbol{h}+|\boldsymbol{h}|\left[\frac{\mathbf{df}}{\mathbf{dg}}(\boldsymbol{g}(\boldsymbol{x}))\,\varphi_g(\boldsymbol{h})+\left(\frac{\mathbf{dg}}{\mathbf{dy}}(\mathbf{x})\,\frac{\boldsymbol{h}}{|\boldsymbol{h}|}+\varphi_g(\boldsymbol{h})\right)\varphi_f(\boldsymbol{k}(\boldsymbol{h}))\right]$

with $\boldsymbol{k}(\boldsymbol{h})=\boldsymbol{g}(\boldsymbol{x}+\boldsymbol{h})-\boldsymbol{g}(\boldsymbol{x})=\frac{\mathbf{dg}}{\mathbf{dy}}(\mathbf{x})\,\boldsymbol{h}+|\boldsymbol{h}|\,\varphi_g(\boldsymbol{h})=|\boldsymbol{h}|\left(\frac{\mathbf{dg}}{\mathbf{dy}}(\mathbf{x})\,\frac{\boldsymbol{h}}{|\boldsymbol{h}|}+\varphi_g(\boldsymbol{h})\right)$**.**

For $|\boldsymbol{h}|\to 0$ we have $\varphi_g(\boldsymbol{h})\to 0$ resp. $\boldsymbol{k}(\boldsymbol{h})\to 0$. Since for $|\boldsymbol{k}|\to 0$ we have $\varphi_f(\boldsymbol{k})\to 0$ the bracket vanishes and the formula is proven.

1.7 The mean value theorem for vector spaces

For every **continuously differentiable** $\boldsymbol{f}: U\to\mathbb{R}^n$ on an **open** and **convex** set $U\subset\mathbb{R}^m$ such that the straight path $\{(1-\tau)\cdot\boldsymbol{a}+\tau\cdot\boldsymbol{b}: 0\leq\tau\leq 1\}\subset U$ there is a $0\leq t\leq 1$ with $\boldsymbol{f}(\boldsymbol{b})-\boldsymbol{f}(\boldsymbol{a})=\frac{d\boldsymbol{f}}{d\boldsymbol{\xi}}((1-t)\cdot\boldsymbol{a}+t\cdot\boldsymbol{b})\cdot(\boldsymbol{b}-\boldsymbol{a})$.

Proof: We combine the **mean value theorem** [8, 12.11] for vector valued functions with the **chain rule** 1.6 for $\boldsymbol{g}:[0;1]\to\mathbb{R}^m$ with $\mathbf{g}(\tau)=(1-\tau)\cdot\boldsymbol{a}+\tau\cdot\boldsymbol{b}$ to obtain $\boldsymbol{f}(\boldsymbol{b})-\boldsymbol{f}(\boldsymbol{a})=\boldsymbol{f}(\boldsymbol{g}(1))-\boldsymbol{f}(\boldsymbol{g}(0))=\frac{\mathbf{df}\circ\boldsymbol{g}}{d\tau}(t)=\frac{\mathbf{df}}{\mathbf{dg}}(\boldsymbol{g}(t))\circ\frac{\mathbf{dg}}{d\tau}(t)=\frac{d\boldsymbol{f}}{d\boldsymbol{y}}((1-t)\cdot\boldsymbol{a}+t\cdot\boldsymbol{b})\cdot(\boldsymbol{b}-\boldsymbol{a})$ for some $0\leq t\leq 1$.

1.8 The inversion lemma

For every **continuously differentiable** $\mathbf{f}: U\to\mathbb{R}^n$ on an **open** set $U\subset\mathbb{R}^n$ with $\frac{\mathbf{df}}{d\boldsymbol{\xi}}(\mathbf{a})=\mathrm{id}\in(\mathbb{R}^n)^*$ at some $\mathbf{a}\in E$ such that $\frac{\mathbf{df}}{d\boldsymbol{\xi}}\left[\overline{B}_\delta(\mathbf{a})\right]\subset\overline{B}_\epsilon(\mathrm{id})$ for an $r>0$ with $\overline{B}_\delta(\mathbf{a})\subset E$ there is an **inverse** $\mathbf{f}^{-1}:\overline{B}_{(1-\epsilon)\delta}(\mathbf{b})\to\overline{B}_\delta(\mathbf{a})$ for $\mathbf{b}=\mathbf{f}(\mathbf{a})$.

Note: The condition $\left\|\frac{\mathbf{df}}{d\boldsymbol{\xi}}(\mathbf{x})-\mathrm{id}\right\|<\epsilon$ refers to the **euclidean norm** $\|\|$ on the **dual space** $(\mathbb{R}^n)^*$.

Proof: For every $\mathbf{y}\in\overline{B}_{(1-\epsilon)\delta}(\mathbf{b})$ we consider the function $\mathbf{g_y}:\mathbb{R}^n\to\mathbb{R}^n$ with $\mathbf{g_y}(\mathbf{x})=\mathbf{x}-\mathbf{f}(\mathbf{x})+\mathbf{y}$ such that $\mathbf{g_y}(\mathbf{a})=\mathbf{a}-\mathbf{b}+\mathbf{y}\in\overline{B}_{(1-\epsilon)\delta}(\mathbf{a})$, $\frac{\mathbf{dg_y}}{d\boldsymbol{\xi}}(\mathbf{a})=\mathbf{0}$ and $\left\|\frac{\mathbf{dg_y}}{d\boldsymbol{\xi}}(\mathbf{x})\right\|\leq\epsilon$ for $\mathbf{x}\in\overline{B}_\delta(\mathbf{a})$. By the preceding **mean value theorem** 1.7 we conclude $\|\mathbf{g_y}(\mathbf{x})-\mathbf{a}+\mathbf{b}-\mathbf{y}\|\leq\epsilon\cdot\|\boldsymbol{x}-\mathbf{a}\|<\epsilon r$ for $\mathbf{x}\in\overline{B}_\delta(\mathbf{a})$, i.e. $\mathbf{g_y}:\overline{B}_\delta(\mathbf{a})\to\overline{B}_{\epsilon\delta}(\mathbf{a}-\mathbf{b}+\mathbf{y})\subset\overline{B}_\delta(\mathbf{a})$ such that we may apply the **contraction principle** [10, 14.12] to infer the existence of an $\mathbf{x}\in\overline{B}_\delta(\mathbf{a})$ with $\mathbf{x}=\mathbf{g_y}(\mathbf{x})$, i.e. $\mathbf{f}(\mathbf{x})=\mathbf{y}$ which proves the theorem.

1.9 The inverse function theorem

For every **continuously differentiable** $\mathbf{f}: E\to\mathbb{R}^n$ on an **open** set $E\subset\mathbb{R}^n$ with **invertible derivative** $\frac{\mathbf{df}}{d\boldsymbol{\xi}}(\mathbf{a})$ at some $\mathbf{a}\in E$ there exist open $U,V\subset\mathbb{R}^n$ with $\mathbf{a}\in U$ and $\mathbf{b}=\mathbf{f}(\mathbf{a})\in V$ such that $\mathbf{f}: U\to V$ is **bijective** and the **inverse** $\mathbf{f}^{-1}: V\to U$ is again **continuously differentiable** on V.

Proof: Since the derivative is **continuous** and **invertible** for $\mathbf{h}=\left(\frac{\mathbf{df}}{d\boldsymbol{\xi}}(\mathbf{a})\right)^{-1}\circ\mathbf{f}$ and every $\epsilon>0$ there is a $\delta>0$ such that $\left\|\frac{\mathbf{dh}}{d\boldsymbol{\xi}}(\mathbf{x})-\mathrm{id}_n\right\|<\epsilon$ for every $\mathbf{x}\in\overline{B}_\delta(\mathbf{a})\subset E$. By the preceding **inversion lemma** 1.8 there is an inverse $\mathbf{h}^{-1}: V\to U$ with for every $\mathbf{y}\in V=B_{(1-\epsilon)\delta}(\mathbf{b})$ and $U=\left(\frac{\mathbf{df}}{d\boldsymbol{\xi}}(\mathbf{a})\right)^{-1}[B_\delta(\mathbf{a})]$. Hence we have an inverse $\mathbf{f}^{-1}=\mathbf{h}^{-1}\circ\left(\frac{\mathbf{df}}{d\boldsymbol{\xi}}(\mathbf{a})\right)^{-1}: B_\delta(\mathbf{a})\to B_{(1-\epsilon)\delta}(\mathbf{b})$.

The inverse $\mathbf{h}^{-1}$ and consequently $\mathbf{f}^{-1}$ are **continuous** since for any $\mathbf{x}_1,\mathbf{x}_2\in\overline{B}_\delta(\mathbf{a})$ another recourse at the **mean value theorem** 1.7 yields

$$\begin{aligned}\|\mathbf{x}_1-\mathbf{x}_2\| &\leq \|\mathbf{h}(\mathbf{x}_1)-\mathbf{h}(\mathbf{x}_2)\|+\|\mathbf{h}(\mathbf{x}_1)-\mathbf{x}_1-(\mathbf{h}(\mathbf{x}_2)-\mathbf{x}_2)\|\\ &\leq \|\mathbf{h}(\mathbf{x}_1)-\mathbf{h}(\mathbf{x}_2)\|+\epsilon\,\|\mathbf{x}_1-\mathbf{x}_2\|\end{aligned}$$

whence

$$\|\mathbf{x}_1-\mathbf{x}_2\| \le \frac{1}{1-\epsilon}\|\mathbf{h}(\mathbf{x}_1)-\mathbf{h}(\mathbf{x}_2)\|$$

The inverse $\mathbf{f}^{-1}$ is **differentiable** with $\frac{d\mathbf{f}^{-1}}{d\boldsymbol{\eta}}(\mathbf{y}) = \left(\frac{d\mathbf{f}}{d\boldsymbol{\xi}} \circ \mathbf{f}^{-1}(\mathbf{y})\right)^{-1}$ since for every $\mathbf{y}, \mathbf{y}_1 \in B_{(1-\epsilon)\delta}(\mathbf{b})$ the **differentiability** and the **continuity** of $\mathbf{f}^{-1}$ give

$$\begin{aligned}
&\left\|\mathbf{f}^{-1}(\mathbf{y}_1)-\mathbf{f}^{-1}(\mathbf{y})-\left(\tfrac{d\mathbf{f}}{d\boldsymbol{\xi}}\circ\mathbf{f}^{-1}(\mathbf{y})\right)^{-1}(\mathbf{y}_1-\mathbf{y})\right\| \\
&=\left\|\mathbf{x}_1-\mathbf{x}-\left(\frac{d\mathbf{f}}{d\boldsymbol{\xi}}(\mathbf{x})\right)^{-1}(\mathbf{f}(\mathbf{x}_1)-\mathbf{f}(\mathbf{x}))\right\| \\
&=\left\|\mathbf{x}_1-\mathbf{x}-\left(\frac{d\mathbf{f}}{d\boldsymbol{\xi}}(\mathbf{x})\right)^{-1}\left(\frac{d\mathbf{f}}{d\boldsymbol{\xi}}(\mathbf{x})(\mathbf{x}_1-\mathbf{x})+o(\|\mathbf{x}-\mathbf{x}_1\|)\right)\right\| \\
&=\left\|\left(\frac{d\mathbf{f}}{d\boldsymbol{\xi}}(\mathbf{x})\right)^{-1}(o(\|\mathbf{x}-\mathbf{x}_1\|))\right\| \\
&=\left\|\left(\frac{d\mathbf{f}}{d\boldsymbol{\xi}}(\mathbf{x})\right)^{-1}\left(o\left(\left\|\mathbf{f}^{-1}(\mathbf{y}_1)-\mathbf{f}^{-1}(\mathbf{y})\right\|\right)\right)\right\| \\
&=\left\|\left(\frac{d\mathbf{f}}{d\boldsymbol{\xi}}(\mathbf{x})\right)^{-1}\right\|\cdot o(\|\mathbf{y}-\mathbf{y}_1\|).
\end{aligned}$$

Finally the derivative $\frac{d\mathbf{f}^{-1}}{d\boldsymbol{\eta}}(\mathbf{y}) = \left(\frac{d\mathbf{f}}{d\boldsymbol{\xi}} \circ \mathbf{f}^{-1}(\mathbf{y})\right)^{-1}$ is **continuous** since its components $\mathbf{f}^{-1} : \mathbb{R}^n \to \mathbb{R}^n$, $\frac{d\mathbf{f}}{d\boldsymbol{\xi}} : \mathbb{R}^n \to (\mathbb{R}^n)^*$ and the inversion $(\mathbb{R}^n)^* \to (\mathbb{R}^n)^*$ with $\|\mathbf{f}^{-1}\| = \frac{1}{\|\mathbf{f}\|}$ are all continuous.

1.10 The implicit function theorem

For every **continuously differentiable** $\mathbf{f} : U \times V \to \mathbb{R}^n$ on **open** $U \subset \mathbb{R}^m$ resp. $V \subset \mathbb{R}^n$ with $\mathbf{f}(\mathbf{a};\mathbf{b}) = 0$ for some $(\mathbf{a};\mathbf{b}) \in U \times V$ and an **invertible partial derivative** $\frac{\partial\mathbf{f}}{\partial\boldsymbol{\eta}}(\mathbf{a};\mathbf{b}) : V \to \mathbb{R}^n$ there exist open $\mathbf{a} \in U_o \subset U$ resp. $\mathbf{b} \in V_o \subset V$ such that there is a **continuously differentiable** $\mathbf{g} : U_0 \to \mathbb{R}^n$ with $\mathbf{g}(\mathbf{a}) = \mathbf{b}$ and $\mathbf{f}(\mathbf{x};\mathbf{g}(\mathbf{x})) = \mathbf{0}$ for every $\mathbf{x} \in U_0$ and $\frac{d\mathbf{g}}{d\boldsymbol{\xi}} = -\left(\frac{\partial\mathbf{f}}{\partial\boldsymbol{\eta}}\right)^{-1} \circ \frac{\partial\mathbf{f}}{\partial\boldsymbol{\xi}}$.

Proof: The function $\mathbf{F} : U \times V \to \mathbb{R}^m \times \mathbb{R}^n$ with $\mathbf{F}(\mathbf{x};\mathbf{y}) = \begin{pmatrix} \mathbf{x} \\ \mathbf{f}(\mathbf{x};\mathbf{y}) \end{pmatrix}$ is **continuously differentiable** with $\frac{d\mathbf{F}}{d(\boldsymbol{\xi};\boldsymbol{\eta})} = \begin{pmatrix} \mathrm{id}_m & \mathbf{0} \\ \frac{\partial\mathbf{f}}{\partial\boldsymbol{\xi}} & \frac{\partial\mathbf{f}}{\partial\boldsymbol{\eta}} \end{pmatrix}$ and **inverse** $\left(\frac{d\mathbf{F}}{d(\boldsymbol{\xi};\boldsymbol{\eta})}\right)^{-1} = \begin{pmatrix} \mathrm{id}_m & \mathbf{0} \\ -\left(\left(\frac{\partial\mathbf{f}}{\partial\boldsymbol{\eta}}\right)^{-1} \circ \frac{\partial\mathbf{f}}{\partial\boldsymbol{\xi}}\right) & \left(\frac{\partial\mathbf{f}}{\partial\boldsymbol{\eta}}\right)^{-1} \end{pmatrix}$ such that we can apply the preceding **inverse function theorem** 1.9 to obtain an **inverse** $\mathbf{F}^{-1}$: $U_0 \times W \to U_0 \times V_0$ with $\mathbf{F}^{-1}(\mathbf{a};\mathbf{0}) = (\mathbf{a};\mathbf{b})$ and $\mathbf{F}^{-1}(\mathbf{x};\mathbf{y}) = \left(\mathbf{x};\mathbf{f}_{\eta}^{-1}(\mathbf{x};\mathbf{y})\right)$ such that $\mathbf{f}_{\eta}^{-1}(\mathbf{x};\mathbf{f}(\mathbf{x};\mathbf{y})) = \mathbf{f}\left(\mathbf{x};\mathbf{f}_{\eta}^{-1}(\mathbf{x};\mathbf{y})\right) = \mathbf{y}$ for $(\mathbf{a};\mathbf{0}), (\mathbf{x};\mathbf{y}) \in U_0 \times W \subset U \times \mathbb{R}^n$ resp. $\mathbf{b} \in V_0 \subset V \subset \mathbb{R}^m$. With $\mathbf{g}(\mathbf{x}) = \mathbf{f}_{\eta}^{-1}(\mathbf{x};\mathbf{0})$ we obtain $(\mathbf{x};\mathbf{f}(\mathbf{x};\mathbf{g}(\mathbf{x}))) = \mathbf{F}(\mathbf{x};\mathbf{g}(\mathbf{x})) = \mathbf{F}\left(\mathbf{x};\mathbf{f}_{\eta}^{-1}(\mathbf{x};\mathbf{0})\right) = (\mathbf{F} \circ \mathbf{F}^{-1})(\mathbf{x};\mathbf{0}) = (\mathbf{x};0)$, i.e. every $\mathbf{x} \in U_0$ is a solution of the equation $\mathbf{f}(\mathbf{x};\mathbf{g}(\mathbf{x})) = \mathbf{0}$. The function $\mathbf{g}$ is **uniquely determined** since from $\mathbf{0} = \mathbf{f}(\mathbf{x};\mathbf{g}(\mathbf{x})) - \mathbf{f}(\mathbf{x};\mathbf{g}'(\mathbf{x})) \in \mathbb{R}^n$ follows $\mathbf{0} = (\mathbf{x};\mathbf{f}(\mathbf{x};\mathbf{g}(\mathbf{x}))) - (\mathbf{x};\mathbf{f}(\mathbf{x};\mathbf{g}'(\mathbf{x}))) \in \mathbb{R}^{m+n}$ whence $\mathbf{0} = \mathbf{F}^{-1}(\mathbf{0}) = (\mathbf{x};\mathbf{g}(\mathbf{x})) - (\mathbf{x};\mathbf{g}'(\mathbf{x}))$ and consequently $\mathbf{g}(\mathbf{x}) = \mathbf{g}'(\mathbf{x})$ for every $\mathbf{x} \in U_0$.

Note: In the **linear case** with $\mathbf{f} = (\mathbf{f}_{\xi};\mathbf{f}_{\eta}) \in M(n \times m+n)$ and $\mathbf{f}_{\xi} \in M(n \times m)$; $\mathbf{f}_{\eta} \in GL(n)$ the equation $\mathbf{f}(\mathbf{x};\mathbf{g}(\mathbf{x})) = \mathbf{0}$ holde for **every** $\mathbf{x} \in \mathbb{R}^m$ with $\mathbf{g} = -\mathbf{f}_{\eta}^{-1} \circ \mathbf{f}_{\xi} : \mathbb{R}^m \to \mathbb{R}^n$.

1.11 Second-order partial derivatives

For every **twice continuously differentiable** $f : U \to \mathbb{R}$ on an **open** set $U \subset \mathbb{R}^2$ we have $\frac{\partial^2 f}{\partial\xi\partial\eta}(x;y) = \frac{\partial^2 f}{\partial\eta\partial\xi}(x;y)$ for every $(x;y) \in U$.

Proof: For $B_\epsilon(x;y) \subset U$ due to **Fubini's theorem** [8, 8.5] and the **continuity** of the two mixed derivatives $\frac{\partial^2 f}{\partial\xi\partial\eta}$ resp. $\frac{\partial^2 f}{\partial\eta\partial\xi}$ on the one hand we have

$$
\begin{aligned}
&f(x+\Delta x, y+\Delta y) - f(x+\Delta x, y) - f(x, y+\Delta y) + f(x;y) \\
&= \int_y^{y+\Delta y} \frac{\partial}{\partial\eta}\left(f(x+\Delta x;\eta) - f(x;\eta)\right) d\eta \\
&= \int_y^{y+\Delta y}\int_x^{x+\Delta x} \frac{\partial^2 f}{\partial\xi\partial\eta} f(\xi;\eta)\, d\xi d\eta \\
&= \int_{[0;1]^2} \frac{\partial^2 f}{\partial\xi\partial\eta} f(x+s\cdot\Delta\xi; y+t\cdot\Delta\eta)\cdot\Delta\xi\cdot\Delta\eta\cdot d\lambda^2 \\
&= \Delta\xi\cdot\Delta\eta\cdot\left(\frac{\partial^2 f}{\partial\xi\partial\eta}(x;y) + \int_{[0;1]^2}\left(\frac{\partial^2 f}{\partial\xi\partial\eta} f(x+s\cdot\Delta\xi; y+t\cdot\Delta\eta) - \frac{\partial^2 f}{\partial\xi\partial\eta} f(x;y)\right) d\lambda^2\right) \\
&= \Delta\xi\cdot\Delta\eta\cdot\left(\frac{\partial^2 f}{\partial\xi\partial\eta}(x;y) + O\left(|\Delta\xi\cdot|\Delta\eta||\right)\right)
\end{aligned}
$$

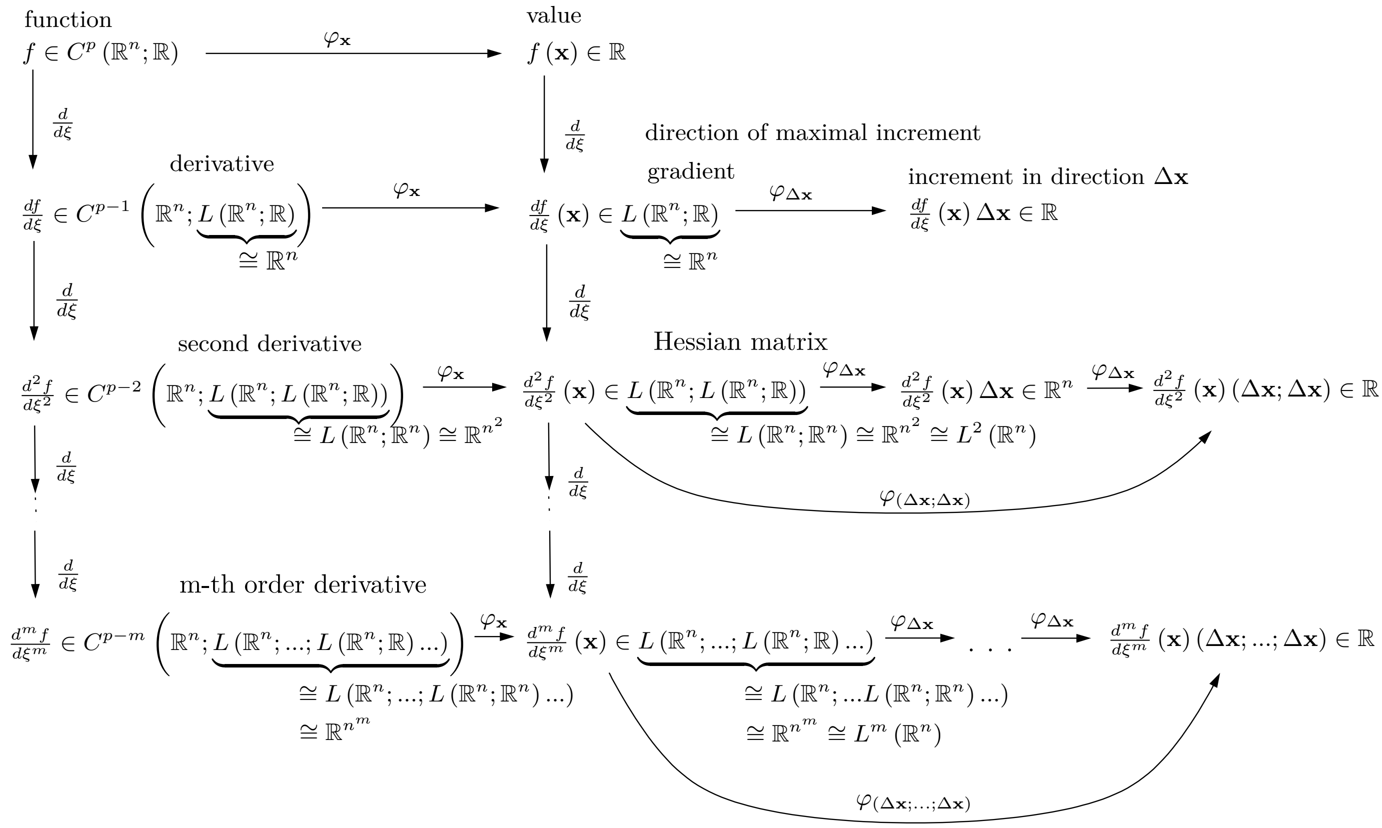
but on the other hand

$$
\begin{aligned}
&f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y) - f(x+\Delta x, y) + f(x;y) \\
&= \int_x^{x+\Delta\xi} \frac{\partial}{\partial\xi}\left(f(\xi;y+\Delta y) - f(\xi;y)\right) d\xi \\
&= \int_x^{x+\Delta\xi}\int_y^{y+\Delta\eta} \frac{\partial^2 f}{\partial\eta\partial\xi} f(\xi;\eta)\, d\eta d\xi \\
&= \int_{[0;1]^2} \frac{\partial^2 f}{\partial\eta\partial\xi} f(x+s\cdot\Delta\xi; y+t\cdot\Delta\eta)\cdot\Delta\xi\cdot\Delta\eta\cdot\lambda^2 \\
&= \Delta\xi\cdot\Delta\eta\cdot\left(\frac{\partial^2 f}{\partial\eta\partial\xi}(x;y) + O\left(|\Delta\xi\cdot|\Delta\eta||\right)\right)
\end{aligned}
$$

Since the two expressions have equal value we conclude $\frac{\partial^2 f}{\partial\eta\partial\xi}(x;y) - \frac{\partial^2 f}{\partial\xi\partial\eta}(x;y) = O\left(|\Delta\xi\cdot|\Delta\eta||\right)$ and hence the assertion.

1.12 Higher derivatives

The repeated differentiation of a p times **continuously differentiable** function $f \in C^p(\mathbb{R}^n;\mathbb{R})$ results in **symmetric** and **mulitlinear** mappings $\frac{df^m}{d\boldsymbol{\xi}^m}(\mathbf{x}) \in L^m\left(\mathbb{R}^{n^m};\mathbb{R}\right)$ with $\frac{df^m}{d\boldsymbol{\xi}^m}(\mathbf{x})(\Delta x_1;...;\Delta x_m) = \frac{df^m}{d\boldsymbol{\xi}^m}(\mathbf{x})\left(\Delta x_{\sigma(1)};...;\Delta x_{\sigma(m)}\right)$ for every permutation $\sigma : \{1;...;m\} \to \{1;...;m\}$. The **gradient** $\frac{df}{d\boldsymbol{\xi}}(\mathbf{x}) \in L(\mathbb{R}^n;\mathbb{R})$ and the **Hessian matrix** $\frac{d^2 f}{d\boldsymbol{\xi}^2}(\mathbf{x}) \in L^2\left(\mathbb{R}^{n^2};\mathbb{R}\right)$ have a special geometric interpretation as direction of **maximal increment** resp. measure for the **curvature** of the plane $\{\mathbf{x}; f(\mathbf{x})\} \subset \mathbb{R}^n \times \mathbb{R}$:



1.13 The Taylor expansion

For every p times **continuously differentiable** function $f \in C^p(U;\mathbb{R})$ on an open $U \subset \mathbb{R}^n$ with $\mathbf{a} \in \{\mathbf{a} + t \cdot \mathbf{x} : 0 \le t \le 1\} \subset U$ and the k-tuple $\mathbf{x}^{(k)} = (\mathbf{x}; ...; \mathbf{x})$ we have the **Taylor expansion**

$$f(\mathbf{a}+\mathbf{x}) = f(\mathbf{a}) + \sum_{k=1}^{p-1} \frac{1}{k!} \cdot \frac{d^k f}{d\boldsymbol{\xi}^k}(\mathbf{a})\,\mathbf{x}^{(k)} + \int_0^1 \frac{(1-t)^{p-1}}{(p-1)!} \cdot \frac{d^p f}{d\boldsymbol{\xi}^p}(\mathbf{a}+t\cdot\mathbf{x})\,\mathbf{x}^{(p)} dt.$$

Proof: By the **fundamental theorem of calculus** [8, 12.10] and **integration by parts** 1.5 with we obtain

$$\begin{aligned}
f(\mathbf{a}+\mathbf{x}) &= f(\mathbf{a}) + \int_0^1 \frac{df}{d\boldsymbol{\xi}}(\mathbf{a}+t\cdot\mathbf{x})\,\mathbf{x}\,dt \\
&= f(\mathbf{a}) + \int_{-1}^0 \frac{df}{d\boldsymbol{\xi}}(\mathbf{a}+(1+s)\cdot\mathbf{x})\,\mathbf{x}\,ds \\
&= f(\mathbf{a}) + \frac{df}{d\boldsymbol{\xi}}(\mathbf{a})\,\mathbf{x} - \int_{-1}^0 s \cdot \frac{d^2 f}{d\boldsymbol{\xi}^2}(\mathbf{a}+(1+s)\cdot\mathbf{x})\,\mathbf{x}\mathbf{x}\,ds \\
&= f(\mathbf{a}) + \frac{df}{d\boldsymbol{\xi}}(\mathbf{a})\,\mathbf{x} + \frac{1}{2}\frac{d^2 f}{d\boldsymbol{\xi}^2}(\mathbf{a})\,\mathbf{x}^{(2)} + \int_{-1}^0 \frac{s^2}{2} \cdot \frac{d^3 f}{d\boldsymbol{\xi}^3}(\mathbf{a}+(1+s)\cdot\mathbf{x})\,\mathbf{x}^{(3)} ds \\
&\vdots \\
&= f(\mathbf{a}) + \sum_{k=1}^{p-1} \frac{1}{k!} \cdot \frac{d^k f}{d\boldsymbol{\xi}^k}(\mathbf{a})\,\mathbf{x}^{(k)} + \int_{-1}^0 \frac{(-s)^{p-1}}{(p-1)!} \cdot \frac{d^p f}{d\boldsymbol{\xi}^p}(\mathbf{a}+(1+s)\cdot\mathbf{x})\,\mathbf{x}^{(p)} ds
\end{aligned}$$

whence follows the assertion.

1.14 Continuity under the integral sign

Let $f : \mathbb{R}^n \times I \to \mathbb{R}$ be a function and $I \subset \mathbb{R}$ an interval such that

1. $t \mapsto f(\boldsymbol{x}; t)$ is **continuous** for every $\boldsymbol{x} \in \mathbb{R}^n$
2. $x \mapsto f(\boldsymbol{x}; t)$ is **integrable** for every $t \in I$
3. There exists a **Lebesgue integrable majorant** $F \in L^1(\mathbb{R}^n; \mathbb{R})$ with $|f(\boldsymbol{x}; t)| \le F(\boldsymbol{x})$ for every $(\boldsymbol{x}; t) \in \mathbb{R}^n \times I$.

Then $g : I \to \mathbb{R}$ with $g(t) = \int f(\boldsymbol{x}; t)\, d\boldsymbol{x}$ is **continuous** on I.

Proof: For every sequence $(t_n)_{n\in\mathbb{N}}$ with $\lim\limits_{n\to\infty} t_n = t$ we have $\lim\limits_{n\to\infty} f(\boldsymbol{x}; t_n) = f(\boldsymbol{x}; t)$ on account of 1. and due to 2. and 3. the **dominated convergence** theorem [8, 5.14] yields

$$\lim_{n\to\infty} g(t_n) = \lim_{n\to\infty} \int f(\boldsymbol{x}; t_n)\, dx = \int \lim_{n\to\infty} f(\boldsymbol{x}; t_n)\, dx = \int f(\boldsymbol{x}; t)\, dx = g(t)$$

and hence the assertion due to [10, 3.2].

1.15 Differentiability under the integral sign

Let $f : \mathbb{R}^n \times I \to \mathbb{R}$ be a function and $I \subset \mathbb{R}$ an interval such that

1. $t \mapsto f(\boldsymbol{x}; t)$ is **differentiable** for every $\boldsymbol{x} \in \mathbb{R}^n$
2. $x \mapsto f(\boldsymbol{x}; t)$ is **integrable** for every $t \in I$
3. There exists a **Lebesgue integrable majorant** $F \in L^1(\mathbb{R}^n; \mathbb{R})$ with $\left|\frac{\delta f}{\delta t}(\boldsymbol{x}; t)\right| \le F(\boldsymbol{x})$ **uniformly in** t for every $(\boldsymbol{x}; t) \in \mathbb{R}^n \times I$.

Then $g : I \to \mathbb{R}$ with $g(t) = \int f(\boldsymbol{x}; t)\, d\boldsymbol{x}$ is **differentiable** on I with $g'(t) = \int \frac{\delta f}{\delta t}(\boldsymbol{x}; t)\, dx$

Proof: For every sequence $(t_n)_{n\in\mathbb{N}}$ with $\lim\limits_{n\to\infty} t_n = t$ we have $\lim\limits_{n\to\infty} \frac{f(\boldsymbol{x};t)-f(\boldsymbol{x};t_n)}{t-t_n} = \frac{\delta f}{\delta t}(\boldsymbol{x}; t)$ on account of 1. and a sequence $(\tau_n)_{n\in\mathbb{N}}$ with $\tau_n \in [t_n; t]$ resp. $[t; t_n]$ and in particular $\lim\limits_{n\to\infty} \tau_n = t$ such that $\frac{f(\boldsymbol{x};t)-f(\boldsymbol{x};t_n)}{t-t_n} = \frac{\delta f}{\delta t}(\boldsymbol{x}; \tau_n)$ according to the **mean value theorem** [8, 12.11]. Owing to 3. we can apply the **dominated convergence** theorem [8, 5.14] to obtain $g'(t) = \lim\limits_{n\to\infty} \frac{g(\boldsymbol{x};t)-g(\boldsymbol{x};t_n)}{t-t_n} = \lim\limits_{n\to\infty} \int \frac{f(\boldsymbol{x};t)-f(\boldsymbol{x};t_n)}{t-t_n} d\boldsymbol{x} = \lim\limits_{n\to\infty} \int \frac{\delta f}{\delta t}(\boldsymbol{x}; \tau_n)\, d\boldsymbol{x} = \int \lim\limits_{n\to\infty} \frac{\delta f}{\delta t}(\boldsymbol{x}; \tau_n)\, d\boldsymbol{x} = \int \frac{\delta f}{\delta t}(\boldsymbol{x}; t)\, d\boldsymbol{x}$ and hence the assertion.

2 Holomorphic functions

2.1 The Taylor expansion for complex functions

A function $f : U \to \mathbb{C}$ on some **open** $U \subset \mathbb{C}$ is **$\mathbb{C}$-differentiable** at $z \in U$ iff there exists a **$\mathbb{C}$-linear** $\frac{df}{d\zeta}(z) : \mathbb{C} \to \mathbb{C}$ with $\lim\limits_{|h|\to 0} \frac{1}{|h|} \left| f(z+h) - f(z) - \frac{df}{d\zeta}(z)\, h \right| = 0$ or equivalently $f(z+h) = f(z) + \frac{df}{d\zeta}(z)\, h + o(|h|)$. If we omit the argument z and write $\frac{df}{d\zeta}$ for $\frac{df}{d\zeta}(z)$ the $\mathbb{C}$-linearity implies $\frac{df}{d\zeta}(c \cdot u) = a \cdot \frac{df}{d\zeta} u$ for arbitrary $c; u \in \mathbb{C}$, in particular $\frac{df}{d\zeta} c = c \cdot \frac{df}{d\zeta} = \frac{df}{d\zeta} \cdot c$ in the case of $u = 1$. Hence the derivative must be a **complex number** $\frac{df}{d\zeta} \in \mathbb{C}$ and the **isomorphism** $\mathbb{C} \to \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} : a; b \in \mathbb{R} \right\} \subset M(2 \times 2; \mathbb{R})$ with $a + ib \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ applied to the derivative yields

$$\frac{df}{d\zeta} \mapsto \begin{pmatrix} \frac{\partial \mathrm{Re} f}{\partial \mathrm{Re} \zeta} & \frac{\partial \mathrm{Re} f}{\partial \mathrm{Im} \zeta} \\ \frac{\partial \mathrm{Im} f}{\partial \mathrm{Re} \zeta} & \frac{\partial \mathrm{Im} f}{\partial \mathrm{Im} \zeta} \end{pmatrix}$$

which implies the **Cauchy-Riemann differential equations**

$$\frac{\partial \mathrm{Re} f}{\partial \mathrm{Re}\zeta} = \frac{\partial \mathrm{Im} f}{\partial \mathrm{Im}\zeta} \text{ and } \frac{\partial \mathrm{Re} f}{\partial \mathrm{Im}\zeta} = -\frac{\partial \mathrm{Im} f}{\partial \mathrm{Re}\zeta}$$

for the **partial derivatives** in the **Jacobian matrix**. With the **differential operators**

$$\partial = \frac{1}{2}\left(\frac{\partial}{\partial \mathrm{Re}\zeta} - i\frac{\partial}{\partial \mathrm{Im}\zeta}\right) \text{ and } \overline{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial \mathrm{Re}\zeta} + i\frac{\partial}{\partial \mathrm{Im}\zeta}\right)$$

they take the form

$$\overline{\partial} f = 0$$

and the total differential becomes

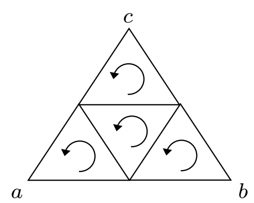
$$\begin{aligned}
\frac{df}{d\zeta} &= \partial f \\
&= \frac{\partial \mathrm{Re} f}{\partial \mathrm{Re}\zeta} + \frac{\partial i \mathrm{Im} f}{\partial \mathrm{Re}\zeta} = \frac{\partial f}{\partial \mathrm{Re}\zeta} \\
&= \frac{1}{2}\left(\frac{\partial \mathrm{Re} f}{\partial \mathrm{Re}\zeta} + \frac{\partial i \mathrm{Im} f}{\partial \mathrm{Re}\zeta} + \frac{\partial \mathrm{Re} f}{\partial i \mathrm{Im}\zeta} + \frac{\partial i \mathrm{Im} f}{\partial i \mathrm{Im}\zeta}\right) = \frac{1}{2}\left(\frac{\partial f}{\partial \mathrm{Re}\zeta} + \frac{\partial f}{\partial i \mathrm{Im}\zeta}\right) \\
&= \frac{\partial \mathrm{Re} f}{\partial i \mathrm{Im}\zeta} + \frac{\partial i \mathrm{Im} f}{\partial i \mathrm{Im}\zeta} = \frac{\partial f}{\partial i \mathrm{Im}\zeta} \\
&\vdots
\end{aligned}$$

i.e. a **directionless complex number** providing $\mathbb{C}$-linear approximations in every direction. Of the preceding theorems the **integration by parts** 1.5 , the **product rule** 1.4, the **chain rule** 1.6, **inverse function theorem** 1.9 and the **implicit function theorem** 1.10 all carry over from $\mathbb{R}^{2n}$ to $\mathbb{C}^n$ under conservation of the $\mathbb{C}$-differentiability. The multilinear higher derivatives in the **Taylor expansion** 1.13 on the **vector space** $\mathbb{R}^2$ are reduced to simple **powers** on the **field** $\mathbb{C}$: For every p times **continuously differentiable** function $f \in C^p(U;\mathbb{C})$ on an open $U \subset \mathbb{C}$ with $a \in \{a + t \cdot z : 0 \le t \le 1\} \subset U$ we have the **Taylor expansion**

$$f(a+z) = f(a) + \sum_{k=1}^{p-1} \frac{1}{k!} \cdot \frac{d^k f}{d\zeta^k}(a)\, z^k + \int_0^1 \frac{(1-t)^{p-1}}{(p-1)!} \cdot \frac{d^p f}{d\zeta^p}(a + t \cdot z)\, z^p dt.$$

2.2 Goursat's theorem

For every $\mathbb{C}$-**differentiable** function $f : U \to \mathbb{C}$ on an **open** set $U \subset \mathbb{C}$and every **triangle** $\Delta \subset U$ we have $\int_{\delta\Delta} f d\lambda = 0$.



Proof: The triangle Δ with the vertices $a; b; c$ may be parametrized by $\Delta(a; b; c) = \{a + s(b-a) + t(c-a) : s; t \in \mathbb{R}^+ \wedge s + t \le 1\}$ such that the integral assumes the form $\int_{\delta\Delta} f d\lambda = \int_0^1 f(a + s(b-a)) \cdot (b-a)\, ds + \int_0^1 f(b + s(c-b)) \cdot (c-b)\, ds + \int_0^1 f(c + s(a-c)) \cdot (a-c)\, ds$. By induction and according to the drawing we decompose the triangle $\Delta_n = \bigcup_{i=1}^{4} \Delta_{n+1}^i$ into four congruent parts Δ_{n+1}^i and choose the component with the maximal integral, i.e. $\Delta_{n+1} = \Delta_{n+1}^j$ such that $\left|\int_{\delta\Delta_{n+1}} f d\lambda\right| \ge \left|\int_{\delta\Delta_{n+1}^i} f d\lambda\right|$ resp. $\left|\int_{\delta\Delta_n} f d\lambda\right| \le 4\left|\int_{\delta\Delta_{n+1}} f d\lambda\right|$. Hence we obtain a **decreasing sequence** $(\Delta_n)_{n \ge 0}$ of closed sets with $\Delta_0 = \Delta$, $\left|\int_{\delta\Delta} f d\lambda\right| \le 4^n \left|\int_{\delta\Delta_n} f d\lambda\right|$, **diameter** $\sup\{d(x;y) : x; y \in \Delta_n\} = \delta(\Delta_n) = \frac{|\delta\Delta|}{2^n}$ and **path length** $\int_{z(t) \in \delta\Delta_n} \left|\frac{dz}{dt}\right| dt = |\delta\Delta_n| = \frac{|\delta\Delta|}{2^n}$. **Cantor's intersection theorem** [10, 14.9.2] yields $\bigcap_{n \ge 0} \Delta_n = z_0 \in \Delta$. Since f has a derivative at z_0 we can find a $\delta > 0$ such that $\left|\frac{f(z) - f(z_0)}{z - z_0} - \frac{df}{dz}(z_0)\right| < \epsilon$ for every

$z \in B_\delta(z_0) \subset U$. For $\delta(\Delta_n) = \frac{\delta(\Delta)}{2^n} < \delta$ we have $z_0 \in \Delta_n \subset B_d(z_o)$ and since due to the **fundamental theorem of calculus** [8, 12.10.2] the integrals $\int_{\delta\Delta_n} 1dz = \int_{\delta\Delta_n} \zeta dz = 0$ vanish over closed paths such that

$$\begin{aligned}
\left|\int_{\delta\Delta} f d\lambda\right| &\leq 4^n \cdot \left|\int_{\delta\Delta_n} f d\lambda\right| \\
&= 4^n \cdot \left|\int_{\delta\Delta_n} \left(f(z) - f(z_0) - \frac{df}{dz}(z_0)(z - z_0)\right) dz\right| \\
&\leq 4^n \cdot \int_{z(t)\in\delta\Delta_n} \left|f(z) - f(z_0) - \frac{df}{dz}(z_0)(z - z_0)\right| \left|\frac{dz}{dt}\right| dt \\
&\leq 4^n \cdot \epsilon \cdot \int_{z(t)\in\delta\Delta_n} |z - z_0| \left|\frac{dz}{dt}\right| dt \\
&\leq 4^n \cdot \epsilon \cdot \delta(\Delta_n) \cdot |\delta\Delta_n| \\
&= \epsilon \cdot \delta(\Delta) \cdot |\delta\Delta|.
\end{aligned}$$

2.3 Morera's theorem

Every **continuous** $f : U \to \mathbb{C}$ on an **open** set $U \subset \mathbb{C}$with $\int_{\delta\Delta} f d\lambda = 0$ for every **triangle** $\Delta \subset U$ has a **primitive** $F : U \to \mathbb{C}$ with $\frac{dF}{dz} = f$.

Proof: Since f is **continuous** for every $\epsilon > 0$ there is a $\delta > 0$ such that $|f(\zeta) - f(z_0)| < \epsilon$ for $\zeta; z_0 \in B_\delta(a) \subset U$. For $z \in B_\delta(a) \subset U$ and the path $[a; z] = \{a + t(z - a) : 0 \leq t \leq 1\}$ define $F(z) = \int_{[a;z]} f d\lambda = \int_0^1 f(a + t(z - a)) \cdot (z - a) dt = \int_{[a;z_0]} f d\lambda + \int_{[z_0;z]} f d\lambda$. Hence

$$\begin{aligned}
\left|\frac{F(z) - F(z_0)}{z - z_0} - f(z_0)\right| &= \left|\frac{1}{z - z_0} \int_{[z_0;z]} (f(\zeta) - f(z_0)) d\zeta\right| \\
&\leq \frac{1}{|z - z_0|} \cdot \sup\{|f(\zeta) - f(z_0)| : \zeta \in [z_0; z]\} \cdot |z - z_0| \\
&= \sup\{|f(\zeta) - f(z_0)| : \zeta \in [z_0; z]\} \\
&< \epsilon.
\end{aligned}$$

2.4 Holomorphic power series

For every **power series**

$$f(a + z) = \sum_{n\in\mathbb{N}} c_n z^n$$

there is a **radius of convergence** $R \geq 0$ with $\frac{1}{R} = \limsup_{n\to\infty} |c_n|^{1/n}$ such that the series converges **absolutely and uniformly** for $z \in \overline{B}_r(0)$ with $r < R$ and **diverges** for any $z \in \mathbb{C} \setminus \overline{B}_R(0)$. In the case of a **positive** radius of convergence $R > 0$ the function $f : B_R(0) \to \mathbb{C}$ is **holomorphic**, i.e. **infinitely often $\mathbb{C}$-differentiable** with

$$\frac{d^k f}{d\zeta^k}(a + z) = \sum_{n\geq k} \frac{n!}{(n-k)!} c_n z^{n-k}, \text{ in particular } \frac{d^k f}{d\zeta^k}(a) = k! \cdot c_k$$

for $z \in B_R(0)$. In the case of $R = \infty$ we speak of an **entire** function.

Proof: The convergence inside and divergence outside of $\overline{B}_r(a)$ are obvious by comparison with the **geometric series** $\sum_{n=0}^{N} x^n = \frac{x^{N+1}-1}{x-1}$ for $x = \frac{z}{|c_n|^{1/n}}$. We show the power series expansion of the derivatives by **induction**: For $k = 0$ there is nothing to prove. For arbitrary $k \geq 1$ assume $g(a + z) = \frac{d^k f}{d\zeta^k}(z) = \sum_{n\geq k} \frac{n!}{(n-k)!} c_n z^{n-k}$ for $z \in B_R(a)$. Then the power series $h(a + z) = \sum_{n\geq k+1} \frac{n!}{(n-k-1)!} c_n z^{n-k-1} = \sum_{n\geq k} (n + 1 - k) \frac{(n+1)!}{(n+1-k)!} c_{n+1} z^{n-k}$ has the same radius of convergence R because $\lim_{n\to\infty} n^{1/n} = \lim_{n\to\infty} e^{1/n\cdot\ln(n)} = 1$. Furthermore for $z, w \in B_r(o)$ and $r < R$ we note that in the following estimate

- the summand $n = k$ vanishes since the two c_k cancel each other out
- the summand $n = k + 1$ vanishes since the bracket n this case assumes the value 0
- the summands $n \geq k + 2$ by **polynomial long division** are divisible by $(z - w)^2$

such that

$$\begin{aligned}\left|\frac{g(a+z)-g(a+w)}{z-w} - h(a+w)\right| &= \left|\sum_{n\geq k+1}\frac{n!}{(n-k)!}c_n\left(\frac{z^{n-k}-w^{n-k}}{z-w} - (n-k)\,z^{n-k-1}\right)\right| \\ &= \left|\sum_{n\geq k+2}\left(\frac{n!}{(n-k)!}c_n\,(z-w)\cdot\sum_{i=1}^{n-k-1} i\cdot w^{i-1}\cdot z^{n-k-i-1}\right)\right| \\ &\leq |z-w|\cdot\left|\sum_{n\geq k+1}\frac{n!}{(n-k)!}\cdot c_n\cdot\frac{(n-k)(n-k-1)}{2}\cdot r^{n-k-2}\right| \\ &= O\left(|z-w|\right)\end{aligned}$$

since due to the induction hypothesis $\sum_{n\geq k+1}\frac{n!}{(n-k)!}\cdot c_n\cdot r^{n-k-2} < \infty$ and the convergence extends to the series above by the **root test** as before.

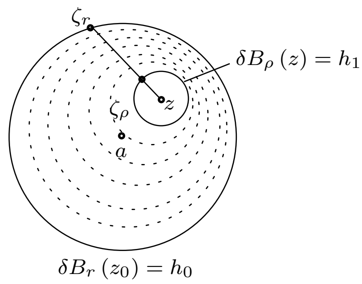
2.5 The local Cauchy formula

For every $\mathbb{C}$-**continuously differentiable** $f : U \to \mathbb{C}$ and every $r > 0$ such that $z \in \overline{B}_r(a) \subset U$ we have $f(z) = \frac{1}{2\pi i}\int_{\delta B}\frac{f(\zeta)}{\zeta - z}d\zeta$ with $\delta B = \overline{B} \setminus B$.

Proof:

1. According to 1.15 the function $g : [0;1] \to \mathbb{C}$ defined by $g(t) = \int_0^{2\pi}\frac{re^{is}}{re^{is}+t(a-z)}ds$ is **continuously differentiable** with $\frac{dg}{d\tau}(t) = \int_0^{2\pi}\frac{(a-z)re^{is}}{(re^{is}+t(a-z))^2}ds = \left[\frac{(a-z)ire^{is}}{re^{is}+t(a-z)}\right]_0^{2\pi} = 0$ and $g(0) = 2\pi$ hence $g(1) = \int_0^{2\pi}\frac{re^{is}}{a+re^{is}-z}ds = 2\pi$.
2. Analogously to 1. and due to the hypothesis $h(t) = \int_0^{2\pi}\frac{f\left(z+t\left(a+re^{is}-z\right)\right)\cdot re^{is}}{a+re^{is}-z}ds$ is **continuously differentiable** and by the **chain rule**1.6 we obtain $\frac{dh}{d\tau}(t) = \int_0^{2\pi}\left(\frac{df}{d\zeta}\left(z+t\left(a+re^{is}-z\right)\right)\cdot re^{is}\right)ds = -\frac{i}{t}\left[f\left(z+t\left(a+re^{is}-z\right)\right)\right]_0^{2\pi} = 0$ for $0 < t \leq 1$. According to 1. we have $h(0) = 2\pi f(z)$ and since $\frac{dh}{d\tau}$ is **continuous** this implies $h(1) = \int_0^{2\pi}\frac{f\left(a+re^{is}\right)\cdot re^{is}}{a+re^{is}-z}ds = 2\pi f(z)$.
3. According to 1. resp. 2. we have $\frac{1}{2\pi i}\int_{\delta B}\frac{f(\zeta)-f(z)}{\zeta-z}d\zeta = h(1) - g(1)\cdot f(z) = 0$ hence $\int_{\delta B}\frac{f(\zeta)}{\zeta-z}d\zeta = \int_{\delta B}\frac{f(z)}{\zeta-z}d\zeta = f(z)\cdot\int_0^{2\pi}\frac{ire^{-is}}{a+re^{-is}-z}ds = 2\pi i\cdot f(z)$.

Note:



The proof is a special case of the **homotopy** principle. For example the circles $\delta B_r(a)$ and $\delta B_\rho(z)$ are **homotopic**, i.e. there is a **continuous**

$$h : [0;1] \times [0;2\pi] \to \overline{B}_r(a) \setminus B_\rho(z)$$

with the starting path

$$h_0(t) = b_{r;a}(t) = a + r\cdot e^{it}$$

continuously shrinking via

$$h_s(t) = s\cdot b_{\rho;z}(t) + (1-s)\cdot b_{r;z_o}(t)$$

to the end path

$$h_1(t) = b_{\rho;z}(t) = z + \rho\cdot\frac{b_{r;a}(t) - z}{|b_{r;a}(t) - z|}.$$

2.6 $\mathbb{C}$-differentiable functions are holomorphic

Every $\mathbb{C}$-**differentiable** $f : U \to \mathbb{C}$ is **holomorphic** with $\frac{d^n f}{d\zeta^n}(z) = \frac{n!}{2\pi \cdot r^n} \int_0^{2\pi} \frac{f\left(z+re^{it}\right)}{e^{int}} dt$ for every $z \in U$ and $r > 0$ with $B_r(z) \subset U$.

Proof: At first we prove the assertiounder the assumption of **continuous** $\mathbb{C}$-**differentiability**. For a fixed $z \in B_r(a) \subset U$ and every $\zeta \in \delta B_r(a)$ we have $\left|\frac{z-a}{\zeta-a}\right| = \frac{|z-a|}{r} < 1$ whence according to 2.1 the **Taylor series** $\frac{1}{\zeta-z} = \frac{1}{(\zeta-a)-(z-a)} = \sum_{n\in\mathbb{N}} \frac{(z-a)^n}{(\zeta-a)^{n+1}}$ converges uniformly and absolutely. Considering again $f(\zeta) \le B$ for $\zeta \in \delta B_r(a)$ and by the **local Cauchy formula** 2.5 (hence the provisional assumption) and **dominated convergence** [10, 5.14] we obtain $f(z) = \frac{1}{2\pi i} \int_{\delta B_r(a)} \frac{f(\zeta)}{\zeta-z} d\zeta = \sum_{n\in\mathbb{N}} c_n (z-a)^n$ with coefficients $c_n = \frac{1}{2\pi i} \int_{\delta B_r(a)} \frac{f(\zeta)d\zeta}{(\zeta-a)^{n+1}} = \frac{1}{2\pi} \int_0^{2\pi} \frac{f\left(a+re^{it}\right)}{re^{int}} \cdot dt$ and convergence radius r. The assertion then follows from 2.4 with $\frac{d^n f}{d\zeta^n}(a) = n! \cdot c_n$.

Due to the theorems of **Goursat** 2.2 and **Morera** 2.3 **every** $\mathbb{C}$-**differentiable function has a continuously** $\mathbb{C}$-**differentiable primitive** which according to the above proved assertion is **holomorphic**. Due to 2.4 the holomorphy extends to every derivative and in particular to f.

2.7 The binomial series

For every $\alpha \in \mathbb{R}$ we define the **generalized binomial coefficients** by $\left(\begin{array}{c}\alpha\\0\end{array}\right) = 1$ and

$$\left(\begin{array}{c}\alpha\\n\end{array}\right) = \frac{\alpha \cdot (\alpha-1) \cdot \ldots \cdot (\alpha-n+1)}{n!}.$$

Then we have $(1+z)^\alpha = \sum_{n=0}^{\infty} \left(\begin{array}{c}\alpha\\n\end{array}\right) z^n$ for $|z| < 1$ and in the case of $\alpha > 0$ also for $|z| = 1$.

Proof: For nonnegatives integers $\alpha \in \mathbb{N}$ the series condenses to the finite binomial sum. Otherwise for $a_n = \left(\begin{array}{c}\alpha\\n\end{array}\right)$ and $n \ge [\alpha] + 1$ we have $\left|\frac{a_{n+1}}{a_n}\right| = \frac{|\alpha-n|}{n+1} = \frac{n-\alpha}{n+1}$, i.e. $na_n - (n+1)a_{n+1} = \alpha a_n > 0$ such that $(na_n)_{n\ge[\alpha]+1}$ is a positive decreasing sequence with $\lim_{n\to\infty}(na_n) < \infty$ whence for $|z| \le 1$ we have $\left|\overset{+}{\underset{n=[\alpha}{1}}]\infty\sum \left(\begin{array}{c}\alpha\\n\end{array}\right) z^n\right| \le \overset{+}{\underset{n=[\alpha}{1}}]\infty\sum a_n = \lim_{n\to\infty} \overset{+}{\underset{k=[\alpha}{1}}]n-1\sum\frac{1}{\alpha}(ka_k - (k+1)a_{k+1}) = \frac{[\alpha]+1}{\alpha} a_{[\alpha]+1} - \lim_{n\to\infty}(na_n) < \infty$. Thus the convergence radius of $f_\alpha(z) = \sum_{n=0}^{\infty} \left(\begin{array}{c}\alpha\\n\end{array}\right) z^n$ is $r \ge 1$ and according to 2.4 for $|z| < 1$ we can differentiate term by term to obtain $\frac{df_\alpha}{d\zeta}(z) = \sum_{n=1}^{\infty} \left(\begin{array}{c}\alpha\\n\end{array}\right) nz^{n-1} = \sum_{n=0}^{\infty} \left(\begin{array}{c}\alpha\\n+1\end{array}\right)(n+1)z^n = \alpha \sum_{n=0}^{\infty} \left(\begin{array}{c}\alpha-1\\n\end{array}\right) z^n = \alpha f_{\alpha-1}(z)$. But we also have $(1+z) f_{\alpha-1}(z) = (1+z) \sum_{n=0}^{\infty} \left(\begin{array}{c}\alpha-1\\n\end{array}\right) z^n = 1 + \sum_{n=1}^{\infty} \left[\left(\begin{array}{c}\alpha-1\\n\end{array}\right) + \left(\begin{array}{c}\alpha-1\\n-1\end{array}\right)\right] z^n = \sum_{n=0}^{\infty} \left(\begin{array}{c}\alpha\\n\end{array}\right) z^n = f_\alpha(z)$ whence $(1+z)\frac{df_\alpha}{d\zeta}(z) = \alpha f_\alpha(z)$. From $\frac{d}{d\zeta}\left(\frac{f_\alpha(z)}{(1+z)^\alpha}\right) = \frac{(1+z)\frac{df_\alpha}{d\zeta}(z) = \alpha f_\alpha(z)}{(1+z)^{\alpha+1}} = 0$ and $\frac{f_\alpha(0)}{(1+0)^\alpha} = 1$ we conclude that $\frac{f_\alpha(z)}{(1+z)^\alpha} = 1$ for $|z| < 1$ resp. $|z| \le 1$ due to [10, 7.13].

2.8 The Leibniz rule

By induction one ca easily prove the following useful extensions of the **product rule**1.4 for higher derivatives of n- th-order $\mathbb{C}$-differentiable functions $f; g : \mathbb{C} \to \mathbb{C}$:

1. $\frac{d^n}{d\zeta^n}(f\cdot g)=\sum_{k=0}^{n}\left(\begin{array}{c}n\\k\end{array}\right)\frac{d^n f}{d\zeta^n}\cdot\frac{d^{n-k}g}{d\zeta^{n-k}}$

2. $f\cdot\frac{d^n g}{d\zeta^n}=\sum_{k=0}^{n}(-1)^k\left(\begin{array}{c}n\\k\end{array}\right)\frac{d^{n-k}f}{d\zeta^{n-k}}\left(\frac{d^k f}{d\zeta^k}\cdot g\right)$

2.9 Liouville's theorem

Every bounded entire function is constant.

Proof: Due to the hypothesis there is an $M>0$ such that $|f|\leq M<\infty$ and owing to the Taylor expansion 2.4we have $\left|\frac{d^n f}{d\zeta^n}\right|\leq\frac{n!\cdot M}{2\pi\cdot r^n}$ for every $r>0$ hence $\frac{d^n f}{d\zeta^n}=0$ whence the assertion follows.

2.10 The fundamental theorem of algebra

Every non constant **polynomial** $p(z)\in\mathbb{C}(z)$ has a complex **root** $a\in\mathbb{C}$ with $p(a)=0$.

Proof: On the one hand $p(z)=\sum_{k=0}^{n}a_k z^k=z^n\cdot\sum_{k=0}^{n}a_k z^{k-n}$ with $a_k\in\mathbb{C}$, $a_n\neq 0$, $n\geq 1$ and $0\leq k\leq n$ is **unbounded**, i.e. $\forall M>0\,\exists R>0:|p(z)|\geq M\,\forall|z|\geq R$ whence $\frac{1}{|p(z)|}\leq\frac{1}{M}\,\forall|z|\geq R$. On the other hand assuming $0\notin p[B_R(0)]$ and since $p:\mathbb{C}\to\mathbb{C}$ is **continuous** there is an $\epsilon<0$ with $|p(z)|>\epsilon$ resp. $\frac{1}{|p(z)|}<\frac{1}{\epsilon}\,\forall|z|<R$ such that the **entire non constant** function $\frac{1}{p}$ is bounded in contradiction to **Liouville's theorem.**

2.11 Conformal maps

A $\mathbb{C}$**-differentiable** function $f:G\to\mathbb{C}$ is **conformal** on a **connected open** set G if $\frac{df}{d\zeta}(z)\neq 0$ for every $z\in G$ and if it **preserves angles** on G, i.e. for any differentiable paths $\gamma_1;\gamma_2:I\to G$ on an interval $I\subset\mathbb{R}$ crossing in $z_0=\gamma_1(t_1)=\gamma_2(t_2)$ we have

$$\arg(f\circ\gamma_1)'(t_1)-\arg(f\circ\gamma_2)'(t_2)=\arg\gamma_1'(t_1)-\arg\gamma_2'(t_2).$$

Theorem: Every **holomorphic** function $f:G\to\mathbb{C}$ with $\frac{df}{d\zeta}(z)\neq 0$ for every $z\in G$ is a **conformal** on G.

Proof: Assuming $\gamma_1'(t_1)\neq\gamma_2'(t_2)\neq 0$ and $z_0=\gamma_1(t_1)=\gamma_2(t_2)$ we have

$$\begin{aligned}\arg(f\circ\gamma_1)'(t_1)-\arg(f\circ\gamma_2)'(t_2)&=\arg\left(f'(\gamma_1(t_1))\cdot\gamma_1'(t_1)\right)-\arg\left(f'(\gamma_2(t_2))\cdot\gamma_2'(t_2)\right)\\&=\arg\left(f'(\gamma_1(t_1))\right)+\arg\left(\gamma_1'(t_1)\right)-\arg\left(f'(\gamma_2(t_2))\right)-\arg\left(\gamma_2'(t_2)\right)\\&=\arg\gamma_1'(t_1)-\arg\gamma_2'(t_2).\end{aligned}$$

Note: The **converse** is also true, i.e. every $\mathbb{C}$-differentiable conformal function $f:\to\mathbb{C}$ on a **connected open** set G if $\frac{df}{d\zeta}(z)\neq 0$ for every $z\in G$ is holomorphic.

2.12 Zeros of a holomorphic function

For a **holomorphic** function $f:G\to\mathbb{C}$ on a **connected open** set $G\subset\mathbb{C}$ the following properties are equivalent:

1. $\{f=0\}$ has a limit point in G.
2. There is an $a\in G$ with $\frac{d^n f}{d\zeta^n}(a)=0$ for every $n\in\mathbb{N}$.
3. $f\equiv 0$.

Proof:

1. $\Rightarrow$ 2.: Since f is **continuous** and G is **open** we have $f(a) = 0$ for the limit point $a \in B_R(a) \subset G$ of $\{f = 0\}$ and some $R > 0$. Hence with the smallest integer n such that $\frac{d^n f}{d\zeta^n}(a) \neq 0$ the **Taylor expansion** in $B_R(a)$ yields $f(z) = \sum_{k \geq n} a_k (z-a)^k$. Then $g(z) = \sum_{k \geq n} a_k (z-a)^{k-n}$ is holomorphic in $B_R(a)$ with $f(z) = (z-a)^n g(z)$ and $g(a) = a_n \neq 0$ such that there is an $r > 0$ with $g(z) \neq 0$ in $B_r(a)$ in contradiction to $f(z) = 0$ for some $z \in B_r(a)$.

2. $\Rightarrow$ 3.: The set $A = \bigcap_{n \in \mathbb{N}} \left\{ \frac{d^n f}{d\zeta^n} = 0 \right\}$ is **closed** in G since for every $z \in \overline{A}$ there is a sequence $(z_k)_{k \in \mathbb{N}} \subset A$ with $\lim_{k \to \infty} z_k = z$ whence $z \in A$ due to the **continuity** of the $\frac{d^n f}{d\zeta^n}$. But A is also **open** in G since for every $a \in A$ there is an $R > 0$ with $B_R(a) \subset G$ such that $f(z) = \sum_{k \geq 0} a_k (z-a)^k$ for $z \in B_R(a)$ with $a_k = \frac{1}{k!} \frac{d^k f}{d\zeta^k}(a) = 0$ for $k \in \mathbb{N}$ whence $B_R(a) \subset \{f = 0\}$ and by repeating the argument from the first part we obtain $B_R(a) \subset A$. Since G is **connected** we conclude $A = G$.

3. $\Rightarrow$ 1. is obvious.

4. **Corollary**: Each **zero** of a **holomorphic** function $f : G \to \mathbb{C}$ on a **connected open** set $G \subset \mathbb{C}$ **not vanishing on** G has **finite multiplicity**, i.e. for $a \in G$ with $f(a) = 0$ there is an $n \in \mathbb{N}$such that $g : G \to \mathbb{C}$ with $g(z) = (z-a)^{-n} \cdot f(z)$ is **holomorphic** on G with $g(a) \neq 0$.

Proof: According to 1 there is an $r > 0$ with $f(z) \neq 0$ for $z \in B_r(a) \setminus \{a\}$ and due to 2 a smallest $n \in \mathbb{N}$with $\frac{d^n f}{d\zeta^n}(a) \neq 0$ whence the argument from 1. $\Rightarrow$ 2. from the preceding proof yields the desired function g.

2.13 The maximum modulus theorem

A **holomorphic** function $f : G \to \mathbb{C}$ on a **connected open** set $G \subset \mathbb{C}$is **constant** iff there is a **maximum modulus** point $a \in G$, i.e. $|f(z)| \leq |f(a)|$ for every $z \in G$.

Proof: According to the hypothesis there is an $R > 0$ with $B_R(a) \subset G$ such that by the **local Cauchy formula** 2.5 we have $|f(a)| = \frac{1}{2\pi i} \int_{\delta B} \frac{f(\zeta)}{\zeta - z} d\zeta = \left| \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{it})\, dt \right| \leq \sup_{z \in \delta B} |f(z)|$ for every $B = B_r(a)$ and $0 < r \leq R$ whence $|f(a)| \leq |f(z)|$ for every $z \in B_R(a)$. Hence $|f(z)| = |f(a)|$ on $B_R(a)$

2.14 Montel's theorem

A family $\mathcal{F}$ of **holomorphic** functions on an **open** set $U \subset \mathbb{C}$ is **normal** iff it is **locally bounded**.

Proof:

$\Rightarrow$: Assume $\mathcal{F}$ fails to be locally bounded, i.e. there is a compact set $K \subset G$ and a sequence $(f_n)_{n \in \mathbb{N}} \subset \mathcal{F}$ with $\sup\{|f_n(z)| : z \in K\} \geq n$. Since $\mathcal{F}$ is **normal** there is a subsequence $(f_{n_k})_{k \in \mathbb{N}}$ converging uniformly on K to a $f \in H(G; \mathbb{C})$ which due to [10, 9.8] resp. **Heine-Borel** [10, 9.10] must be bounded on K contrary to the assumption.

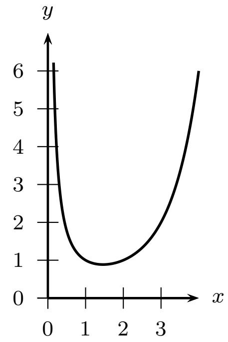
$\Leftarrow$: Owing to **Heine-Borel** [10, 9.10] and the **Arzela-Ascoli theorem** [10, 19.6] we only have to show that $\mathcal{F}$ is **equicontinuous** in every $a \in G$: From the hypothesis there is an $r > 0$ and $m > 0$ such that $\overline{B}_r(a) \subset G$ and $\sup\left\{|f(z)| : z \in \overline{B}\right\} \leq M$ for all $f \in \mathcal{F}$. For $|z - a| < \frac{1}{2}r$, $f \in \mathcal{F}$, $\gamma(t) = a + re^{it}$, $0 \leq t \leq 2\pi$ **Cauchy's integral formula** 2.5 yields $|f(a) - f(z)| \leq \frac{1}{2\pi} \left| \int_\gamma \frac{f(w)(a-z)}{(w-a)(w-z)} dw \right| \leq \frac{2M}{r} |a - z|$, i.e. **Lipschitz continuity** which according to [10, 19.2.1] implies equicontinuity and hence the proposition.

Note: Montel's theorem is used in the proof of the **Riemann mapping theorem** which states that every connected open set in $\mathbb{C}$ is analytically homeomorphic to the unit circle.

3 Some special functions

3.1 The Gamma function

The **Gamma function** $\Gamma : \mathbb{R}^+ \to \mathbb{R}^+$ with $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt = \lim\limits_{n\to\infty} \frac{n!n^x}{x\cdot(x+1)\cdot\ldots\cdot(x+n)}$ is well defined and **uniquely determined** by the following three properties: (**Bohr-Mollerup theorem**)



1. $\Gamma(x+1) = x\cdot\Gamma(x)$
2. $\Gamma(1) = 1$
3. $\ln\Gamma$ is **convex.**

Proof: The three porperties determine a unique function $\Gamma : \mathbb{R}^+ \to \mathbb{R}^+$ with $\Gamma(x) = \frac{n!n^x}{x\cdot(x+1)\cdot\ldots\cdot(x+n)}$: Because of 1. we can confine the argument to $0 < x < 1$. We examine $\ln f$ with $\ln f(1) = 0$ and $\ln f(x+1) = \ln x + \ln f(x)$ resp. $\ln f(n+1+x) = \ln f(x) + \ln[x\cdot(x+1)\cdot\ldots\cdot(x+n)]$ (*). Due to the convexity of $\ln f$ the **difference quotients** $\frac{\ln f(x+h)-\ln f(x)}{h}$ are nondecreasing, whence $\ln n \leq \frac{\ln f(n+1+x)-\ln f(n+1)}{x} \leq \ln n+1$. Substituting (*) yields $0 \leq \ln f(x) - \ln\left[\frac{n!n^x}{x\cdot(x+1)\cdot\ldots\cdot(x+n)}\right] \leq x\ln\left(1+\frac{1}{n}\right)$. Since the last expression tends to 0 as $n\to\infty$ the Function f is uniquely determined by the expression in the middle.

The integral $I(x) = \int_0^\infty t^{x-1}e^{-t}dt$ is well defined for $0 < x < \infty$: We split the domain as well as the range of the integral into $[0;1]$ and $[1;\infty]$: In the case of $x < 1$ we consider $\int_0^1 t^{x-1}e^{-t}dt \overset{5.7}{\leq} \int \lim\limits_{n\to\infty}\chi_{[1/n;0]}\cdot t^{x-1}\cdot e\cdot dt \overset{5.12}{=} \lim\limits_{n\to\infty}\int_{1/n}^1 e\cdot t^{x-1}dt = \lim\limits_{n\to\infty}\frac{e}{x}\left(1-\frac{1}{n^x}\right) = \frac{e}{x} < \infty$ whereas for $x \geq 1$ we have $\int_0^1 t^{x-1}e^{-t}dt \overset{5.7}{\leq} \int_0^1 e^{-t}dt = 1-\frac{1}{e} < \infty$. The remainder converges in any case since $\int_1^\infty t^{x-1}e^{-t}dt \overset{5.7}{\leq} \int_1^\infty \frac{[x+1]!}{t^2}dt = \frac{[x+1]!}{3} < \infty$.

An **integration by parts** [8, 13.5] delivers the **functional equation** $I(x+1) = x\cdot I(x)$ for $0 < x < \infty$. Since $I(1) = 1$ we have $I(n+1) = n!$ for $n \geq 1$. Finally **Hölder's inequality** [8, 6.4.1] with $f(t) = \left(t^{x-1}e^{-t}\right)^{1/p}$ resp. $g(t) = \left(t^{x-1}e^{-t}\right)^{1/q}$ delivers $I\left(\frac{x}{p}+\frac{y}{q}\right) \leq I^{1/p}(x)\cdot I^{1/q}(y)$ for $\frac{1}{p}+\frac{1}{q} = 1$ such that $\ln I$ is **convex** for $0 < x < \infty$. Owing to the first part we have $I = \Gamma$.

3.2 The Beta function

For $x, y > 0$ the **Beta function** is $B(x;y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt = \frac{\Gamma(x)\cdot\Gamma(y)}{\Gamma(x+y)}$.

Proof: As above for each fixed y we find $B\left(\frac{x_1}{p}+\frac{x_2}{q}, y\right) \leq B^{1/p}(x_1)\cdot B^{1/q}(x_2)$ for $\frac{1}{p}+\frac{1}{q} = 1$, i.e. the function $\ln B(x,y)$ is **convex** in x. Also we have $B(1,y) = \frac{1}{y}$ and $B(x+1,y) = \int_0^1\left(\frac{t}{1-t}\right)^x(1-t)^{x+y-1}dt = \frac{x}{x+y}\int_0^1\left(\frac{1}{1-t}\right)^2\left(\frac{t}{1-t}\right)^{x-1}(1-t)^{x+y}dt = \frac{x}{x+y}B(x,y)$. Hence the function $f(x) = \frac{\Gamma(x+y)}{\Gamma(y)}\cdot B(x,y)$ satisfies the three characteristic properties of Γ and we can apply the **Bohr-Mollerup theorem** 3.1 to conclude that $f = \Gamma$.

3.3 Further useful formulae

We have

1. $\frac{\Gamma(x)\cdot\Gamma(y)}{\Gamma(x+y)} \overset{t=\sin^2\vartheta}{=} 2\int_0^{\pi/2}(\sin\vartheta)^{2x-1}\cdot(\cos\vartheta)^{2y-1}d\vartheta$ which in the case of $x = y = \frac{1}{2}$ gives
2. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ and by another application of the Bohr-Mollerup theorem
3. $\Gamma(x) = \frac{2^{x-1}}{\sqrt{\pi}}\Gamma\left(\frac{x}{2}\right)\cdot\Gamma\left(\frac{x+1}{2}\right)$. The original integral yields the identity

4. $\Gamma(x) \overset{t=s^2}{=} 2\int_0^\infty s^{2x-1}e^{-s^2}ds$ which in the case of $x=\frac{1}{2}$ gives
5. **Gauss' integral** $\int e^{-s^2}ds=\sqrt{\pi}$ resp $\int e^{-s^2/2}ds=\sqrt{2\pi}$.
6. **Stirling's formula**: $\lim\limits_{x\to\infty}\frac{\Gamma(x+1)}{\sqrt{2\pi x}\left(\frac{x}{e}\right)^x}=1$

Proof: The substitution $t=x+s\sqrt{2x}$ gives

$\Gamma(x+1)$

$$=\int_0^\infty t^x e^{-t}dt=\sqrt{2x}\left(\frac{x}{e}\right)^x=\int_{-\sqrt{x/2}}^\infty\left(1+s\sqrt{2/x}\right)^x e^{-s\sqrt{2x}}ds=\sqrt{2x}\left(\frac{x}{e}\right)^x\int_{-\sqrt{x/2}}^\infty e^{-s^2\cdot h_x(s)}ds$$

with $h_x(s)=\frac{1}{s^2}\left(s\sqrt{2x}-x\ln\left(1+s\sqrt{2/x}\right)\right)=\frac{2}{\left(s\sqrt{2/x}\right)^2}\left(s\sqrt{2/x}-\ln\left(1+s\sqrt{2/x}\right)\right)$. Thus we have $\lim\limits_{x\to\infty}h_x(s)=\lim\limits_{u\to0}\frac{2}{u^2}(u-\ln(1+u))=\lim\limits_{u\to0}\frac{2}{u^2}\sum\limits_{k=2}^\infty\frac{u^k}{k}=1$. Furthermore $h_x(s)>0$ for every $u>-1$ resp. $s>-\sqrt{x/2}$ and $h_x(s)>1$ for every $u>0$ resp. $s>0$. Hence we can apply **dominated convergence** [8, 5.14] with the **Lebesgue integrable majorant**

$$m_x(s)=\begin{cases}0 & \text{for } s\le-\sqrt{x/2}\\ 1 & \text{for } -\sqrt{x/2}<s<0\\ e^{-s^2} & \text{for } s\ge0\end{cases}$$

to obtain

$$\lim_{x\to\infty}\int_{-\sqrt{x/2}}^\infty e^{-s^2\cdot h_x(s)}ds=\int\lim_{x\to\infty}\chi_{\left[-\sqrt{x/2};\infty\right[}e^{-s^2\cdot h_x(s)}ds=\int e^{-s^2}ds=\sqrt{\pi}$$

and hence the assertion.

3.4 Rotational symmetric functions

For a **compact sphere** $K=\{\boldsymbol{x}\in\mathbb{R}^n:\rho\le|\boldsymbol{x}|\le R\}\subset\mathbb{R}^n$ with $0\le\rho<R$ and every **continuous** $f\in L^1([\rho;R];\mathbb{R})$ we have $\int_K f(|\boldsymbol{x}|)d\boldsymbol{x}=n\cdot\tau_n\cdot\int_\rho^R f(r)\cdot r^{n-1}dr$ with the **volume** $\tau_n=\frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)}$ **of the unit sphere** (cf [8, 8.13]) .

Proof: For $N\ge2$ and $0\le k\le N$ we define $r_k=\rho\frac{k}{N}(R-\rho)$ and corresponding **partitions** of the interval $[\rho;R[$ by $A_k=\{x\in\mathbb{R}^n:r_{k-1}\le|x|<r_k\}$ for $1\le k\le N$. Due to [8, 8.13] we have $\lambda^n(A_k)=\lambda^n(B_{r_k})-\lambda^n(B_{r_{k-1}})=\tau_n\left(r_k^n-r_{k-1}^n\right)$ and according to the **mean value theorem** [8, 12.9] there is a $\xi_k\in]r_{k-1};r_k[$ such that $\lambda^n(A_k)=n\cdot\tau_n\cdot\xi_k^{n-1}\cdot(r_k-r_{k-1})=n\cdot\tau_n\cdot\xi_k^{n-1}\cdot\lambda([r_k;r_{k-1}[)$. For the **step functions** $\psi_N=\sum\limits_{k=1}^N f(\xi_k)\cdot\chi_{A_k}$ resp. $\varphi_N=n\cdot\tau_n\cdot\sum\limits_{k=1}^N f(\xi_k)\cdot\xi_k^{n-1}\cdot\chi_{[r_k;r_{k-1}[}$ and considering the λ^n-null set $\{|\boldsymbol{x}|=R\}$ we then have $\int_K\psi_N d\lambda^n=\int_{[\rho;R]}\varphi_N d\lambda$. Furthermore since $f(|\boldsymbol{x}|)$ resp. $f(r)\cdot r^{n-1}$ are **uniformly continuous** on the **compact** interval $[\rho;R]$ we have **pointwise** $\lim\limits_{N\to\infty}\psi_N(|\boldsymbol{x}|)=f(|\boldsymbol{x}|)$ resp. $\lim\limits_{N\to\infty}\varphi_N(r)=n\cdot\tau_n\cdot f(r)\cdot r^{n-1}$. Since all concerned functions are bounded on the compact set K resp. $[\rho;R]$ we can apply the **dominated convergence theorem** [8, 5.14] twice to obtain $\int_K f(|\boldsymbol{x}|)d\boldsymbol{x}=\int_K\lim\limits_{N\to\infty}\psi_N(|\boldsymbol{x}|)d\boldsymbol{x}=\lim\limits_{N\to\infty}\int_K\psi_N(|\boldsymbol{x}|)d\boldsymbol{x}=\lim\limits_{N\to\infty}\int_{[\rho;R]}\varphi_N(r)dr=\int_{[\rho;R]}\lim\limits_{N\to\infty}\varphi_N(r)dr=n\cdot\tau_n\cdot\int_\rho^R f(r)\cdot r^{n-1}dr$.

3.5 The multidimensional Gauss integral

By 3.4 and **monotone convergence** [8, 5.12] we have $\int_{\mathbb{R}^n}e^{-|\boldsymbol{x}|^2}d\boldsymbol{x}\overset{5.12}{=}\lim\limits_{R\to\infty}\int_{|\boldsymbol{x}|\le R}e^{-|\boldsymbol{x}|^2}d\boldsymbol{x}\overset{2.4}{=}n\cdot\tau_n\cdot\lim\limits_{R\to\infty}\int_0^R e^{-|\boldsymbol{x}|^2}r^{n-1}dr\overset{t=r^2}{=}\frac{n}{2}\cdot\tau_n\cdot2^{n/2}\cdot\lim\limits_{R\to\infty}\int_0^{R^2/2}e^{-t}t^{n/2-1}dt=\frac{n}{2}\cdot\tau_n\cdot\Gamma\left(\frac{n}{2}\right)=\pi^{n/2}$ with the definition resp. functional equation 3.1 of the **Gamma function** as well as the volume $\tau_n = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)} = \frac{\pi^{n/2}}{\frac{n}{2}\cdot\Gamma\left(\frac{n}{2}\right)}$ of the unit sphere according to [8, 8.13].

3.6 Moments of inertia

We compute some moments of inertia $\Theta_{x_1} = \int_K \left(x_2^2 + x_3^2\right) \cdot \rho \cdot d\boldsymbol{x}$ of **compact** bodies $K \subset \mathbb{R}^3$ with constant **densities** $\rho > o$ for rotation around the x_1-axis and in relation thier **mass** $m_K = \int_K \rho \cdot d\boldsymbol{x}$:

1. For the **hollow sphere** $K = \left\{\boldsymbol{x} \in \mathbb{R}^3 : \rho \leq |\boldsymbol{x}| \leq R\right\}$ with mass $m_K = \rho\tau_3\left(R^3 - \rho^3\right)$ we have $\Theta_{x_1} = \int_K \left(x_2^2 + x_3^2\right) \cdot \rho \cdot d\boldsymbol{x} = \frac{2}{3}\int_K \left(x_1^2 + x_2^2 + x_3^2\right) \cdot \rho \cdot d\boldsymbol{x} = \frac{2}{3}\int_K |\boldsymbol{x}|^2 \cdot \rho \cdot d\boldsymbol{x} \overset{15.5}{=} 2\rho\tau_3 \int_\rho^R r^4 dr = \frac{2}{5} m_K \frac{R^5-\rho^5}{R^3-\rho^3}$.
2. For the **ellipsoid** $K = \left\{\boldsymbol{x} \in \mathbb{R}^3 : \left(\frac{x_1}{a_1}\right)^2 + \left(\frac{x_2}{a_2}\right)^2 + \left(\frac{x_3}{a_3}\right)^2 \leq 1\right\}$ with mass $m_K = \int_K \rho \cdot d\boldsymbol{x} \overset{x_i'=x_i/a_i}{=} \rho \int_{|\boldsymbol{x}'|\leq 1} a_1a_2a_3 d\boldsymbol{x}' = a_1a_2a_3\rho\tau_3$ we have
$$\begin{aligned}\Theta_{x_1} &= \int_K \left(x_2^2 + x_3^2\right) \cdot \rho \cdot d\boldsymbol{x} \\ &= a_1a_2a_3\rho \int_{|\boldsymbol{x}'|\leq 1} \left(a_2^2\left(x'\right)_2^2 + a_3^2\left(x'\right)_2^2\right) d\boldsymbol{x}' \\ &= a_1a_2a_3\rho\frac{a_2^2+a_3^2}{3}\int_{|\boldsymbol{x}'|\leq 1} |\boldsymbol{x}'|^2 d\boldsymbol{x}' \\ &\overset{15.5}{=} a_1a_2a_3\rho\tau_3\left(a_2^2+a_3^2\right)\int_0^1 r^4 dr \\ &\frac{1}{5} m_K\left(a_2^2+a_3^2\right).\end{aligned}$$
3. For the **tube** $K = \left\{\boldsymbol{x} \in \mathbb{R}^3 : 0 \leq x_1 \leq a; \rho \leq x_2^2 + x_3^2 \leq R\right\}$ with mass $m_K = \int_K \rho \cdot d\boldsymbol{x} \overset{14.14}{=} a\rho\pi\left(R^2 - \rho^2\right)$ we have $\Theta_{x_1} = \int_K \left(x_2^2 + x_3^2\right) \cdot \rho \cdot d\boldsymbol{x} \overset{14.14}{=} \int_0^a \int_0^{2\pi} \int_\rho^R r^2 \rho dr d\varphi dx_1 = \frac{2}{3} a\rho\pi\left(R^3 - \rho^3\right) = \frac{2}{5} m_K \frac{R^3-\rho^3}{R^2-\rho^2}$.
4. **Steiner's theorem**: For an axis L' with distance d to the axis L through the **center $\boldsymbol{s}$ of gravity** we have $\Theta_{L'} = \Theta_L + md^2$: W.l.o.g we set the center of mass at the **origin** such that $\boldsymbol{s} = \int_K \boldsymbol{x} d\boldsymbol{x} = \boldsymbol{0}$ and particularly $\int_{K_{x_2x_3}} x_1 dx_1 = 0$ on every **cut** $K_{x_2x_3} = \{\boldsymbol{y} \in K : y_2 = x_2; y_3 = x_3\}$. Also we choose $L = \{x_2 = x_3 = 0\}$ resp. $L' = \{x_2 = d_2; x_3 = d_3\}$ such that $\Theta_{L'} \overset{x_i'=x_i+d_i}{=} \int_K \left((x_2+d_2)^2 + (x_3+d_3)^2\right) \cdot \rho \cdot d\boldsymbol{x} = \int_K \left(x_2^2 + x_3^2\right) \cdot \rho \cdot d\boldsymbol{x} + 2\int_K \left(d_2x_2 + d_3x_3\right) \cdot \rho \cdot d\boldsymbol{x} + \int_K \left(d_2^2 + d_3^2\right) \cdot \rho \cdot d\boldsymbol{x} = \Theta_L + 0 + md^2$.

4 Differential equations

In this section we restrict ourselves to one variable and write $\varphi'(t) = \frac{d\varphi}{d\tau}(t)$ resp. $\varphi^{(n)}(t) = \frac{d^n\varphi}{d\tau^n}(t)$.

4.1 The Picard-Lindelöf theorem

For every **first order differential equation** $\mathbf{x}' = \mathbf{f}(t; \mathbf{x})$ with a locally **Lipschitz** (cf. [10, 19.2.1]) **continuous** $\mathbf{f} : U \to \mathbb{C}^n$ on an open $U \subset \mathbb{R} \times \mathbb{C}^n$ and every $(t_0; \mathbf{x}_0) \in U$ there is an $\epsilon > 0$ and a **unique solution** $\mathbf{x} : [t_0 - \epsilon; t_0 + \epsilon] \to \mathbb{C}^n$ with **initial value** $\mathbf{x}(t_0) = \mathbf{x}_0$.

Proof:

Existence: According to the hypothesis there is a $r > 0$ and a **Lipschitz constant** $L > 0$ such that $|\mathbf{f}(\tau; \mathbf{y}) - \mathbf{f}(\tau; \mathbf{x})| \leq L \cdot |(\tau; \mathbf{y}) - (\tau; \mathbf{x})| = L \cdot |\mathbf{y} - \mathbf{x}|$ for every $(\tau; \mathbf{x}), (\tau; \mathbf{y}) \in \overline{B}_r(t_0; \mathbf{x}_0) \subset U$ and also an upper bound $M > 0$ for the continuous function $\mathbf{f}$ on the compact set $\overline{B}_r(t_0; \mathbf{x}_0)$. For

$\epsilon = \min\left\{r; \frac{r}{M}\right\}$ and $I = [t_0 - \epsilon; t_0 + \epsilon]$ the functions $\mathbf{x}_k : I \to \mathbb{C}^n$ with $\mathbf{x}_k(t) = \mathbf{x}_0$ resp. $\mathbf{x}_{k+1}(t) = \mathbf{x}_0 + \int_{t_0}^{t} \mathbf{f}(\tau; \mathbf{x}_k)\, d\tau$ for $k \geq 1$ are **well defined** and **converge to the desired solution** $\mathbf{x} = \lim_{k\to\infty} \mathbf{x}_k$. Both assertions can be proved by **induction**: Concerning the **integrability** we only have to show that $(t; \mathbf{x}_k(t)) \in \overline{B}_r(t_0; \mathbf{x}_0) \subset U$. This is trivial for $k = 0$ and assuming the claim for $k \geq 0$ we have $|\mathbf{x}_{k+1}(t) - \mathbf{x}_0| \leq \left|\int_{t_0}^{t} \mathbf{f}(\tau; \mathbf{x}_k)\, d\tau\right| \leq M \cdot |t - t_0| \leq M\epsilon \leq r$. As to the **convergence** we show that $|\mathbf{x}_{k+1}(t) - \mathbf{x}_k(t)| \leq ML^k \frac{|t-t_0|^{k+1}}{(k+1)!}$ for $t \in I$: The induction start is provided by the estimate above and assuming the assertion for $k \geq 0$ we have $|\mathbf{x}_{k+1}(t) - \mathbf{x}_k(t)| = \left|\int_{t_0}^{t} (\mathbf{f}(\tau; \mathbf{x}_k) - \mathbf{f}(\tau; \mathbf{x}_{k-1}))\, d\tau\right| \leq \int_{t_0}^{t} L \cdot |\mathbf{x}_k(\tau) - \mathbf{x}_{k-1}(\tau)|\, d\tau \leq \frac{ML^k}{k!} \int_{t_0}^{t} |\tau - t_0|^k\, d\tau = ML^k \frac{|t-t_0|^{k+1}}{(k+1)!}$. Owing to this **majorant** the limit $\mathbf{x}(t) = \lim_{k\to\infty} \mathbf{x}_k(t) = \lim_{k\to\infty} \sum_{j=1}^{k} (\mathbf{x}_{j+1}(t) - \mathbf{x}_j(t))$ of this **absolutely and uniformly convergent** series of continuous functions is again a **continuous function** $\mathbf{x} : I \to \mathbb{C}^n$ with $\mathbf{x}(t_0) = \mathbf{x}_0$ and by the **dominated convergence** theorem [8, 5.14] resp. the **continuity** of $\mathbf{f}$ we obtain $\mathbf{x}(t) = \mathbf{x}_0 + \lim_{k\to\infty} \int_{t_0}^{t} \mathbf{f}(\tau; \mathbf{x}_k)\, d\tau = \mathbf{x}_0 + \int_{t_0}^{t} \lim_{k\to\infty} \mathbf{f}(\tau; \mathbf{x}_k)\, d\tau = \mathbf{x}_0 + \int_{t_0}^{t} \lim_{k\to\infty} \mathbf{f}\left(\tau; \lim_{k\to\infty} \mathbf{x}_k\right) d\tau = \mathbf{x}_0 + \int_{t_0}^{t} \lim_{k\to\infty} \mathbf{f}(\tau; \mathbf{x})\, d\tau$. The assertion then follows from the **fundamental theorem of calculus** [8, 12.10].

Uniqueness: By applying the above comparison of subsequent approximations $\mathbf{x}_{k+1}; \mathbf{x}_k$ to two solutions $\mathbf{x}; \mathbf{y}$ we obtain $|\mathbf{y}(t) - \mathbf{x}(t)| = \left|\int_{t_0}^{t} (\mathbf{f}(\tau; \mathbf{y}) - \mathbf{f}(\tau; \mathbf{x}))\, d\tau\right| \leq \int_{t_0}^{t} L \cdot |\mathbf{y}(\tau) - \mathbf{x}(\tau)|\, d\tau \leq L \cdot |t - t_0| \cdot M(t)$ with $M(t) = \sup\left\{|\mathbf{y}(\tau) - \mathbf{x}(\tau)| : \tau \in \overline{B}_t(t_0)\right\}$ for $|t - t_0| < r$. Hence we have $M(t) \leq L \cdot |t - t_0| \cdot M(t)$ whence $M(t) = 0$ for $|t - t_0| < \min\left\{r; \frac{1}{L}\right\}$. It remains to show that $M(t) = 0$ for $|t - t_0| < \epsilon$. Since $\mathbf{x}$ and $\mathbf{y}$ are **continuous** we have $\mathbf{y}(t_1) = \mathbf{x}(t_1)$ for $t_1 = \sup\{t > t_0 : M(t) = 0\}$ and assuming $t_1 < t_0 + \epsilon$ there exists some $r_1 > 0$ such that the **Lipschitz condition** holds in $\overline{B}_{r_1}(t_1; \mathbf{x}(t_1)) \subset U$. But then a repetition of the preceding argument yields $M(t) = 0$ for $|t - t_1| < \min\left\{r_1; \frac{1}{L_1}\right\}$ in contradicition to the definition of x_1. Hence $t_1 \geq t_0 + \epsilon$ which proves the uniqueness.

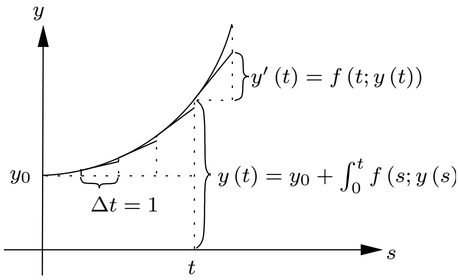
4.2 Examples

1. The **Gauss equation** $x'(t) = -t \cdot x(t)$ for $(t; x) \in \mathbb{R}^2$ with initial value $x(0) = 1$ yields the Picard approximation $x_1(t) = 1 - \int_0^x \tau \cdot 1 d\tau = 1 - \frac{t^2}{2}$, $x_2(t) = 1 - \int_0^x \tau \cdot \left(1 - \frac{\tau^2}{2}\right) d\tau = 1 - \frac{t^2}{2} + \frac{t^4}{2\cdot 4}$,...whence by induction we obtain $x(t) = \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k}}{2\cdot k!} = e^{-t^2/2}$.
2. The **harmonic oscillator equation** $x''(t) = -x(t)$ for $(t; x) \in \mathbb{R}^2$ with initial values $x(0) = 0$ resp. $x'(0) = 1$ is reduced to first order by $\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} x_2(t) \\ -x_1(t) \end{pmatrix}$ with $\mathbf{x}_0(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ so that $\mathbf{x}_1(t) = \begin{pmatrix} t \\ 1 \end{pmatrix}$, $\mathbf{x}_2(t) = \begin{pmatrix} t \\ 1 - \frac{t^2}{2} \end{pmatrix}$, $\mathbf{x}_3(t) = \begin{pmatrix} t - \frac{t^3}{6} \\ 1 - \frac{t^2}{2} \end{pmatrix}$, $\mathbf{x}_4(t) = \begin{pmatrix} t - \frac{t^3}{6} \\ 1 - \frac{t^2}{2} + \frac{t^4}{24} \end{pmatrix}$,...whence by induction we obtain $\mathbf{x}(t) = \begin{cases} \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k+1}}{(2k+1)!} = \sin t \\ \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k}}{(2k)!} = \cos t \end{cases}$.

4.3 Peano's theorem

For every **first order differential equation** $\mathbf{x}' = \mathbf{f}(t; \mathbf{x})$ with a **bounded** and **continuous** $\mathbf{f} : U \to \mathbb{R}^n$ on an open $U \subset \mathbb{R}^{n+1}$and every $(t_0; \mathbf{x}_0) \in U$ there is an $t_1 > t_0$ and **at least one solution** $\mathbf{x} : [t_0; t_1] \to \mathbb{R}^n$ with **initial value** $\mathbf{x}(t_0) = \mathbf{x}_0$.

Proof: The corresponding integral equation $\mathbf{x}(t) = \mathbf{x}_0 + \int_{t_0}^{t} \mathbf{f}(\tau; \mathbf{x}(\tau))\, d\tau$ leads to a **recursive approximation** $\mathbf{x}_n(t) = \mathbf{x}_0 + \int_{t_0}^{t} \mathbf{f}\left(\tau; \mathbf{x}_n\left(\tau - \frac{1}{n}\right)\right) d\tau$ with **stepwidth** $\Delta t = \frac{1}{n}$: Due to the hypothesis there is a $K < \infty$ with $|\mathbf{f}(t; \mathbf{x})| < K$ for $(t; \mathbf{x}) \in U$ and an $\epsilon > 0$ with $B_\epsilon(t_0; \mathbf{x}_0) \subset U$. Choose a $t_1 > t_0$ such that $[t_0; t_1] \times B_{\epsilon K}(\mathbf{x}_0) \subset U$. Then start with $\mathbf{x}_n(t) = \mathbf{x}_0$ for $t \leq t_0$.



For the first step $t_0 \leq \tau \leq t_0 + \frac{1}{n}$ we have $\mathbf{x}_n\left(\tau - \frac{1}{n}\right) = \mathbf{x}_0$ and hence obtain $\mathbf{x}_n(t) = \mathbf{x}_0 + \int_{t_0}^{t} \mathbf{f}(\tau; \mathbf{x}_0)\, d\tau$ for $t_0 \leq t \leq t_0 + \frac{1}{n}$. For the $k+1$-th step we use the previously calculated values $\mathbf{x}_n\left(\tau - \frac{1}{n}\right)$ for $\tau - \frac{1}{n} \leq t_0 + \frac{k}{n}$ to compute $\mathbf{x}_n(t) = \mathbf{x}_0 + \int_{t_0}^{t} \mathbf{f}\left(\tau; \mathbf{x}_n\left(\tau - \frac{1}{n}\right)\right) d\tau = \mathbf{x}_n\left(t_0 + \frac{k}{n}\right) + \int_{t_0 + k/n}^{t} \mathbf{f}\left(\tau; \mathbf{x}_n\left(\tau - \frac{1}{n}\right)\right) d\tau$. Hence we obtain a family $H = (\mathbf{x}_n : [t_0; t_1] \to \mathbb{R}^n)_{n \geq 1}$ which according to [10, 19.2.1] is **equicontinuous** since due to the hypothesis there is a $K \in \mathbb{R}$ with $|\mathbf{f}(t; \mathbf{x})| < K$ for $(t; \mathbf{x}) \in U$ yielding a **Lipschitz condition** $|\mathbf{x}_n(\tau_2) - \mathbf{x}_n(\tau_1)| \leq \left|\int_{\tau_1}^{\tau_2} \mathbf{f}\left(\tau; \mathbf{x}_n\left(\tau - \frac{1}{n}\right)\right) d\tau\right| \leq K \cdot |\tau_2 - \tau_1|$ for $t_0 \leq \tau_1 < \tau_2 \leq t_1$. Since the preceding estimate does not depend on n the family $H(t) \subset \mathbb{R}$ is **bounded** for every $0 \leq t \leq 1$ whence by **Heine-Borel** [10, 9.10] its closure $\overline{H(t)}$ is **compact** such that by **Arzela-Ascoli** [10, 19.6] there is a subsequence $(\mathbf{x}_k)_{k \in I}$ with $I \subset \mathbb{N}$ **uniformly converging** on the compact set $[t_0; t_1]$ to a $\mathbf{x} : [t_0; t_1] \to \mathbb{R}^n$. Hence $\left|\mathbf{x}_k\left(\tau - \frac{1}{n}\right) - \mathbf{x}(\tau)\right| \leq \left|\mathbf{x}_n\left(\tau - \frac{1}{n}\right) - \mathbf{x}_k(\tau)\right| + |\mathbf{x}_n(\tau) - \mathbf{x}(\tau)| \leq \frac{K}{n} + \epsilon$ for large enough k, i.e., $\mathbf{x}_n\left(\tau - \frac{1}{n}\right)$ also converges **uniformly** to $\mathbf{x}(\tau)$ in $[t_0; t_1]$ whence the **continuity** of $\mathbf{f}$ with the **dominated convergence theorem** [9, 5.14] yields $\int_{t_0}^{t} \mathbf{f}(\tau; \mathbf{x}(\tau))\, d\tau = \lim_{n \to \infty} \int_{t_0}^{t} \mathbf{f}\left(\tau; \mathbf{x}_n\left(\tau - \frac{1}{n}\right)\right) d\tau = \lim_{n \to \infty} \mathbf{x}_n(t) - \mathbf{x}_0 = \mathbf{x}(t) - \mathbf{x}_0$.

Note: Peano's theorem only requests a **bounded f** as opposed to a **Lipschitz continuous** one and hence does only yield **compact convergence of a subsequence** instead of **uniform and absolute convergence.** Thus **uniqueness** cannot be guaranteed any more: The differential equation $x'(t) = \sqrt{x(t)}$ for $t \in [0; 1]$ with initial value $x(0) = 0$ according to Peano's theorem has solutions but the Picard-Lindelöf theorem does not apply since $f(t; x) = \sqrt{x(t)}$ is not Lipschitz continuous in any neighbourhood containing 0. Actually there are infinitely many solutions $x_{t_0}(t) = \begin{cases} 0 & t \leq t_0 \\ \frac{(t - t_0)^2}{4} & t > t_0 \end{cases}$ for every $t_0 \in [0; 1]$.

4.4 Linear differential equations

1. For every **linear differential equation** $\mathbf{x}'(t) = A(t)\mathbf{x} + \mathbf{b}(t)$ with **continuous** $A : I \to M(n \times n; \mathbb{C})$ resp. $\mathbf{b} : I \to \mathbb{C}^n$ on an interval $I \subset \mathbb{R}$ and every $(t_0; \mathbf{x}_0) \in I \times \mathbb{C}^n$ there is a **unique solution** $\mathbf{x} : I \to \mathbb{C}^n$ with **initial value** $\mathbf{x}(t_0) = \mathbf{x}_0$.
2. The set $S_h = \{\mathbf{x} : I \to \mathbb{C}^n : \mathbf{x}'(t) = A(t)\mathbf{x}\}$ of solutions for the **homogenous equation** is an n-dimensional vector space over $\mathbb{C}$. A family $(\mathbf{x}_j)_{1 \leq j \leq k} \subset S_h$ is **linearly independent** iff $(\mathbf{x}_j(t_0))_{1 \leq j \leq k} \subset \mathbb{C}^n$ is **linearly independent** for some $t_0 \in I$. A **fundamental system of solutions** is a **basis** $(\mathbf{x}_j)_{1 \leq j \leq n} \subset S_h = \operatorname{span}\{\mathbf{x}_j : 1 \leq j \leq n\}$ of S_h.
3. For every solution $\mathbf{y} \in S_{ih} = \{\mathbf{y} : I \to \mathbb{C}^n : \mathbf{y}'(t) = A(t)\mathbf{y} + \mathbf{b}(t)\}$ of the **inhomogenous equation** we have $S_{ih} = \mathbf{y} + S_h$, i.e. S_{ih} is an **affine space**.
4. **Variation of constants**: For any **fundamental system of solutions** $X = (\mathbf{x}_1; ...; \mathbf{x}_n)$ with **inverse** X^{-1} and every differentiable solution $\mathbf{u} : I \to \mathbb{C}^n$ of the equation $X\mathbf{u}' = \mathbf{b}$ with $\mathbf{u}(t) = \int_{t_0}^{t} X^{-1}(\tau)\, \mathbf{b}(\tau)\, dt + c$ and $c \in \mathbb{C}$ we have $\mathbf{y} = X\mathbf{u} \in S_{ih}$. In the one-dimensional case we simply multiply the given fundamental solution with a "**variable constant**" u to obtain $(ux)' = u'x + ux' = Aux + b$ with $Ax = x'$ whence $u' = \frac{b}{x}$ resp. $u(t) = \int_{t_0}^{t} \frac{b}{x(t)} dt + c$.
5. **Constant coefficients**: For every **linear differential equation** $\mathbf{x}'(t) = A\mathbf{x} + \mathbf{b}$ with **constant coefficients** $A = (a_{ij})_{1 \leq i; j \leq n} \in M(n; \mathbb{C})$ and $\mathbf{b} \in \mathbb{C}^n$ there is a unique solution $x_i(t) = \sum_{j=1}^{n}$

$x_{0i} \cdot e^{a_{ij}t}$ with **initial value** $\mathbf{x}(t_0) = \mathbf{x}_0$.

Proof:

1. Follows from the **Picard-Lindelöf theorem** 4.1 since due to the **continuity** of A on the **compact** set I we have a **global Lipschitz constant** $L = \sup\{|A(t)| : t \in I\} < \infty$ with $|\mathbf{f}(\tau;\mathbf{y}) - \mathbf{f}(\tau;\mathbf{x})| \leq L \cdot |(\tau;\mathbf{y}) - (\tau;\mathbf{x})| = L \cdot |\mathbf{y} - \mathbf{x}|$ for $\mathbf{f}(\tau;\mathbf{y}) = A(\tau)\mathbf{y} + \mathbf{b}(\tau)$. Hence in the proof of the theorem we can choose $r = \sup\{|t - t_0| : t \in I\}$ and since the continuity resp. integrability of $\mathbf{f}$ extends to the entire domain $U = I \times \mathbb{C}^n$ the approximation sequence converges for $\epsilon = r$, i.e. on the whole interval I.
2. Owing to the linearity of the operators $\frac{d}{d\tau}$ on the left hand side and $A(t)$ on the right hand side of the equation its solutions S_h obviously form a **vector space**. If the equation $\sum_{j=1}^{k} \alpha_j \mathbf{x}_j(t_0)$ has a **nontrivial solution** with $\alpha_i \neq 0$ for some $1 \leq i \leq k$ according to 1. the solutions $\mathbf{x}_i(t) = \sum_{1 \leq j \leq k; j \neq i} \frac{\alpha_j}{\alpha_i} \mathbf{x}_j(t)$ must conincide for all $t \in I$. Since there are at most n linearly independent **initial values** $(\mathbf{x}_j(t_0))_{1 \leq j \leq k} \subset \mathbb{C}^n$ it follows that there are at most n linearly independent **solutions** $(\mathbf{x}_j)_{1 \leq j \leq k} \subset S_h$. Hence S_h is of dimension n.
3. $S_{ih} \subset \mathbf{y} + S_h$ since for every $\mathbf{z} \in S_{ih}$ we obviously have $\mathbf{z} - \mathbf{y} \in S_h$ and conversely $\mathbf{y} + S_h \subset S_{ih}$ since for every $\mathbf{x} \in S_h$ clearly $\mathbf{y} + \mathbf{x} \in S_{ih}$.
4. Since $X' = AX$ we have $A\mathbf{y} + \mathbf{b} = AX\mathbf{u} + \mathbf{b} = X'\mathbf{u} + \mathbf{b} = X'\mathbf{u} + X\mathbf{u}' = \mathbf{y}'$.

4.5 Reduction to first order

1. For an interval $I = [a;b] \subset \mathbb{R}$ and $f : I \times \mathbb{C}^n \to \mathbb{C}$ the function $x \in \mathcal{C}^p(I;\mathbb{C})$ is a solution of the n**-th order differential equation** $x^{(n)}(t) = f\left(t; x; x^{(1)}...; x^{(n-1)}\right)$ with initial conditions $\left(x^{(i-1)}(t_0)\right)_{1 \leq i \leq n} = \mathbf{x}_0$ iff the function $\mathbf{x} = (x_1;...;x_n) \in C^p(I;\mathbb{C}^n)$ is a solution of the **first order system of differential equations**

$$\begin{pmatrix} x_1' = x_2 \\ \vdots \\ x_{n-1}' = x_n \\ x_n' = f(t; x_1; ...; x_n) \end{pmatrix},$$

   i.e. $\mathbf{x}' = \mathbf{f}(t;\mathbf{x})$ with $\mathbf{f} : I \times \mathbb{C}^n \to \mathbb{C}^n$ defined by $\mathbf{f}(t;\mathbf{x}) = (x_2;...;x_n; f(t;\mathbf{x}))$ and initial value $\mathbf{x}(t_0) = \mathbf{x}_0$.
2. In the case of a **linear** $f\left(t; x; x^{(1)}...; x^{(n-1)}\right) = \sum_{i=0}^{n} a_i(t) x^{(i)}(t) + b(t)$ the solutions $S_h = \left\{y \in \mathcal{C}^p(I;\mathbb{C}) : y^{(n)} = \sum_{i=0}^{n} a_i y^{(i)}\right\}$ of the **homogenous system** form an n-dimensional vector space over $\mathbb{C}$. A family $(x_j)_{1 \leq j \leq k} \subset S_h$ is **linearly independent** iff the vectors $(\mathbf{x}_j(t_0))_{1 \leq j \leq k} \subset \mathbb{C}^n$ with $\mathbf{x}_j = \left(x_j; x_j^{(1)}; ...; x_j^{(n-1)}\right)$ are **linearly independent** for some $t_0 \in I$. A **fundamental system of solutions** is a **basis** $(\mathbf{x}_j)_{1 \leq j \leq n} \subset S_h = span\{\mathbf{x}_j : 1 \leq j \leq n\}$ of S_h with **invertible Wronskian matrix** $\left(x_j^{(i-1)}\right)_{ij} \in GL(n;\mathbb{C})$.
3. For every solution $y \in S_{ih} = \left\{y \in \mathcal{C}^p(I;\mathbb{C}) : y^{(n)} = \sum_{i=0}^{n} a_i y^{(i)} + b\right\}$ of the **inhomogenous equation** we have $S_{ih} = y + S_h$, i.e. S_{ih} is an **affine space**.

Proof:

1. $\Rightarrow$ is obvious with $x_1 = x$ resp. $x_k = x^{(k-1)}$ for $2 \leq k \leq n$ and $\Leftarrow$ is equally obvious with $x = x_1$.
2. Follows from 1. and 2.
3. Follows from 1. and 3.

4.6 Separation of variables

For every differential equation $x'(t) = f(t) \cdot g(x)$ with **continuous** $f : I \to \mathbb{R}$ resp. $0 \neq g : J \to \mathbb{R}$ on intervals $I; J \subset \mathbb{R}$ and $(t_0; x_0) \in I \times J$ there exists a **unique solution** $x : I' \to \mathbb{R}$ with $x(t_0) = x_0$ and we have $\int_{t_0}^{t} f(\tau)\, d\tau = \int_{x_0}^{x(t)} \frac{d\xi}{g(\xi)} = \int_{t_0}^{t} \frac{x'(\tau)}{g(x(\tau))} d\tau$ for every $t \in I'$ resp. in simplified notation $\frac{dx}{g(x)} = f(t)\, dt$.

Proof: Since $G' = \frac{1}{g} \neq 0$ the function $G : J \to \mathbb{R}$ with $G(x) = \int_{x_0}^{x(t)} \frac{d\xi}{g(\xi)}$ is **strictly monotone** whence by the **inverse function theorem** 1.9 there is a **continuously differentiable inverse** $G^{-1} : G[J] \to \mathbb{R}$. According to 1.15 the function $F : I \to \mathbb{R}$ with $F(t) = \int_{t_0}^{t} f(\tau)\, d\tau$ is also **continuously differentiable**. From $x'(t) = f(t) \cdot g(x)$ follows $F(t) = G(x(t))$ such that we have a unique **continuously differentiable solution** $x = G^{-1} \circ F : I' = (F^{-1} \circ G)[J] \to \mathbb{R}$ with $\frac{1}{g(x)} = G'(x(t)) \cdot x'(t) = F'(t) = f(t)$, i.e. $x'(t) = f(t) \cdot g(x)$.

4.7 Reduction of order

For every **second-order differential equation** $x'' + ax' + bx = 0$ with continuous $a; b : I \to \mathbb{R}$ on an interval $I \subset \mathbb{R}$ and a given solution $0 \neq x : I \to \mathbb{R}$ a **second linearly independent solution** is given by $y = x \cdot u$ with $u(t) = \int_{t_0}^{t} \left(\frac{1}{x^2(\tau)} \exp\left(-\int_{t_0}^{\tau} a(\vartheta)\, d\vartheta \right) \right) d\tau$.

Proof: For non constant u the functions $y = x \cdot u$ and x are linearly independent. We have $u'(t) = \frac{1}{x^2(t)} \exp\left(-\int_{t_0}^{t} a(\tau)\, d\tau \right)$ resp. $u'' = -\left(\frac{2x'}{x} + a \right) \cdot u'$ whence $y' = x' \cdot u + x \cdot u'$ resp. $y'' = x'' \cdot u + 2x' \cdot u' + x \cdot u'' = x'' \cdot u - a \cdot x \cdot u'$. Thus we obtain $y'' + ay' + by = \underline{x'' \cdot u} + a \cdot x \cdot u' + \underline{a \cdot x' \cdot u} - a \cdot x \cdot u' + \underline{b \cdot x \cdot u} = 0$.

4.8 Examples

1. For $t > 0$ and $p = 0$ the **Bessel function** $J_0(t) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{(k!)^2} \left(\frac{t}{2}\right)^{2k}$ and the **Neumann function** $N_0(t) = \frac{2}{\pi} \left(\ln \frac{t}{2} + C\right) J_0(t) + \frac{2}{\pi} \sum_{k=1}^{\infty} (-1)^k \frac{c_k}{(k!)^2} \left(\frac{t}{2}\right)^{2k}$ with $c_k = \sum_{i=1}^{k} \frac{1}{i}$ and the **Euler Mascheroni-constant** $C = \lim_{n \to \infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \ln n \right) \approx 0.5772...$ are solutions of the **Bessel differential equation** $t^2 x'' + tx' + (t^2 - p^2)\, x = 0$.
2. For $t > 0$ the **Laguerre polynomials** $L_n(t) = e^t \frac{d^n}{d\tau^n} \left(\frac{t^n}{e^t} \right)$ are solutions of the **Laguerre differential equation** $tx'' + (1 - t)\, x' + nx = 0$.
3. For $-1 < t < 1$ the **Legendre polynomials** $P_n(t) = \frac{1}{n! 2^n} \frac{d^n}{d\tau^n} (t^2 - 1)^n$ are solutions of the **Legendre differential equation** $(1 - t^2)\, x'' - 2tx' + n(n+1)\, x = 0$.
4. For $t \in \mathbb{R}$ the **Hermite polynomials** $H_n(t) = (-1)^n e^{t^2} \frac{d^n}{d\tau^n} e^{-t^2}$ are solutions of the **Hermite differential equation** $(1 - t^2)\, x'' - 2tx' + n(n+1)\, x = 0$.

4.9 Differential equations with constant coefficients

The linear differential equation

Proof:

4.10 Spectral theorem for linear differential equations

Proof:

5 Fourier transforms

In this section $L^p; C_C^\infty$, etc. stands for $L^p(\mathbb{R}^n;\mathbb{C})$, etc. if not specified otherwise.

5.1 Convolutions

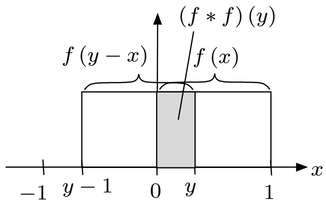
For every $f \in L^1$ and $g \in L^p$ with $1 \le p \le \infty$ their **convolution** $f * g : \mathbb{R}^n \to \mathbb{C}$ defined by $(f*g)(\boldsymbol{x}) = \int f(\boldsymbol{x}-\boldsymbol{y})\, g(\boldsymbol{y})\, d\boldsymbol{y}$ is also p- **integrable** with $\|f*g\|_p \le \|f\|_1 \cdot \|g\|_p$. Since the convolution is **associative**, **commutative** and **bilinear** the space $L^1(\mathbb{R}^n;\mathbb{C};+;*)$ is a **Banach algebra**.

Proof: The algebraic properties of the convolution are obvious resp. (in the case of associativity) tedious. According to the hypothesis resp. the **translation invariance** [8, 8.8] of λ for every $\boldsymbol{x},\boldsymbol{y} \in \mathbb{R}^n$ the functions $y \mapsto |g(\boldsymbol{y}-\boldsymbol{x})|^p$ with $\int |g(\boldsymbol{y}-\boldsymbol{x})|^p\, dy = \|g\|_p^p$ and $\boldsymbol{x} \mapsto \|g\|_p^p \cdot f(\boldsymbol{x})$ are **Lebesgue integrable**. Hence we may apply **Hölder's inequality** [8, 6.4.1] with $\frac{1}{p}+\frac{1}{q} = 1$ and **Fubini's theorem** [8, 8.5] to obtain

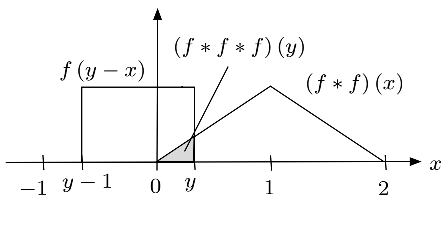
$$
\begin{aligned}
\|f*g\|_p^p &= \int \left| \int |f(\boldsymbol{x})|^{\frac{1}{p}} \cdot |g(\boldsymbol{y}-\boldsymbol{x})| \cdot |f(\boldsymbol{x})|^{\frac{1}{q}}\, d\boldsymbol{x} \right|^p d\boldsymbol{y} \\
&= \int \left| \int f(\boldsymbol{x}) \cdot g(\boldsymbol{y}-\boldsymbol{x})\, dx \right|^p d\boldsymbol{y} \\
&\overset{6.4.1}{\le} \int \left(\int |f(\boldsymbol{x})| \cdot |g(\boldsymbol{y}-\boldsymbol{x})|^p\, d\boldsymbol{x} \cdot \left(\int |f(\boldsymbol{x})|\, d\boldsymbol{x} \right)^{\frac{p}{q}} \right) d\boldsymbol{y} \\
&\overset{8.5}{=} \int \left(\int |g(\boldsymbol{y}-\boldsymbol{x})|^p\, d\boldsymbol{y} \right) \cdot |f(\boldsymbol{x})|\, d\boldsymbol{x} \cdot \|f\|^{\frac{p}{q}} \\
&= \|g\|_p^p \cdot \|f\|_1 \cdot \|f\|^{\frac{p}{q}} \\
&= \|g\|_p^p \cdot \|f\|_1^p .
\end{aligned}
$$

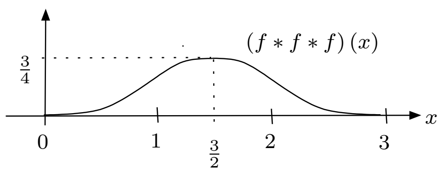
5.2 Examples

We examine the effect of a convolution on $f = \chi_{[0;1]}$:
With $\chi_{[0;1]}(x-y) = \chi_{[x-1;x]}(y)$ we compute



$$
\begin{aligned}
(f*f)(x) &= \int \chi_{[x-1;x]}(y) \cdot \chi_{[0;1]}(y) dy \\
&= \int \chi_{[x-1;x] \cap [0;1]}(y)\, dy \\
&= \begin{cases} 0 & : x < 0 \\ \int \chi_{[0;x]}(y)\, dy = x & : 0 \le x < 1 \\ \int \chi_{[x-1;1]}(y) dy = 2 - x & : 1 \le x < 2 \\ 0 & : 2 \le x \end{cases} \\
&= \sup\{0; 2 - |x|\}
\end{aligned}
$$

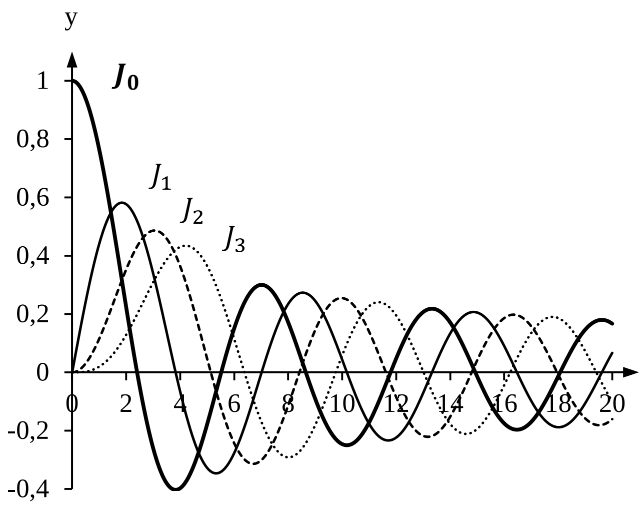




$$(f*f*f)(x)=\begin{cases}0 & : x<0\\ \int \chi_{[x-1;x]\cap[0;1]}(y)\cdot y\cdot dy & : 0\leq x<1\\ \int \chi_{[x-1;x]\cap[1;2]}(y)\cdot(2-y)\cdot dy & : 1\leq x<2\\ 0 & : 2\leq x\end{cases}$$

$$=\begin{cases}0 & : x<0\\ \int \chi_{[0;x]}(y)\cdot y\cdot dy=\frac{1}{2}x^2 & : 0\leq x<1\\ \int \chi_{[x-1;1]}(y)\cdot y\cdot dy+\int \chi_{[1;x]}(2-y)\cdot dy=-\left(x-\frac{3}{2}\right)^2+\frac{3}{4} & : 1\leq x<2\\ \int \chi_{[x-1;2]}(2-y)\cdot dy=\frac{1}{2}(x-3)^2 & : 2\leq x<3\\ 0 & : 3\leq x\end{cases}$$

5.3 Bessel functions



For $p\geq 0$ the **Bessel functions of the first kind** $J_p(x)=\frac{1}{\sqrt{\pi}}\frac{\left(\frac{x}{2}\right)^p}{\Gamma\left(p+\frac{1}{2}\right)}\int_0^\pi \sin^{2p}t\cdot e^{-ix\cdot\cos t}dt$ are generalized trigonometric functions solving the radial component of the **Laplace equation** in cylindrical coordinates as well as the radial component of the **Helmholtz equation** in spherical coordinates, i.e. the **Bessel differential equation**

$$\frac{\delta^2 J_p}{\delta r^2}(r)+\frac{1}{r}\frac{\delta J_p}{\delta r}(r)+\left(1-\frac{p^2}{r^2}\right)J_p(r)=0.$$

We have the explicit formula

$$J_{1/2}(x)=\sqrt{\frac{2}{\pi}}\cdot\frac{\sin x}{\sqrt{x}}$$

and the recursive formula

$$J_{p+1}(x)=-\frac{\delta J_p}{\delta x}(x)+\frac{p}{x}J_p(x)$$

.

Proof: Preliminarily we examine the functions $f_p(x)=\int_0^\pi \sin^{2p}t\cdot e^{-x\cdot\cos t}dt$ with $\frac{\delta^k f_p}{\delta r^k}(x)=(-i)^k\cdot\int_0^\pi \cos^k t\cdot\sin^{2p}t\cdot e^{-x\cdot\cos t}dt$. With $\cos^2 t=1-\sin^2 t$ and omitting the argument x for brevity we obtain $\frac{\delta^2 f_p}{\delta x^2}=f_{p+1}-f_p$ (*). In order to express f_{p+1} in terms of $\frac{\delta f_p}{\delta x}$ we compute

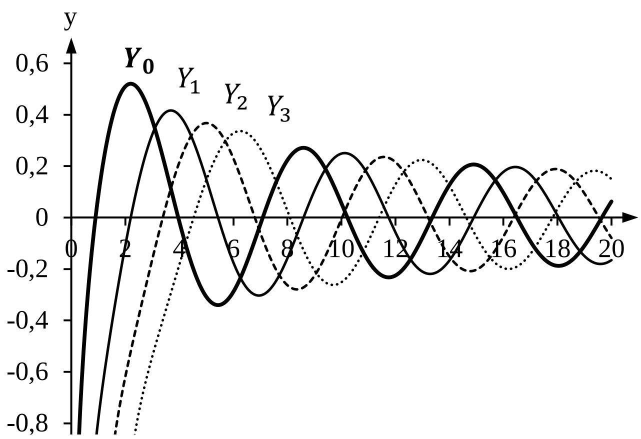
$$\begin{aligned}\frac{\delta f_p}{\delta x}&=-i\cdot\int_0^\pi \cos t\cdot\sin^{2p}t\cdot e^{-x\cdot\cos t}dt\\ &=0+i\cdot\int_0^\pi \sin t\cdot\left(\cos t\cdot 2p\cdot\sin^{2p-1}t+\sin^{2p}t\cdot x\cdot\sin t\right)\cdot e^{-x\cdot\cos t}dt\\ &=-2p\cdot\frac{\delta f_p}{\delta x}-x\cdot f_{p+1}\end{aligned}$$

resp. $(2p+1)\frac{\delta f_p}{\delta x}=-x\cdot f_{p+1}$ (**). Hence we can substitute f_{p+1} in equation (*) to obtain $\frac{\delta^2 f_p}{\delta x^2}+\frac{2p+1}{x}\cdot\frac{\delta f_p}{\delta x}+f_p=0$. Since $J_p=c_p\cdot x^p\cdot f_p$ and dividing by $c_p=\frac{1}{\sqrt{\pi}}\frac{1}{2^p\cdot\Gamma\left(p+\frac{1}{2}\right)}$ we have

$$\frac{\delta^2 J_p}{\delta x^2} + \frac{1}{x}\frac{\delta J_p}{\delta x} + \left(1 - \frac{p^2}{x^2}\right) J_p = \left(p\,(p-1)\cdot x^{p-2}\cdot f_p + 2p\cdot x^{p-1}\cdot \frac{\delta f_p}{\delta x} + x^p \cdot \frac{\delta^2 f_p}{\delta x^2}\right)$$
$$+ \frac{1}{x}\cdot\left(p\cdot x^{p-1}\cdot f_p + x^p\cdot\frac{\delta f_p}{\delta x}\right) + \left(1-\frac{p^2}{x^2}\right)\cdot x^p\cdot f_p$$
$$= x^p\cdot\frac{\delta^2 f_p}{\delta x^2} + \left(2p\cdot x^{p-1} + x^{p-1}\right)\cdot\frac{\delta f_p}{\delta x}$$
$$+\left(p\,(p-1)\cdot x^{p-2} + p\cdot x^{p-2} + x^p - p^2 x^{p-2}\right)\cdot f_p$$
$$= x^p\cdot\left(\frac{\delta^2 f_p}{\delta x^2} + \frac{2p+1}{x}\cdot\frac{\delta f_p}{\delta x} + f_p\right)$$
$$= 0$$

For $p = \frac{1}{2}$ an elementary integration yields $J_{1/2}(x) = \sqrt{\frac{x}{2\pi}}\cdot\int_0^\pi \sin t\cdot e^{-ix\cdot\cos t}dt = -\sqrt{\frac{x}{2\pi}}\cdot\int_0^\pi e^{-ix\cdot\cos t}d\cos t = -\sqrt{\frac{1}{2\pi x}}\cdot i\left[e^{-ix\cdot\cos t}\right]_0^\pi = -\sqrt{\frac{1}{2\pi x}}\cdot i\left[e^{ix\cdot} - e^{-ix\cdot}\right]_0^\pi = \sqrt{\frac{1}{2\pi x}}\cdot 2\sin x = \sqrt{\frac{2}{\pi}}\cdot\frac{\sin x}{\sqrt{x}}$. The recusive formula is derived from (**) considering the **functional equation of the Gamma function** $\Gamma\left(p+\frac{3}{2}\right) = \left(p+\frac{1}{2}\right)\cdot\Gamma\left(p+\frac{1}{2}\right)$ by substituting $(2p+1)\left(2^p\cdot\Gamma\left(p+\frac{1}{2}\right)\right)\frac{\delta}{\delta x}\left(x^{-p}\cdot J_p\right) = -x^{-p}\cdot\left(2^{p+1}\cdot\Gamma\left(p+\frac{3}{2}\right)\right)J_{p+1}$ whence $(2p+1)\left(-p\cdot x^{-p-1}\cdot J_p + x^{-p}\cdot\frac{\delta J_p}{\delta x}\right) = -x^{-p}\cdot 2\left(p+\frac{1}{2}\right)\cdot J_{p+1}$ resulting in the desired formula.

Note: For **noninteger** $p \in \mathbb{R}\setminus\mathbb{Z}$ the functions J_p and J_{-p} are linearly independent and form a basis for the solutions of the second order differential Bessel equation. In the case of integer $n \in \mathbb{Z}$ we have $J_{-n} = (-1)^n J_n$ and need the **Bessel functions of the second kind** resp. **Weber** resp. **Neumann functions** $Y_p(x) = \frac{J_p(x)\cdot\cos(p\pi) - J_{-p}(x)}{\sin(p\pi)}$ for noninteger $p\in\mathbb{R}\setminus\mathbb{Z}$ and $Y_n(x) = \lim_{p\to n} Y_p(x)$ to define the vector space of solutions of the Bessel equation.

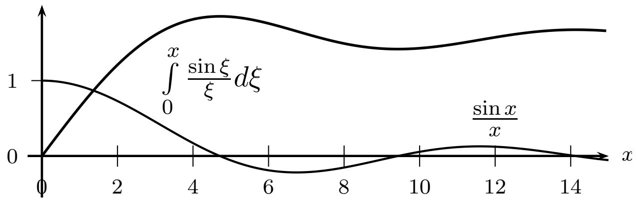


5.4 Fourier transforms

Due to [8, 5.18] for every **(Lebesgue) integrable** $f:\mathbb{R}^n\to\mathbb{C}$ the product $\boldsymbol{x}\mapsto f(\boldsymbol{x})\cdot e^{-i\langle\boldsymbol{x},\boldsymbol{\xi}\rangle}$ is again **(Lebesgue) integrable** and the **Fourier transform** $\hat{f}(\boldsymbol{\xi}) = \frac{1}{(2\pi)^{n/2}}\int f(\boldsymbol{x})\cdot e^{-i\langle\boldsymbol{x},\boldsymbol{\xi}\rangle}d\boldsymbol{x}$ is **continuous** due to 1.14 and in the case of $f\in L^1$ **bounded** since $\left|\hat{f}(\boldsymbol{\xi})\right| = \frac{\|f\|_1}{(2\pi)^{n/2}}$ according to [8, 5.15].

5.5 Examples

1. For $f(x) = \chi_{[-1;1]}$ we have $\hat{f}(\xi) = \sqrt{\frac{2}{\pi}}\frac{\sin\xi}{\xi}$ whence $\frac{1}{\pi}\int\frac{\sin\xi}{\xi}d\xi = f^{\hat{}\hat{}}(0) = 1$.



2. For $f(x) = e^{-|x|}$ we have $\hat{f}(\xi) = \sqrt{\frac{2}{\pi}}\frac{1}{1+\xi^2}$

3. For $f(x) = e^{-x^2/2}$ we have $\hat{f}(\xi) = f(\xi)$
4. For $f(\boldsymbol{x}) = \chi_{B_1^n(\mathbf{0})}(\boldsymbol{x})$ we have $\hat{f}(\boldsymbol{\xi}) = \frac{J_{n/2}(|\boldsymbol{\xi}|)}{|\boldsymbol{\xi}|^{n/2}}$
5. For $f(\boldsymbol{x}) = e^{-|\boldsymbol{x}|^2/2}$ we have $\hat{f}(\boldsymbol{\xi}) = f(\boldsymbol{\xi})$
6. For $f(\boldsymbol{x}) = e^{-\langle \boldsymbol{x}, A\boldsymbol{x}\rangle}$ with any **symmetric** and **positive definite** $A \in \mathbb{R}^{n\times n}$ we have $\hat{f}(\boldsymbol{\xi}) = \prod_{k=1}^{n} \frac{1}{\sqrt{2d_k}} \cdot e^{-x_k^2/4d_k}$ with positive **eigenvalues** $d_k > 0$ of A.

Proof:

1. $\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int \chi_{[-1;1]} \cdot e^{-ix\xi} dx \overset{13.10}{=} \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-ix\xi}}{-i\xi}\right]_{x=-1}^{x=1} = \frac{1}{\sqrt{2\pi}} \cdot \frac{e^{-ix\xi} - e^{-ix\xi}}{i\xi} = \sqrt{\frac{2}{\pi}} \frac{\sin \xi}{\xi}$.
2. By splitting the integral into two parts we get

$$\begin{aligned}
\hat{f}(\xi) &= \frac{1}{\sqrt{2\pi}} \int e^{-|x|} \cdot e^{-ix\xi} dx \\
&= \frac{1}{\sqrt{2\pi}} \int \lim_{R\to\infty} \chi_{[0;R]} \cdot e^{-x} \cdot \left(e^{-ix\xi} + e^{-ix\xi}\right) dx \\
&\overset{5.14}{=} \frac{1}{\sqrt{2\pi}} \lim_{R\to\infty} \int \chi_{[0;R]} \cdot e^{-x} \cdot \left(e^{-ix\xi} + e^{-ix\xi}\right) dx \\
&\overset{13.10}{=} \frac{1}{\sqrt{2\pi}} \lim_{R\to\infty} \left[\frac{e^{-x(1+i\xi)}}{-1-i\xi} + \frac{e^{-x(1-i\xi)}}{-1+i\xi}\right]_{x=0}^{x=R} \\
&= \sqrt{\frac{2}{\pi}} \frac{1}{1+\xi^2}
\end{aligned}$$

3. Due to $\left|\frac{\delta f}{\delta \xi}(x;\xi)\right| = \left|-ix \cdot e^{-x^2/2} \cdot e^{-ix\xi}\right| = |x| \cdot e^{-x^2/2} \in L^1(\mathbb{R};\mathbb{R})$ for every $\xi \in \mathbb{R}$ and according to 1.15 the function $\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int e^{-x^2/2} \cdot e^{-ix\xi} dx$ is differentiable with

$$\begin{aligned}
\frac{\delta \hat{f}}{\delta \xi}(\xi') &= -\frac{i}{\sqrt{2\pi}} \int x \cdot e^{-x^2/2} \cdot e^{-ix\xi} dx \\
&= -\frac{i}{\sqrt{2\pi}} \int \lim_{R\to\infty} \chi_{[-R;R]} \cdot x \cdot e^{-x^2/2} \cdot e^{-ix\xi} dx \\
&\overset{5.14}{=} -\frac{i}{\sqrt{2\pi}} \lim_{R\to\infty} \int_{-R}^{R} x \cdot e^{-x^2/2} \cdot e^{-ix\xi} dx \\
&= 0 - \frac{\xi}{\sqrt{2\pi}} \lim_{R\to\infty} \int_{-R}^{R} e^{-x^2/2} \cdot e^{-ix\xi} dx \\
&= -\xi \hat{f}(\xi)
\end{aligned}$$

making use of **dominated convergence** [8, 5.14] resp. **integration by parts** [8, 13.5]. Considering 3.5 we have the initial value $\hat{f}(0) = 1$ so that the differential equation is solved by $\hat{f}(\xi) = e^{-\xi^2/2} = f(\xi)$.

4. On account of the **rotational symmetry** of the integrand we can restrict the computation to $\boldsymbol{\xi} = (0, ..., 0, \xi_n)$ such that

$$\begin{aligned}
\hat{f}(\boldsymbol{\xi}) &= \frac{1}{(2\pi)^{n/2}} \int \chi_{B_1^n(\mathbf{0})}(\boldsymbol{x}) \cdot e^{-i\langle \boldsymbol{x},\boldsymbol{\xi}\rangle} d\boldsymbol{x} \\
&= \frac{1}{(2\pi)^{n/2}} \int \chi_{B_1^n(\mathbf{0})}(x_1, ..., x_{1n}) \cdot e^{-ix_n\xi_n} d\boldsymbol{x} \\
&= \frac{1}{(2\pi)^{n/2}} \int \lambda^{n-1}\left(\left(B_1^{n-1}\right)_{x_n}\right) \cdot e^{-ix_n\xi_n} dx_n \\
&\overset{8.13}{=} \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}} \lambda^{n-1}\left(\left(B^{n-1}_{\sqrt{1-x_n^2}}\right)\right) \cdot e^{-ix_n\xi_n} dx_n \\
&= \frac{\tau_{n-1}}{(2\pi)^{n/2}} \cdot \int_{-1}^{1} \left(1 - x_n^2\right)^{(n-1)/2} \cdot e^{-ix_n\xi_n} dx_n \\
&\overset{x_n=\cos t}{=} \frac{1}{(2\pi)^{n/2}} \cdot \frac{\pi^{(n-1)/2}}{\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)} \int_0^{\pi} \sin^n t \cdot e^{-i\xi_n \cos t} dt \\
&\overset{3.5}{=} \frac{J_{n/2}(|\boldsymbol{\xi}|)}{|\boldsymbol{\xi}|^{n/2}}
\end{aligned}$$

using the **Bessel functions** of 5.3. A direct calculation results in
$\hat{f}(\mathbf{0}) = \frac{1}{(2\pi)^{n/2}} \lambda^n(B_1^n) = \frac{1}{2^{n/2}\Gamma\left(\frac{n}{2}+1\right)}$.

5. $\hat{f}(\boldsymbol{\xi}) = \frac{1}{(2\pi)^{n/2}} \int e^{-|\boldsymbol{x}|^2/2} \cdot e^{-i\langle \boldsymbol{x},\boldsymbol{\xi}\rangle} d\boldsymbol{x} = \prod_{k=1}^{n} \left(\frac{1}{\sqrt{2\pi}} \int e^{-x_k^2/2} \cdot e^{-ix_k\xi_k} dx_k\right) \overset{3.}{=} \prod_{k=1}^{n} e^{-x_k^2/2} = e^{-|\boldsymbol{\xi}|^2/2}$.
6. According to [2, S. 312] we have the decomposition $A = O^{-1}DO$ into an **orthogonal** $O \in \mathcal{O}(n;\mathbb{R})$ with $O^{-1} = O^T$ resp. $|\det O| = 1$ and a **diagonal** $D = (d_k\delta_{ki})_{1\le k;i\le n}$ containing the **positive eigenvalues** $d_k > 0$ for $1 \le k \le n$. Hence with $\sqrt{D} := (\sqrt{d_k}\delta_{ki})_{1\le k;i\le n}$ and $\left(\sqrt{D}\right)^{-1} = \left(\frac{\delta_{ki}}{\sqrt{d_k}}\right)_{1\le k;i\le n}$ we obtain

$$\begin{aligned}
\hat{f}(\boldsymbol{\xi}) &= \frac{1}{(2\pi)^{n/2}} \int e^{-\langle \boldsymbol{x}, A\boldsymbol{x}\rangle} \cdot e^{-i\langle \boldsymbol{x},\boldsymbol{\xi}\rangle} d\boldsymbol{x} \\
&\overset{\boldsymbol{y}=O\boldsymbol{x}}{=} \frac{1}{(2\pi)^{n/2}} \int e^{-\langle \boldsymbol{y}, D\boldsymbol{y}\rangle} \cdot e^{-i\langle \boldsymbol{y}, O\boldsymbol{\xi}\rangle} d\boldsymbol{y} \\
&\overset{\boldsymbol{z}=\sqrt{2D}\boldsymbol{y}}{=} \frac{1}{(2\pi)^{n/2}} \int e^{-|\boldsymbol{z}|^2/2} \cdot \exp\left(-i\left\langle \boldsymbol{z}, \left(\sqrt{2D}\right)^{-1} O\boldsymbol{\xi}\right\rangle\right) \cdot \frac{1}{\det\sqrt{2D}} d\boldsymbol{z} \\
&\overset{5.}{=} \frac{1}{\det\sqrt{2D}} \cdot \exp\left(-\frac{1}{2}\left|\left(\sqrt{2D}\right)^{-1} O\boldsymbol{\xi}\right|^2\right) \\
&= \frac{1}{\det\sqrt{2D}} \cdot \exp\left(-\frac{1}{4}\left\langle\left(\sqrt{D}\right)^{-1}\boldsymbol{\xi}; \left(\sqrt{D}\right)^{-1}\boldsymbol{\xi}\right\rangle\right) \\
&= \prod_{k=1}^{n} \frac{1}{\sqrt{2d_k}} \cdot e^{-x_k^2/4d_k}.
\end{aligned}$$

5.6 Properties of the Fourier transforms

For **integrable** $f, g : \mathbb{R}^n \to \mathbb{C}$ we have

1. $(f \circ A)^\wedge(\boldsymbol{\xi}) = \frac{1}{|\det A|} \cdot \hat{f} \circ \left(A^T\right)^{-1}(\boldsymbol{\xi})$ for every **linear transformation** $A(\boldsymbol{x}) = A\boldsymbol{x}$ with $A \in GL(n;\mathbb{R})$
2. $(f \circ \sigma_\alpha)^\wedge(\boldsymbol{\xi}) = \frac{1}{\alpha^n} \cdot \hat{f}\left(\frac{\boldsymbol{\xi}}{\alpha}\right)$ for every **homothety** $\sigma_\alpha(\boldsymbol{x}) = \alpha\boldsymbol{x}$ with $\alpha \in \mathbb{R}$
3. $(f \circ \tau_a)^\wedge(\boldsymbol{\xi}) = e^{-i\langle \boldsymbol{a},\boldsymbol{\xi}\rangle} \cdot \hat{f}(\boldsymbol{\xi})$ for every **translation** $\tau_{\boldsymbol{a}}(\boldsymbol{x}) = \boldsymbol{x} - \boldsymbol{a}$ with $\boldsymbol{a} \in \mathbb{R}^n$

4. $(f * g)^{\wedge}(\boldsymbol{\xi}) = (2\pi)^{n/2} \cdot \hat{f}(\boldsymbol{\xi}) \cdot \hat{g}(\boldsymbol{\xi})$
5. $\left(\frac{\delta f}{\delta x_i}\right)^{\wedge}(\boldsymbol{\xi}) = i\xi_i \cdot \hat{f}(\boldsymbol{\xi})$ for $f \in \mathcal{C}_c^1$
6. $(x_i \cdot f)^{\wedge}(\boldsymbol{\xi}) = i\frac{\delta \hat{f}}{\delta \xi_i}(\boldsymbol{\xi})$ for $x_i f \in L^1$
7. $\int \hat{f}(\boldsymbol{x}) \cdot g(\boldsymbol{x})\, d\boldsymbol{x} = \int f(\boldsymbol{y}) \cdot \hat{g}(\boldsymbol{y})\, d\boldsymbol{y}$
8. $\left|\widehat{g} - \widehat{f}\right| \leq \frac{1}{(2\pi)^{n/2}} \|g - f\|_1$, in particular for any sequence $(f_n)_{n\in\mathbb{N}} \subset L^1$ converging **in mean** to an $f \in L^1$ the Fourier transforms $\left(\hat{f}_n\right)_{n\in\mathbb{N}}$ converge **uniformly** to $\hat{f}$.

Proof:

1. Follows from the **change of variable theorem** [8, 14.8] with $e^{-i\left\langle \boldsymbol{x}^T, \boldsymbol{\xi}\right\rangle} = e^{-i\left\langle \boldsymbol{x}^T A^T, \left(A^T\right)^{-1}\boldsymbol{\xi}\right\rangle} = e^{-i\left\langle A\boldsymbol{x}, \left(A^T\right)^{-1}\boldsymbol{\xi}\right\rangle}$.
2. As in 1. with $\left|\det\left(\frac{d\boldsymbol{x}}{d\alpha}\right)\right| = \frac{1}{\alpha^n}$ and $e^{-i\langle \boldsymbol{x}, \boldsymbol{\xi}\rangle} = e^{-i\langle \sigma_\alpha(\boldsymbol{x}), \boldsymbol{\xi}/\alpha\rangle}$.
3. As in 1. with $\left|\det\left(\frac{d\boldsymbol{x}}{d\tau_{\boldsymbol{a}}}\right)\right| = 1$ and $e^{-i\langle \boldsymbol{x}, \boldsymbol{\xi}\rangle} = e^{-i\langle \boldsymbol{a}, \boldsymbol{\xi}\rangle} \cdot e^{-i\langle \tau_{\boldsymbol{a}}(\boldsymbol{x}), \boldsymbol{\xi}\rangle}$.
4. By **Fubini's theorem** [8, 8.5] resp. 1. we obtain

$$
\begin{aligned}
(f * g)^{\wedge}(\boldsymbol{\xi}) &= \frac{1}{(2\pi)^{n/2}} \int \int f(\boldsymbol{x} - \boldsymbol{y})\, g(\boldsymbol{y})\, d\boldsymbol{y} \cdot e^{-i\langle \boldsymbol{x}, \boldsymbol{\xi}\rangle} d\boldsymbol{x} \\
&= \frac{1}{(2\pi)^{n/2}} \int \left(\int f(\boldsymbol{x} - \boldsymbol{y}) \cdot e^{-i\langle \boldsymbol{x} - \boldsymbol{y}, \boldsymbol{\xi}\rangle} d(\boldsymbol{x} - \boldsymbol{y})\right) g(\boldsymbol{y})\, d\boldsymbol{y} \cdot e^{-i\langle \boldsymbol{y}, \boldsymbol{\xi}\rangle} d\boldsymbol{x} \\
&= (2\pi)^{n/2} \cdot \hat{f}(\boldsymbol{\xi}) \cdot \hat{g}(\boldsymbol{\xi})
\end{aligned}
$$

5. Since $e^{-i\langle \boldsymbol{x}, \boldsymbol{\xi}\rangle}$ is continuously differentiable we can apply **integration by parts** [8, 13.5] to get $\left(\frac{\delta f}{\delta x_i}\right)^{\wedge}(\boldsymbol{\xi}) = \frac{1}{(2\pi)^{n/2}} \int \frac{\delta f}{\delta x_i}(\boldsymbol{x}) \cdot e^{-i\langle \boldsymbol{x}, \boldsymbol{\xi}\rangle} d\boldsymbol{x} = -\frac{1}{(2\pi)^{n/2}} \int f(\boldsymbol{x}) \cdot \frac{\delta}{\delta x_i} e^{-i\langle \boldsymbol{x}, \boldsymbol{\xi}\rangle} d\boldsymbol{x} = i\xi_i \cdot \hat{f}(\boldsymbol{\xi})$.
6. Since
$\xi_i \mapsto f(\boldsymbol{x}) \cdot e^{-i\langle \boldsymbol{x}, \boldsymbol{\xi}\rangle}$ is **differentiable** for every $\boldsymbol{x} \in \mathbb{R}^n$
$\boldsymbol{x} \mapsto f(\boldsymbol{x}) \cdot e^{-i\langle \boldsymbol{x}, \boldsymbol{\xi}\rangle}$ due to [8, 5.18] is **integrable** for every $\xi_i \in \mathbb{R}$
$\boldsymbol{x} \mapsto \left|f(\boldsymbol{x}) \cdot x_i \cdot e^{-i\langle \boldsymbol{x}, \boldsymbol{\xi}\rangle}\right|$ is a **Lebesgue integrable majorant** for $f(\boldsymbol{x}) \cdot \frac{\delta}{\delta x_i} e^{-i\langle \boldsymbol{x}, \boldsymbol{\xi}\rangle}$
we may **exchange the order of integration and differentiation** according to 1.15 whence

$$
\begin{aligned}
(x_i \cdot f)^{\wedge}(\boldsymbol{\xi}) &= \frac{1}{(2\pi)^{n/2}} \int f(\boldsymbol{x}) \cdot x_i \cdot e^{-i\langle \boldsymbol{x}, \boldsymbol{\xi}\rangle} d\boldsymbol{x} \\
&= \frac{i}{(2\pi)^{n/2}} \int f(\boldsymbol{x}) \cdot \frac{\delta}{\delta x_i} e^{-i\langle \boldsymbol{x}, \boldsymbol{\xi}\rangle} d\boldsymbol{x} \\
&= \frac{i}{(2\pi)^{n/2}} \cdot \frac{\delta}{\delta x_i} \int f(\boldsymbol{x}) \cdot \frac{\delta}{\delta x_i} e^{-i\langle \boldsymbol{x}, \boldsymbol{\xi}\rangle} d\boldsymbol{x} \\
&= i\frac{\delta \hat{f}}{\delta \xi_i}(\boldsymbol{\xi})
\end{aligned}
$$

7. By **Fubini's theorem** [8, 8.5] resp. 1. we obtain

$$
\begin{aligned}
\int \hat{f}(\boldsymbol{x}) \cdot g(\boldsymbol{x})\, d\boldsymbol{x} &= \frac{1}{(2\pi)^{n/2}} \int \left(\int f(\boldsymbol{y}) \cdot e^{-i\langle \boldsymbol{y}, \boldsymbol{x}\rangle} d\boldsymbol{y}\right) \cdot g(\boldsymbol{x})\, d\boldsymbol{x} \\
&= \frac{1}{(2\pi)^{n/2}} \int \int f(\boldsymbol{y}) \cdot g(\boldsymbol{x}) \cdot e^{-i\langle \boldsymbol{y}, \boldsymbol{x}\rangle} d\boldsymbol{y} d\boldsymbol{x} \\
&= \frac{1}{(2\pi)^{n/2}} \int \left(\int g(\boldsymbol{x}) \cdot e^{-i\langle \boldsymbol{y}, \boldsymbol{x}\rangle} d\boldsymbol{x}\right) \cdot f(\boldsymbol{y})\, d\boldsymbol{y} \\
&= \int f(\boldsymbol{y}) \cdot \hat{g}(\boldsymbol{y})\, d\boldsymbol{y}
\end{aligned}
$$

8. Directly follows from the definition subsec:Fourier-transforms.

5.7 Hermite functions

1. For the **Hermite polynomials** $H_n(x) = (-1)^n \cdot e^{x^2} \cdot \frac{d^n}{dx^n}\left(e^{-x^2}\right)$ with $n \in \mathbb{N}$ we have the **recursive formula** $H_{n+2}(x) = 2xH_{n+1}(x) - (2n+2)H_n(x)$ and the **differential equation** $\frac{d^2}{dx^2}H_n(x) - 2x \cdot \frac{d}{dx}H_n(x) + 2nH_n(x) = 0$.
2. The **Hermite functions** $h_n(x) = H_n(x) \cdot e^{-x^2/2}$ are the solutions of the differential equation $\frac{d^2}{dx^2}h_n(x) - x^2 h_n(x) + (2n+1)h_n(x) = 0$.
3. The **Hermite functions** h_n are **othogonal** in L^2.
4. The **Hermite functions** h_n are **Fourier eigenfunctions**, i.e. $\hat{h}_n(x) = \lambda_n \cdot h_n(x)$ with **eigenvalues** $\lambda_n \in \mathbb{C}$.

Note: By **spectral theory** the **eigenvalues** can be determined by $\lambda_n^4 - 1 = 0$ i.e. $\lambda_n \in \{\pm 1; \pm i\}$.

Proof of 5.7.1:

We first prove the **recursive formula** by **induction**:

$n = 0:$

$H_0(x) \cdot e^{-x^2} = e^{-x^2}; H_1(x) \cdot e^{-x^2} = 2x \cdot e^{-x^2}; H_2(x) \cdot e^{-x^2} = \left(-2 + 4x^2\right) \cdot e^{-x^2} = \left(2xH_1(x) - 2H_0(x)\right) \cdot e^{-x^2}$

$n \Rightarrow n+1:$

$$
\begin{aligned}
&H_{n+3}(x) \cdot e^{-x^2}\\
&= -\frac{d}{dx}\left(H_{n+2}(x) \cdot e^{-x^2}\right)\\
&= -\left(\frac{d}{dx}H_{n+2}(x) - 2xH_{n+2}(x)\right) \cdot e^{-x^2}\\
&= -\left(\frac{d}{dx}\left(2xH_{n+1}(x) - (2n+2)H_n(x)\right) - 2xH_{n+2}(x)\right) \cdot e^{-x^2}\\
&= \left(2xH_{n+2}(x) - 2H_{n+1}(x) - 2x\frac{d}{dx}H_{n+1}(x) + (2n+2)\frac{d}{dx}H_n(x)\right) \cdot e^{-x^2}\\
&= \left(2xH_{n+2}(x) - 2H_{n+1}(x) - 2x\left(2xH_{n+1}(x) - H_{n+2}(x)\right) + (2n+2)\left(2xH_n(x) - H_{n+1}(x)\right)\right) \cdot e^{-x^2}\\
&= \left(2xH_{n+2}(x) - (2n+4)H_{n+1}(x) + 2x\left[H_{n+2}(x) + 2xH_{n+1}(x) - (2n+2)H_n(x)\right]\right) \cdot e^{-x^2}\\
&= \left(2xH_{n+2}(x) - (2n+4)H_{n+1}(x) + 0\right) \cdot e^{-x^2}
\end{aligned}
$$

The **differential equation** then follows from

$$H_{n+1}(x) \cdot e^{-x^2} = -\frac{d}{dx}\left(H_n(x) \cdot e^{-x^2}\right) = -\left(\frac{d}{dx}H_n(x) - 2xH_n(x)\right) \cdot e^{-x^2},$$

i.e.

$$\frac{d}{dx}H_n(x) = 2xH_n(x) - H_{n+1}(x)$$

and

$$H_{n+2}(x) \cdot e^{-x^2} = \frac{d^2}{dx^2}\left(H_n(x) \cdot e^{-x^2}\right) = \left(\frac{d^2}{dx^2}H_n(x) - 4x\frac{d}{dx}H_n(x) + \left(4x^2 - 2\right)H_n(x)\right) \cdot e^{-x^2},$$

i.e.

$$\frac{d^2}{dx^2}H_n(x) = H_{n+2}(x) + 4x\frac{d}{dx}H_n(x) - \left(4x^2 - 2\right)H_n(x) = H_{n+2}(x) - 4xH_{n+1}(x) + \left(4x^2 + 2\right)H_n(x)$$

whence by the recursive formula we obtain

$\frac{d^2}{dx^2}H_n(x) - 2x \cdot \frac{d}{dx}H_n(x) + 2nH_n(x)$

$= H_{n+2}(x) - 4xH_{n+1}(x) + \left(4x^2+2\right)H_n(x) - 4x^2H_n(x) + 2xH_{n+1}(x) + 2nH_n(x)$

$= H_{n+2}(x) - 2xH_{n+1}(x) + (2n+2)H_n(x)$

$= 0$

Proof of 5.7.2:

With

$$\frac{d}{dx}h_n(x) = \left(\frac{d}{dx}H_n(x) - x \cdot H_n(x)\right) \cdot e^{-x^2/2}$$

and

$$\frac{d^2}{dx^2}h_n(x) = \left(\frac{d^2}{dx^2}H_n(x) - 2x\frac{d}{dx}H_n(x) + \left(x^2-1\right) \cdot H_n(x)\right) \cdot e^{-x^2/2}$$

we have

$\frac{d^2}{dx^2}h_n(x) + \left(\lambda_n - x^2\right)h_n(x)$

$= \left(\left(\frac{d^2}{dx^2}H_n(x) - 2x\frac{d}{dx}H_n(x) + \left(x^2-1\right) \cdot H_n(x)\right) + \left(2n-1-x^2\right)H_n(x)\right) \cdot e^{-x^2/2}$

$= \left(\frac{d^2}{dx^2}H_n(x) - 2x\frac{d}{dx}H_n(x) + 2nH_n(x)\right) \cdot e^{-x^2/2}$

$= 0$

Proof of 5.7.3:

By **integration by parts** we obtain $\langle h_n; h_m\rangle = \int h_n(x) \cdot h_m(x)\,dx = \int H_n(x) \cdot H_m(x) \cdot e^{-x^2}dx = \delta_{nm}$.

Proof of 5.7.4:

By 5 and 6 the differential equation $\frac{d^2}{dx^2}h_n(x) - x^2h_n(x) + (2n+1)h_n(x) = 0$ transforms to the identical equation $0 = \left(\frac{d^2}{dx^2}h_n - x^2h_n + (2n+1)h_n\right)^\wedge(\xi) = -\xi^2\hat{h}_n(\xi) + \frac{d^2}{d\xi^2}\hat{h}_n(\xi) + (2n+1)\hat{h}_n(\xi) = 0$ having the same solutions up to a complex constant, i.e.$h_n = \lambda_n \cdot \hat{h}_n$ with $\lambda_n \in \mathbb{C}$.

5.8 The Riemann-Lebesgue lemma

Let C_0 the vector space of all **continuous** functions $f : \mathbb{R}^n \to \mathbb{C}$ **vanishing at infinity**, i.e. $\lim_{|\boldsymbol{x}|\to\infty} |f(\boldsymbol{x})| = 0$. Hence we have $C_c \subset C_0 \subset C$ and also the following inclusions:

1. For every $f \in \mathcal{C}_c^k$ with $k \in \mathbb{N}$ there exists an $M > 0$ such that $\left|\hat{f}(\xi)\right| \leq \frac{M}{(1+|\xi|)^k}$ for every $\xi \in \mathbb{R}^n$ and particularly $\hat{f} \in C_0^k \cap L^1$.
2. For every $f \in L^1$ we have $\hat{f} \in C_0$.

Proof:

1. We define **multiindices** $\boldsymbol{\alpha} \in \mathbb{N}^n$ with $|\boldsymbol{\alpha}| = \sum_{i=1}^n \alpha_i$ for use in situations as e.g. $\boldsymbol{x}^\alpha = \prod_{i=1}^n x_i^{\alpha_i}$ and $\frac{\delta^{\boldsymbol{\alpha}} f}{\delta \boldsymbol{x}^{\boldsymbol{\alpha}}} = \frac{\delta^{|\alpha|} f}{\delta^{\alpha_1}x_1 \cdot \ldots \cdot \delta^{\alpha_n}x_n}$. Hence owing to 5 for $|\boldsymbol{\alpha}| \leq k$ we have $\left(\frac{\delta^{\boldsymbol{\alpha}} f}{\delta \boldsymbol{x}^{\boldsymbol{\alpha}}}\right)^\wedge(\boldsymbol{\xi}) = i^{|\boldsymbol{\alpha}|} \cdot \boldsymbol{\xi}^{\boldsymbol{\alpha}} \cdot \hat{f}(\boldsymbol{\xi})$ whence $\left|\boldsymbol{\xi}^{\boldsymbol{\alpha}} \cdot \hat{f}(\boldsymbol{\xi})\right| \leq \frac{1}{(2\pi)^{n/2}}\left\|\frac{\delta^{\boldsymbol{\alpha}} f}{\delta \boldsymbol{x}^{\boldsymbol{\alpha}}}\right\|_1 < \infty$ since $\mathcal{C}_c^k \subset L^1$. Consequently there is an $M > 0$ such that $(1 + |\xi_1| + \ldots + |\xi_n|)^k \left|\hat{f}(\boldsymbol{\xi})\right| \leq M$ for all $\xi \in \mathbb{R}^n$ which proves the assertion.
2. According to [8, 14.4.2] there is a $g \in C_c^1$ with $\|f - g\|_1 < \epsilon$ whence $\left|\hat{f}(\xi) - \hat{g}(\xi)\right| < \frac{\epsilon}{(2\pi)^{n/2}}$ for every $\xi \in \mathbb{R}^n$ so that the assertion follows from 1.

5.9 Integrability of Fourier transfoms

According to 5.15.2 the Fourier transform $\hat{f}(\xi) = \frac{\sin \xi}{\xi}$ from 1 is **integrable** with $\int \frac{\sin \xi}{\xi} d\xi = \pi$ but **not Lebesgue integrable** any more, since for every $k \geq 1$ we have $\int_{k\pi}^{(k+1)\pi} \left|\frac{\sin \xi}{\xi}\right| \geq \frac{2}{k\pi}$ whence $\int_{N\pi}^{(2N+1)\pi} \frac{\sin \xi}{\xi} d\xi \geq \frac{2}{\pi} \sum_{k=N}^{2N} \frac{1}{k} = \frac{2}{\pi N} \sum_{k=1}^{N} \frac{N}{N+k} \geq= \frac{2}{\pi N} \sum_{k=1}^{N} \frac{1}{2} \geq \frac{1}{\pi}$ for every $N \geq 1$.

5.10 Lemma

For any **Lebesgue integrable** functions $f, \psi \in L^1$ with $\int \psi(\boldsymbol{x})\, d\boldsymbol{x} = 1$ and $\psi_\alpha(\boldsymbol{\xi}) = \frac{1}{\alpha^n} \psi\left(\frac{\boldsymbol{\xi}}{\alpha}\right)$ we have $\lim_{\alpha \to 0} \|f - f * \psi_\alpha\|_1 = 0$.

Proof: Due to [8, 14.4.2] it suffices to prove the assertion for $f \in \mathcal{C}_c^\infty$. Hence there is an $R > 0$ such that $\operatorname{supp}(f) < B_R(\mathbf{0})$ and for every $\epsilon > 0$ we can find a $\delta > 0$ with $|f(\boldsymbol{x}) - f(\boldsymbol{y})| < \frac{\epsilon}{2 \cdot \lambda^n (B_{R+1}(\mathbf{0})) \cdot \|\psi\|_1}$ for every $x, y \in \mathbb{R}^n$ with $|\boldsymbol{x} - \boldsymbol{y}| < \delta$. Owing to the **monotone convergence** theorem [8, 5.12] and $|\psi| \in L^1$ exists an $\alpha > 0$ such that $\int_{|\boldsymbol{\xi}| > \delta} |\psi_\alpha(\boldsymbol{\xi})|\, d\boldsymbol{\xi} \overset{\boldsymbol{\xi} = \alpha \boldsymbol{x}}{=} \int_{|\boldsymbol{x}| > \delta/\alpha} |\psi(\boldsymbol{x})|\, d\boldsymbol{x} < \frac{\epsilon}{4\|f\|_1}$ we have

$$\begin{aligned}
\|f - f * \psi_\alpha\|_1 \leq & \int_{|\boldsymbol{x}| \leq R} \left(\int_{|\boldsymbol{\xi}| \leq \delta} |f(\boldsymbol{x}) - f(\boldsymbol{x} - \boldsymbol{\xi})| \cdot |\psi_\alpha(\boldsymbol{\xi})|\, d\boldsymbol{\xi} \right) d\boldsymbol{x} \\
& + \int_{|\boldsymbol{x}| \leq R} \left(\int_{|\boldsymbol{\xi}| > \delta} |f(\boldsymbol{x}) - f(\boldsymbol{x} - \boldsymbol{\xi})| \cdot |\psi_\alpha(\boldsymbol{\xi})|\, d\boldsymbol{\xi} \right) d\boldsymbol{x} \\
\leq & \lambda^n (B_{R+1}(\mathbf{0})) \cdot \sup_{|\boldsymbol{x} - \boldsymbol{y}| < \delta} |f(\boldsymbol{x}) - f(\boldsymbol{y})| \cdot \|\psi\|_1 + 2\|f\|_1 \cdot \int_{|\boldsymbol{\xi}| > \delta} |\psi_\alpha(\boldsymbol{\xi})|\, d\boldsymbol{\xi} \\
\leq & \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
= & \epsilon.
\end{aligned}$$

5.11 Fourier inversion formula

For **Lebesgue integrable** $f, \hat{f} \in L^1$ we have λ^n-a.e. $f(\boldsymbol{x}) = \int \hat{f}(\boldsymbol{\xi}) \cdot e^{i\langle \boldsymbol{\xi}; \boldsymbol{x} \rangle} d\boldsymbol{\xi}$, i.e. $f = \hat{\hat{f}} \circ \sigma_{-1}$ resp. $\overline{f} = \overline{\hat{\hat{f}}}$.

Proof: According to 5.8.2 we have $\hat{\hat{f}} \circ \sigma_{-1} \in C_0$. Also for $\psi(\xi) = \frac{1}{(2\pi)^{n/2}} \cdot e^{-|\boldsymbol{\xi}|^2/2}$ holds $\lim_{\alpha \to 0} \psi(\alpha \boldsymbol{\xi}) = 1$ with monotonically increasing $(\psi(\alpha \boldsymbol{\xi}))_{\alpha > 0}$ for every $\boldsymbol{\xi} \in \mathbb{R}^n$ such that we can use **monotone convergence** to obtain

$$\begin{aligned}
\int \hat{f}(\boldsymbol{\xi}) \cdot e^{i\langle \boldsymbol{\xi}; \boldsymbol{x} \rangle} d\boldsymbol{\xi} & \overset{5.12}{=} \lim_{\alpha \to 0} \left(\int \hat{f}(\boldsymbol{\xi}) \cdot e^{i\langle \boldsymbol{\xi}, \boldsymbol{x} \rangle} \cdot \psi(\alpha \boldsymbol{\xi})\, d\boldsymbol{\xi} \right) \\
& \overset{17.8.1}{=} \lim_{\alpha \to 0} \left(\int (f \circ \tau_{-\boldsymbol{x}})^\wedge (\boldsymbol{\xi}) \cdot (\psi \circ \sigma_\alpha)(\boldsymbol{\xi})\, d\boldsymbol{\xi} \right) \\
& \overset{17.8.6}{=} \lim_{\alpha \to 0} \left(\int (f \circ \tau_{-\boldsymbol{x}})(\boldsymbol{\xi}) \cdot (\psi \circ \sigma_\alpha)^\wedge (\boldsymbol{\xi})\, d\boldsymbol{\xi} \right) \\
& \overset{17.8.2}{=} \lim_{\alpha \to 0} \left(\int f(\boldsymbol{\xi} + \boldsymbol{x}) \cdot \frac{1}{\alpha^n} \cdot \hat{\psi}\left(\frac{\boldsymbol{\xi}}{\alpha}\right) d\boldsymbol{\xi} \right) \\
& \overset{17.7.5}{=} \lim_{\alpha \to 0} \left(\int f(\boldsymbol{\xi} + \boldsymbol{x}) \cdot \frac{1}{\alpha^n} \cdot \psi\left(\frac{\boldsymbol{\xi}}{\alpha}\right) d\boldsymbol{\xi} \right) \\
& = \lim_{\alpha \to 0} \left(\int f(\boldsymbol{x} - \boldsymbol{\xi}) \cdot \frac{1}{\alpha^n} \cdot \psi\left(-\frac{\boldsymbol{\xi}}{\alpha}\right) d\boldsymbol{\xi} \right) \\
& = \lim_{\alpha \to 0} (f * \psi_\alpha)(\boldsymbol{x}) \\
& = f(\boldsymbol{x}) \ \lambda^n\text{-a.e.}
\end{aligned}$$

since due to the preceding lemma 5.10 we have $\lim_{\alpha\to 0}\|f - f * \psi_\alpha\|_1 = 0$ such that according to [8, 5.10] a subsequence converges λ^n-a.e. to f.

5.12 Trigonometric integrals

The fourier inversion yields some convenient formulae for integrals of real valued trigonometric functions with $\mathrm{Re}\left(e^{ixt}\right) = \cos(tx)$:

1. According to 5.2, 1 and 4 we have $\sup\{0; 2 - |x|\} = \frac{2}{\pi}\int\left(\frac{\sin x}{x}\right)^2 \cos(x\xi)\,dx$ and in particular for $\xi = 0$ we obtain $\int\left(\frac{\sin x}{x}\right)^2 dx = \pi$.
2. From 2 follows $\int \frac{\cos(tx)}{1+x^2}dx = \pi e^{-|t|}$.

5.13 Lemma

For every $f \in L^1 \cap L^2$ and every $\epsilon > 0$ there is a $\varphi \in \mathcal{C}_C^\infty$ such that $\|f - \varphi\|_1 < \epsilon$ **and** $\|f - \varphi\|_2 < \epsilon$.

Proof: According to [8, 14.4.2] for $\epsilon > 0$ and $h_a(\boldsymbol{x}) = e^{-\alpha|\boldsymbol{x}|^2}$ there is a $\psi \in \mathcal{C}_C^\infty$ with $\|f - \psi\|_2 < \min\left\{\frac{\epsilon}{2}, \frac{\epsilon}{2\|h_a\|_2}\right\}$. Since $h_\alpha \uparrow 1$ for $\alpha \to 0$ we can invoke **monotone convergence** [8, 5.12] to find an $\alpha > 0$ such that $\|f - f\cdot h_\alpha\|_p^p = \int |f(x)|^p \cdot |1 - h_\alpha(x)|^p\,dx < \frac{\epsilon}{2}$. Hence on the one hand we have $\|f\cdot h_\alpha - \psi\cdot h_\alpha\|_2 = \|f - \psi\|_2 \cdot \|h_\alpha\|_2 < \|f - \psi\|_2 < \frac{\epsilon}{2}$ and owing to **Hölder's inequality** [8, 6.4.1] on the other hand there is $\|f\cdot h_\alpha - \psi\cdot h_\alpha\|_1 \le \|f - \psi\|_2 \cdot \|h_\alpha\|_2 < \frac{\epsilon}{2}$. The assertion now follows with the **triangle inequality** for $p = 1; 2$ for $\varphi = \psi \cdot h_\alpha$.

5.14 Plancherel's theorem

There is a L^2**-norm-preserving isomorphism** $\hat{}: L^2 \to L^2$ with $\|f\|_2 = \left\|\hat{f}\right\|_2$ for $f \in L^2$ and $\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}}\int f(x)\cdot e^{-i\langle x,\xi\rangle}dx$ for $f \in L^1 \cap L^2$.

Proof: According to the preceding lemma 5.13 for every $f \in L^1 \cap L^2$ there is a sequence $(f_k)_{k\in\mathbb{N}} \subset C_C^\infty$ converging in the **first and second mean** to f. Due to 1 we have $\left(\hat{f}_{k\in\mathbb{N}}\right)_{k\in\mathbb{N}} \subset L^1$and with 6 follows $\left\|\hat{f}_k\right\|_2^2 = \int \hat{f}_k\overline{\hat{f}_k}d\lambda^n = \int \hat{f}_k\hat{\overline{\hat{f}}}_k d\lambda^n = \int f_k\overline{f}_k d\lambda^n = \|f_k\|_2^2$. Hence $\left(\hat{f}_k\right)_{k\in\mathbb{N}}$ is L^2-Cauchy, which due to [8, 6.7] converges **in the second mean** to a $g \in L^2$ and according to [8, 6.9] a subsequence converges λ^n-a.e. to g. But owing to 8 the entire sequence $\left(\hat{f}_k\right)_{k\in\mathbb{N}}$ **uniformly** converges to $\hat{f}$ which means λ^n-a.e. $g = \hat{f} \in L^2(\mathbb{R}^n)$. Also we have $\left\|\hat{f}\right\|_2 = \lim_{k\to\infty}\left\|\hat{f}_k\right\|_2 = \lim_{k\to\infty}\|f_k\|_2 = \|f\|_2$. Thus we have shown that the Fourier transform $\hat{}: L^1 \cap L^2 \to L^2$ is L^2-norm preserving. Since L^2 is complete and $C_C^\infty \subset L^1 \cap L^2 \subset L^2$ according to [8, 14.4.2] is L^2- dense in L^2 it can be extended to L^2 as usual by assigning to every $f \in L^2$ as Fourier transform the L^2- resp. λ^n-a.e. limit $\hat{f} := \lim_{k\to\infty}\hat{f}_k \in L^2$ of the L^2-Cauchy sequence $\left(\hat{f}_k\right)_{k\in\mathbb{N}} \subset L^2$ of Fourier transforms of any sequence $(f_k)_{k\in\mathbb{N}} \subset L^1 \cap L^2$ converging in the second mean to f. Owing to the L^2-norm preserving character of the Fourier transform and in particular $\left\|\hat{f}_k\right\|_2 = \|f_k\|_2$ resp. the **positive definiteness** of the norm due to [8, 5.8] the Fourier transform $\hat{f}$ is λ^n-a.e. determined and in this sense independent of the approximating sequence. The same argument applies to show that the Fourier transform is **injective** and finally it is **surjective** since the set of all Fourier transforms is closed with respect to the L^2-norm and includes the L^2- dense subset C_C^∞.

Note: In the proof a sequence $(f_k)_{k\in\mathbb{N}} \subset L^1 \cap L^2$ converging in L^2 and λ^n-a-e- to f is used to define the extended Fourier transform $\hat{f} := \lim_{k\to\infty}\hat{f}_k \in L^2$. In the following example we will use the inverse direction and take a sequence $\left(\hat{f}_k\right)_{k\in\mathbb{N}} \subset L^1 \cap L^2$ converging in L^2 and λ^n-a-e- to $\hat{f}$ to find its inverse $f := \lim_{k\to\infty} \hat{\hat{f}}_k \in L^2$:

5.15 More trigonometric integrals

1. $\int_0^\infty \left|J_{n/2}(r)\right|^2 \frac{dr}{r} = \frac{1}{n}$ for every $n \geq 1$ since on the one hand we have $\left\|\hat{\chi}_{B_1^n}\right\|_2^2 \overset{16.7.4}{=} \frac{J_{n/2}^2(|\boldsymbol{\xi}|)}{|\boldsymbol{\xi}|^n} d\boldsymbol{\xi} \overset{15.5}{=} n \cdot \tau_n \cdot \int_0^\infty \left|J_{n/2}(r)\right|^2 \frac{dr}{r}$ and on the other hand $\left\|\chi_{B_1^n}\right\|_2^2 \overset{8.13}{=} \tau_n$ so that the formula follows from **Plancherel's theorem** 5.14.
2. We want to show that $\hat{\hat{f}} = f$ and thereby compute the integral $\int \frac{\sin\xi}{\xi} d\xi$: The Fourier transform $\hat{f}(\xi) = \sqrt{\frac{2}{\pi}} \frac{\sin\xi}{\xi} \in L^2$ of $f(x) = \chi_{[-1;1]}(x) \in L^2$ from 1 is **integrable** due to [8, 5.24] and **square integrable** owing to 5.12.1 but **not Lebesgue integrable**. Hence we cannot directly apply the **inversion formula** 5.11 and have to invoke **Plancherel's theorem** 5.14: The sequence $\left(\hat{f}_k\right)^2_{k\geq 1} \subset L_1$ with $\hat{f}_k = \hat{f} \cdot \chi_{[-k;k]}$ converges to $\hat{f}^2$ **pointwise** and owing to **monotone convergence** [8, 5.12] the sequence $\left(\hat{f}_k\right)_{k\geq 1} \subset L_1 \cap L_2$ converges to $\hat{f} \in L^2$ in **quadratic mean**. Due to **Plancherel's theorem** subsec:Plancherel's-theorem the sequence $(f_k)_{k\geq 1} = \left(\hat{\hat{f}}_k\right)_{k\geq 1} \subset L_1 \cap L_2$ converges in quadratic mean and due to [8, 6.9] also λ^n-a.e. to $\chi_{[-1;1]}(x) = f(x) = \lim_{k\to\infty} f_k(x) = \lim_{k\to\infty} \hat{\hat{f}}_k(x) = \lim_{k\to\infty} \left(\frac{1}{\sqrt{2\pi}} \int \chi_{[-k;k]}(\xi) \cdot \hat{f}(\xi) \cdot e^{ix\xi} d\xi\right) = \frac{1}{\pi} \cdot \lim_{k\to\infty} \int \chi_{[-k;k]}(\xi) \cdot \frac{\sin\xi}{\xi} \cdot \cos(x\xi)\, d\xi \overset{5.24}{=} \frac{1}{\pi} \cdot \int \frac{\sin\xi}{\xi} \cdot \cos(x\xi)\, d\xi$. For $x = 0$ we obtain $1 = \frac{1}{\pi} \cdot \lim_{k\to\infty} \int \frac{\sin\xi}{\xi} d\xi$ whence $\int \frac{\sin\xi}{\xi} d\xi = \pi$.

6 Umordnung von Vektorsummen

6.1 Einleitung

Nach dem Riemannsche Umordnungssatz lässt sich eine bedingt konvergent Reihe reeller Zahlen so umordnen, dass die Partialsummen gegen jede beliebige reelle Zahl konvergieren. Da die Partialsummen einer bedingt konvergenten Reihe zwar beliebig groß werden, die Beträge der Summanden aber gegen Null konvergieren, kann man eine Umordnung konstruieren, die eine beliebige reelle Zahl approximiert. Die Verallgemeinerung auf endlichdimensionale Banachräume und insbesondere die komplexen Zahlen und den $\mathbb{R}^n$ gelang 1905 P. Lévy bzw. 1913 E. Steinitz [6]. Der elementare und sehr aufwendige Beweis wurde von W. Gross [3], I. Halperin [4] und P. Rosenthal [5] vereinfacht. Der mehr topologisch motivierte aber ebenfalls konstruktive Ansatz von T. Banakh [1] aus dem Jahr 2017 verkürzt den Beweis deutlich.

Nach einem ersten Abschnitt mit bekannten Ausssagen zu bedingt und absolut konvergenten Reihen wird im zweiten Abschnitt der Beweis des ersten Teils des Satzes von Lévy-Steinitz nach P. Rosenthal behandelt: Die durch Umordnung möglichen Grenzwerte einer bedingt konvergenten Vektorreihe bilden einen affiner Unterraum. Der Beweis enthält Ergebnisse zur Umordnung endlicher Vektorketten, die auch für sich von Interesse sind. Im dritten Abschnitt wird der Beweis des Satzes von Lévy-Steinitz einschließlich der Aussagen zur Lage des affinen Unterraums nach T. Banakh dargestellt.

6.2 Absolute Konvergenz von Vektorsummen

Sind die Partialsummen einer Vektorfolge $(v_i)_{i\in\mathbb{N}}$ mit $v_i \in \mathbb{R}^n$ **absolut** konvergent mit $\sum_{i\in\mathbb{N}} |v_i| < \infty$, so ist der Grenzwert $\sum_{j\in\mathbb{N}} v_{p(j)} = v$ unabhängig von der Permutation p.

Beweis: Für jedes $\epsilon > 0$ und jede Permutation $p : \mathbb{N} \to \mathbb{N}$ gibt es ein $i_0 \in \mathbb{N}$ mit $\sum_{i\geq i_0} \|v_i\| < \epsilon$ und für $i_1 = \max\{p^{-1}(j) : 0 \leq j \leq i_0\}$ gilt demnach $\left\|\sum_{i\geq 0} v_i - \sum_{j\geq 0} v_{p(j)}\right\| \leq \left\|\sum_{0<i\leq i_1} v_i - \sum_{0<j\leq i_1} v_{p(j)}\right\| +$

$\sum_{i>i_1} \|v_i\| + \sum_{j>i_1} \left\|v_{p(j)}\right\| \leq \left\|\sum_{i_0<i\leq i_1} v_i - \sum_{\substack{0<j\leq i_1 \\ p(j)>i_0}} v_{p(j)}\right\| + 2\epsilon \leq 4\epsilon$. Dabei wurde im vorletzten Schritt die endliche Summe $\sum_{0<j\leq i_1} v_{p(j)}$ so umgeordnet, dass die „großen“ $v_{p(j)}$ mit $p(j) = i$ für $0 \leq i < i_0$ zu Beginn stehen und sich mit den entsprechenden „großen“ Summanden in $\sum_{0<i\leq i_1} v_i$ aufheben. Die übrigen „kleinen“ Summanden $v_i$mit $i > i_0$ und $v_{p(j)}$ mit $p(j) > i_0$ heben sich nicht gegenseitig auf, lassen sich aber wegen der absoluten Konvergenz durch jeweils ϵ abschätzen.

6.3 Bedingte Konvergenz von Zahlensummen und Riemannscher Umordnungssatz

Sind die Partialsummen einer **Zahlenfolge** $(x_i)_{i\in\mathbb{N}}$ mit $x_i \in \mathbb{R}$ **bedingt** konvergent mit $\sum_{i\in\mathbb{N}} x_i < \infty$, aber $\sum_{i\in\mathbb{N}} |x_i| = \infty$, so gibt es für jedes $x \in \mathbb{R}$ eine Permutation $p : \mathbb{N} \to \mathbb{N}$ mit $\sum_{j\in\mathbb{N}} x_{p(j)} = x$.

Beweis: Wegen $|x_i| = x_i^+ + x_i^-$ und $x_i = x_i^+ - x_i^-$ sind die Reihen der Positivteile $x_i^+ = \frac{1}{2}(|x_i| + x_i) = \max\{x_i; 0\}$ und der Negativteile $x_i^- = \frac{1}{2}(|x_i| - x_i) = \min\{x_i; 0\}$ **divergent**: $\sum_{i\in\mathbb{N}} x_i^+ = \sum_{i\in\mathbb{N}} x_i^- = \infty$. Für o.B.d.A. $x > 0$ und $n \geq 1$ definiere zunächst $i_0 = j_0 = 0$ und $i_1 = \min\left\{i_1' \geq 1 : \sum_{i=1}^{i_1'} x_i^+ \geq v\right\}$ sowie $j_1 = \min\left\{j_1' \geq 1 : \sum_{i=1}^{i_1} x_i^+ - \sum_{j=1}^{j_1'} x_j^- \leq x\right\}$. Anschließend setzt man die Umordnung induktiv fort mit

$$i_{n+1} = \min\left\{i_{n+1}' \geq i_n : \sum_{m=0}^{n-1}\left(\sum_{i=i_m+1}^{i_{m+1}'} x_i^+ - \sum_{j=j_m+1}^{j_{m+1}} x_j^-\right) + \sum_{i=i_n+1}^{i_{n+1}'} x_i^+ \geq x\right\}$$

sowie

$$j_{n+1} = \min\left\{j_{n+1}' \geq j_n : \sum_{m=0}^{n}\left(\sum_{i=i_m+1}^{i_{m+1}} x_i^+ - \sum_{j=j_m+1}^{j_{m+1}'} x_j^-\right) \leq x\right\}$$

Für die neugeordneten Partialsummen gilt

$$\left|\sum_{m=0}^{n}\left(\sum_{i=i_{m-1}+1}^{i_m} x_i^+ - \sum_{j=j_{m-1}+1}^{j_m} x_j^-\right) - x\right| \leq \max\left\{x_{i_n}^+; x_{j_n}^-\right\}$$

Da wegen der Konvergenz der Gesamtreihe $\lim_{i\to\infty} |x_i| = 0$, folgt daraus die Behauptung.

6.4 Korollar

Jeder affine Unterraum $v + \Gamma$ des $\mathbb{R}^n$ mit $v \in \mathbb{R}^n$ und einem Untervektorraum $\Gamma \subset \mathbb{R}^n$ lässt sich als Menge Σ der Grenzwerte der Partialsummen $\sum_{i\in\mathbb{N}} v_{p(i)}$ aller möglichen Permutationen $p : \mathbb{N} \to \mathbb{N}$ einer Folge $(v_i)_{i\in\mathbb{N}} \subset \mathbb{R}^n$ von Vektoren darstellen, denn nach dem Riemannschen Umordnungssatz gibt es für jedes $1 \leq j \leq m$ eine Permutation $p_j : \mathbb{N} \to \mathbb{N}$ der Folge $(x_i)_{i\in\mathbb{N}}$ mit $x_i = \frac{(-1)^i}{i}$, so dass $\sum_{i\in\mathbb{N}} x_{p_j(i)} = x_j$.

6.5 Eingrenzung von Polygonen

Für jede **endliche geschlossene Vektorkette** $(v_i)_{i\in I} \subset \mathbb{R}^n$ für $I = \{1; ...; m\}$ mit $\sum_{i=1}^{m} v_i = 0$ und Beträgen $\|v_i\| \leq 1, i \in I$ gibt es eine Permutation $p : I \to I$ mit $p(1) = 1$, so dass $\forall j \in I$ gilt $\left\|\sum_{i=1}^{j} v_{p(i)}\right\| \leq C_n$ mit $C_1 = 1$ und $C_n \leq \sqrt{4C_{n-1}^2 + 1}$.

Beweis durch Induktion über n: Für $n = 1$ und o.B.d.A $v_1 > 0$ wähle die folgenden $v_{p(2)}, v_{p(3)}, ... < 0$, bis die Summe im Bereich $-1 \leq v_1 + v_{p(2)} + v_{p(3)} + ... < 0$ liegt. Anschließend wähle wieder positive $v_{p(i)}$, bis die Summe wieder im poitiven Bereich liegt, usw., bis alle m Vektoren verbraucht sind. Wegen

$\|v_i\| \leq 1$ kann man die $v_{p(i)}$ so wählen, dass die Sume die Bereich $[-1; 1]$ nicht verlässt, womit sich die Behauptung mit $C_1 = 1$ ergibt. Für $n > 1$ wähle unter den 2^{m-1} möglichen Kombinationen die Summe $v = v_1 + u_1 + ... + u_s$ mit $\{u_1; ...; u_s\} \subset \{v_2; ...; v_m\}$ und maximaler Länge $\|v\|$. Man betrachtet den Vektor v als positive Bezugsrichtung und zeigt zunächst mit Hife des **inneren Produktes** $\langle \ldots, \ldots \rangle$, dass die $v_1, u_1, ..., u_s$ in Richtung v ,d.h. $\langle v_1, v\rangle, \langle u_i, v\rangle \geq 0$, und die übrigen Vektoren $\{w_1; ...; w_t\} := \{v_2; ...; v_m\} \setminus \{u_1; ...; u_s\}$ mit $1 + s + t = m$ und $v = -w_1 - ... - w_t$ in Richtung $-v$, d.h.$\langle w_j, v\rangle \leq 0$, orientiert sind:

Angenommen, $\langle v_1, v\rangle < 0$, dann wäre $\|v_1 + w_1 + w_2 + ...\| \geq \left\langle (v_1 + w_1 + w_2 + ...), \frac{-v}{\|v\|}\right\rangle = \frac{-\langle v_1, v\rangle}{\|v\|} + \|v\| > \|v\|$ und damit $v_1 + w_1 + w_2 + ...$ eine längere Vektorkette als L.

Angenommen, $\langle u_i, v\rangle < 0$, dann wäre $\|v - u_i\| \geq \left\langle (v - u_i), \frac{v}{\|v\|}\right\rangle = \frac{-\langle u_i, v\rangle}{\|v\|} + \|v\| > \|v\|$ und damit $v - u_i$ eine längere Vektorkette als L.

Angenommen, $\langle w_j, v\rangle > 0$, dann wäre $\|v + w_j\| \geq \left\langle (v + w_j), \frac{v}{\|v\|}\right\rangle = \frac{\langle w_j, v\rangle}{\|v\|} + \|v\| > \|v\|$ und damit $v + w_j$ eine längere Vektorkette als L.

Sei nun $u' := u - \left\langle u, \frac{v}{\|v\|}\right\rangle \frac{v}{\|v\|}$ die Komponente des Vektors u in dem $n-1$-dimensionalen Unterraum $\{v\}^{\perp} := \left\{u \in \mathbb{R}^n : \left\langle u, \frac{v}{\|v\|}\right\rangle = 0\right\}$ orthogonal zu $\{v\} := \left\{u \in \mathbb{R}^n : \left\langle u, \frac{v}{\|v\|}\right\rangle = u\right\}$. Wegen $v' = v_1' + u_1' + ... + u_s' = -w_1' - ... - w_t' = 0$ gibt es nach Induktionsvoraussetzung eine Permutation q auf $\{1; ...; s\}$, so dass $\forall 1 \leq j \leq s$ gilt $\left\|v_1' + \sum_{i=1}^{j} u_{q(i)}'\right\| \leq C_{n-1}$ und eine Permutation r auf $\{1; ...; t\}$ mit $q(1) = 1$, so dass $\forall 1 \leq j \leq t$ gilt $\left\|\sum_{i=1}^{j} w_{r(i)}'\right\| \leq C_{n-1}$. Die durch q bzw. r vorgegebene Reihenfolge der $u_{q(i)}$ bzw. $w_{r(i)}$ gewährleistet, dass die Vektorketten **beliebiger Länge** orthogonal zu v nicht länger als C_{n-1} werden. Man sucht nun analog zum Beweis für $n = 1$ passende Teilketten abwechselnd aus den $u_{q(i)}$ bzw. $w_{r(i)}$ so heraus, dass auch ihre Längen **parallel** zu v nicht länger als 1 werden: Man beginnt in positiver Richtung mit v_1 mit $0 \leq \left\langle v_1, \frac{v}{\|v\|}\right\rangle \leq 1$ und findet wegen $\left\langle w_j, \frac{v}{\|v\|}\right\rangle \leq 1$ bzw. $\sum_{j=1}^{t} \left\langle w_j, \frac{v}{\|v\|}\right\rangle = \|v\|$ ein $1 \leq t_1 \leq t$, so dass $-1 \leq \left\langle v_1, \frac{v}{\|v\|}\right\rangle + \sum_{i=1}^{t_1} \left\langle w_{r(i)}, \frac{v}{\|v\|}\right\rangle \leq 0$. Anschließend sucht man ein $1 \leq s_1 \leq s$, so dass $0 \leq \left\langle v_1, \frac{v}{\|v\|}\right\rangle + \sum_{i=1}^{t_1} \left\langle w_{r(i)}, \frac{v}{\|v\|}\right\rangle + \sum_{i=1}^{s_1} \left\langle u_{q(i)}, \frac{v}{\|v\|}\right\rangle \leq 1$. Als nächstes kommt wieder ein $t_1 < t_2 \leq t$, so dass $-1 \leq \left\langle v_1, \frac{v}{\|v\|}\right\rangle + \sum_{i=1}^{t_1} \left\langle w_{r(i)}, \frac{v}{\|v\|}\right\rangle + \sum_{i=1}^{s_1} \left\langle u_{q(i)}, \frac{v}{\|v\|}\right\rangle + \sum_{i=t_1+1}^{t_2} \left\langle w_{r(i)}, \frac{v}{\|v\|}\right\rangle \leq 0$, usw., bis alle $u_{q(i)}$ bzw. $w_{r(i)}$ verbraucht sind. Die gesuchte Anordnung ist also $\left(v_1; w_{r(1)}, ..., w_{r(t_1)}, u_{q(1)}, ..., u_{q(s_1)}, w_{r(t_1+1)}, ..., w_{r(t_2)}, ...\right)$. Die Länge der Vektorkette in Richtung v ist höchstens 1 und orthogonal dazu in Richtung $v^{\perp}$ für die $u_{q(i)}$ bzw. $w_{r(i)}$ höchstens $C_{n-1} + C_{n-1}$. Insgesamt gilt also $C_n \leq \sqrt{4C_{n-1}^2 + 1}$.

6.6 Korollar

Für jede **endliche Vektorkette** $(v_i)_{i \in I} \subset \mathbb{R}^n$ mit $I = \{1; ...; m\}$ und $\left\|\sum_{i=1}^{m} v_i\right\| \leq \epsilon$ sowie Beträgen $\|v_i\| \leq \epsilon, i \in I$ gibt es eine Permutation $p : I \to I$ mit $p(1) = 1$, so dass $\forall j \in I$ gilt $\left\|\sum_{i=1}^{j} v_{p(i)}\right\| \leq \epsilon(C_n + 1)$, denn die v_i lassen sich durch $v_{m+1} := -\sum_{i=1}^{m} v_i$ zu einer geschlossenen Vektorkette gemäß 2.1 ergänzen, so dass $\left\|\sum_{i=1}^{j} v_{p(i)}\right\| \leq \epsilon C_n$ und damit $\left\|\sum_{\substack{1 \leq i \leq j \\ p(i) \neq m+1}} v_{p(i)}\right\| \leq \epsilon C_n + \epsilon$.

6.7 Korollar

Für jede **endliche Vektorkette** $(v_i)_{i\in I} \subset \mathbb{R}^n$ mit $I = \{1;...;m\}$ und Beträgen $\|v_i\| \leq \epsilon, i \in I$ sowie jedes $0 < t < 1$ gibt es eine Permutation $p : I \to I$ mit $p(1) = 1$, und ein $j \in I$, so dass $\left\|\sum\limits_{i=1}^{j} v_{p(i)} - tv\right\| \leq \epsilon\sqrt{C_{n-1}^2 + 1}$ mit $v := \sum\limits_{i=1}^{m} v_i$.

Beweis: Sei zunächst $n = 1$ und o.B.d.A $v > 0$ sowie $1 \leq j \leq m$ der kleinste Index mit $\sum\limits_{i=1}^{j} v_i - tv > 0$, dann ist $\sum\limits_{i=1}^{j-1} v_i - tv < 0$ und wegen $v_j < \epsilon$ folgt $\left|\sum\limits_{i=1}^{j} v_i - tv\right| < \epsilon$, womit die Behauptung für $C_0 := 0$ erfüllt ist. Im Fall $n > 1$ betrachtet man wie im Beweis zu 20.4 die Projektionen $v_i' := v_i - \left(v_i, \frac{v}{\|v\|}\right) \frac{v}{\|v\|}$ auf den orthogonalen Unterraum $\{v\}^\perp$. Wegen $\sum\limits_{i=1}^{j} v_i' = v' = 0$ und $\|v_i'\| \leq \epsilon$ lässt sich 2.1 anwenden und liefert eine Permutation $p : I \to I$ mit $p(1) = 1$, so dass $\forall j \in I$ gilt $\left\|\sum\limits_{i=1}^{j} v'_{p(i)}\right\| \leq \epsilon C_{n-1}$. Für die Komponenten $\left\langle v_i, \frac{v}{\|v\|}\right\rangle \frac{v}{\|v\|}$ parallel zu v gilt $\left\langle v_i, \frac{v}{\|v\|}\right\rangle < \epsilon$ und $\left\|\sum\limits_{i=1}^{m} \left\langle v_{p(i)}, \frac{v}{\|v\|}\right\rangle\right\| = \|v\| = v, \frac{v}{\|v\|}$, so dass sich wie im Beweis für $n = 1$ ein $j \in I$ finden lässt mit $\left\|\sum\limits_{i=1}^{j} \left\langle v_{p(i)}, \frac{v}{\|v\|}\right\rangle - t\|v\|\right\| < \epsilon$. Die Differenz $\sum\limits_{i=1}^{j} v'_{p(i)} - tv$ der beiden Vektorketten und in Richtung v ist höchstens ϵ und orthogonal dazu in Richtung $v^\perp$ höchstens ϵC_{n-1}. Insgesamt gilt also $\left\|\sum\limits_{i=1}^{j} v_{p(i)} - tv\right\| \leq \epsilon\sqrt{C_{n-1}^2 + 1}$.

6.8 Umordnungssatz

Besitzt die Folge $(S_m)_{m\geq 1}$ der Partialsummen $S_m = \sum\limits_{i=1}^{m} v_i$ einer Folge $(v_i)_{i\in\mathbb{N}} \subset \mathbb{R}^n$ von Vektoren eine Teilfolge $(S_{m_k})_{k\geq 1}$, die gegen ein $S \in \mathbb{R}^n$ konvergiert, so lässt sich die Gesamtreihe durch eine Bijektion $p : \mathbb{N} \to \mathbb{N}$ so umordnen, dass sie gegen S konvergiert: $\lim\limits_{m\to\infty} \left\|S_{p(m)} - S\right\| = 0$.

Beweis: Man verwendet 2.2, um die zwischen den Gliedern der konvergierenden Teilfolge liegenden Vektoren $v_{m_k+1}, ..., v_{m_{k+1}-1}$ so umzuordnen, dass ihre Partialsummen und damit die Abweichungen von S_{m_k} minimal werden: Für $\delta_k := \|S_{m_k} - S\|$ und $\epsilon_k := \max\{\delta_k + \delta_{k+1}, \sup\{\|v_i\| : i \geq m_k\}\}$ gilt $\left\|\sum\limits_{i=m_k+1}^{m_{k+1}-1} v_i\right\| = \left\|\left(\sum\limits_{i=1}^{m_{k+1}} v_i - S\right) - \left(\sum\limits_{i=1}^{m_k} v_i - S\right) - v_{m_{k+1}}\right\| < \delta_{k+1} + \delta_k + \|v_{k+1}\| < 2\epsilon_k$. Gemäß 20.5 existiert eine Permutation $p_k$von $\{m_k + 1; ..., ; m - 1\}$ mit $\left\|\sum\limits_{i=m_k+1}^{j} v_{p_k(i)}\right\| \leq 2\epsilon_k (C_n + 1) \forall m_k + 1 \leq j \leq m_{k+1} - 1$. Setzt man $p(m) := p_k(i)$ für $m_k + 1 \leq i \leq m_{k+1} - 1$ und $p(m_k) := m_k$ sonst, so folgt $\left\|S_{p(m)} - S_{m_k}\right\| \leq 2\epsilon_k (C_n + 1) \to 0$ und wegen $\|S_{m_k} - S\| = \delta_k \to 0$ schließlich die Behauptung.

6.9 Satz von Lévy und Steinitz I

Die Menge Σ der Grenzwerte der Partialsummen $\sum_{i\in\mathbb{N}} v_{p(i)}$ aller möglichen Permutationen $p : \mathbb{N} \to \mathbb{N}$ einer gegebenen Folge $(v_i)_{i\in\mathbb{N}} \subset \mathbb{R}^n$ von Vektoren ist ein affiner Unterraum der Gestalt $\Sigma = \Sigma + \Gamma$ mit einem Untervektorraum $\Gamma \subset \mathbb{R}^n$.

Beweis: Für $\Sigma = \emptyset$ ist nichts zu zeigen. Sei also $v \in \Sigma \neq \emptyset$, dann kann o.B.d.A. v_1 durch $v_1 - v$ ersetzt werden und durch diese Verschiebung der gesamten Summe erhält man $0 \in S$. Es genügt nun zu zeigen, dass Σ ein Untervektorraum ist:

$s_1, s_2 \in \Sigma \Rightarrow s_1{+}s_2 \in \Sigma:$ Da es drei verschiedene Anordnungen gibt, die **unabhängig von endlichen Umordnungen** gegen s_1, 0 bzw. s_2 konvergieren, existieren für jedes $m \geq 1$ **endliche** Indexmengen $\{1; ...; m\} \subset I_m \subset J_m \subset K_m \subset I_{m+1} \subset ...$ mit $\left\|\sum_{i\in I_m} v_i - s_1\right\| < 2^{-m}$, $\left\|\sum_{i\in J_m} v_i - 0\right\| < 2^{-m}$ und $\left\|\sum_{i\in K_m} v_i - s_2\right\| < 2^{-m}$. Die Summe bewegt sich also auf I_m in Richtung s_1, dann auf $J_m \setminus I_m$ wieder zurück in Richtung $-s_1$ bzw. 0 und schließlich auf $K_m \setminus J_m$ in Richtung s_2. Wenn man die Summanden im Bereich $J_m \setminus I_m$ entfernt, werden die Restsummen gegen $s_1 + s_2$ streben: Mittels einer endlichen Permutation $p : \mathbb{N} \to \mathbb{N}$ lassen sich die Indexmengen I_m, J_m, K_m und I_{m+1} aufsteigend sortieren, so dass es Indizes $i_m < j_m < k_m < i_{m+1} < ...$ gibt mit $I_m = p\left[\{1; ...; i_m\}\right]$, $J_m = p\left[\{1; ...; j_m\}\right]$, usw. und damit $\left\|\sum\limits_{i=1}^{i_m} v_{p(i)} - s_1\right\| < \frac{1}{m}$, $\left\|\sum\limits_{i=1}^{j_m} v_{p(i)}\right\| < \frac{1}{m}$ und $\left\|\sum\limits_{i=1}^{k_m} v_{p(i)} - s_2\right\| < \frac{1}{m}$. Wegen $\left\|\sum\limits_{i=j_m+1}^{k_m} v_{p(i)} - s_2\right\| = \left\|\sum\limits_{i=1}^{k_m} v_{p(i)} - s_2 - \sum\limits_{i=1}^{j_m} v_{p(i)}\right\| < \frac{2}{m}$ folgt daraus für den Rest $\left\|\sum\limits_{i=1}^{i_m} v_{p(i)} + \sum\limits_{i=j_m+1}^{k_m} v_{p(i)} - (s_1 + s_2)\right\| < \frac{3}{m}$.

Wenn man den hinteren Abschnitt mittels $p'(p(i)) = \begin{cases} p(i + j_m - i_m) & \text{für } i_m < i \leq i_m + k_m - j_m \\ p(i - j_m + i_m) & \text{für } i_m + k_m - j_m < i \leq k_m \\ p(i) & \text{sonst} \end{cases}$

nach vorne in die Lücke schiebt, ergibt sich eine Partialsumme $\left\|\sum\limits_{i=1}^{i_m+k_m-j_m} v_{p'\circ p(i)} - (s_1 + s_2)\right\| < \frac{3}{m}$ und damit eine gegen $s_1 + s_2$ konvergierende Teilfolge, woraus nach 2.4 die Behauptung folgt.

$s \in \Sigma \Rightarrow ts \in \Sigma \forall t \in \mathbb{R}$: Man verwendet o.B.d.A s_2 mit den Anordnungen aus Teil 1. und nutzt die eben bewiesene Additivität, um sich auf $0 < t < 1$ zu beschränken und damit 2.3 anwenden zu können: Mit $\delta_m = \sup\left\{\left\|v_{p(i)}\right\| : i = j_m + 1, ..., k_m\right\}$ gibt es eine Permutation q_m von $\{p(j_m + 1), ..., p(k_m)\}$, so dass $\left\|\sum\limits_{i=j_m+1}^{k_m} v_{q_m(p(i))} - t\cdot \sum\limits_{i=j_m+1}^{k_m} v_{p(i)}\right\| \leq \delta_m\sqrt{C_{n-1}^2 + 1}$. Wegen $\left\|t\cdot \sum\limits_{i=j_m+1}^{k_m} v_{p(i)} - ts_2\right\| < \frac{2}{m}$ und $\left\|\sum\limits_{i=1}^{j_m} v_{p(i)}\right\| < \frac{1}{m}$ folgt $\left\|\sum\limits_{i=1}^{j_m} v_{p(i)} + \sum\limits_{i=j_m+1}^{k_m} v_{q_m(p(i))} - ts_2\right\| \leq \delta_m\sqrt{C_{n-1}^2 + 1} + \frac{3}{m}$. Aus der Annahme $\Sigma \neq \emptyset$ folgt $\lim\limits_{m\to\infty} \delta_m = 0$, so dass die Teilfolge der so umgeordneten Partialsummen gegen ts_2 konvergiert und aus 2.4 folgt wieder die Behauptung.

$s \in \Sigma \Rightarrow -s \in \Sigma$: Man verwendet wieder s_2 mit den Anordnungen aus Teil 1. und betrachtet $\left\|\sum\limits_{i=1}^{j_m} v_{p(i)} + \sum\limits_{i=k_m+1}^{j_{m+1}} v_{p(i)} - (-s_2)\right\| = \left\|\sum\limits_{i=1}^{j_m} v_{p(i)} + \sum\limits_{i=1}^{j_{m+1}} v_{p(i)} - \left(\sum\limits_{i=1}^{k_m} v_{p(i)} - s_2\right)\right\| < \frac{1}{m} + \frac{1}{m+1} + \frac{1}{m}$. Analog zu 1. lässt sich durch die Rückverschiebung der letzten $j_{m+1} - (k_m + 1)$ Glieder eine gegen $-s_2$ konvergierende Teilfolge von Partialsummen bilden, so dass sich mit 2.4 erneut die Behauptung ergibt.

6.10 Komponentenweise bedingte Konvergenz

Eine Reihe $\sum_{i\in\mathbb{N}} v_i$ mit $v_i \in \mathbb{R}^n$ heißt **bedingt konvergent**, wenn es eine Permutation $p : \mathbb{N} \to \mathbb{N}$ gibt mit $\left\|\sum_{i\in\mathbb{N}} v_{p(i)}\right\| < \infty$ und **komponentenweise bedingt konvergent**, wenn die Summen $\sum_{i\in\mathbb{N}} \langle v_i, w\rangle$ der Projektionen in alle Richtungen $w \in \mathbb{R}^n$ und insbesondere die Summen $\sum_{i\in\mathbb{N}} v_{ij}$ der Komponenten $v_{ij} = \langle v_i, e_j\rangle$ bedingt konvergent sind. Aufgrund der (Sesqui-)Linearität des Skalarproduktes und der Schwarz-Ungleichung gilt $\left|\sum\limits_{i=0}^{m} \langle v_i, w\rangle\right| = \left|\left\langle \left(\sum\limits_{i=0}^{m} v_i\right), w\right\rangle\right| \leq \left\|\sum\limits_{i=0}^{m} v_i\right\| \cdot \|w\|$ für alle $m \in \mathbb{N}$, d.h., eine bedingt konvergente Vektorreihe ist insbesondere komponentenweise bedingt konvergent.

6.11 Folge der Summanden

Für eine komponentenweise bedingt konvergente Reihe $\sum_{i\in\mathbb{N}} v_i$ konvergiert die Folge der Summanden gegen Null: $\lim\limits_{i\to\infty} \|v_i\| = 0$.

Beweis: Angenommen, es gibt ein $\epsilon > 0$, so dass die Menge $E = \{n \in \mathbb{N} : \|v_i\| > \epsilon\}$ unendlich ist, und die Folge $\left(\frac{v_i}{\|v_i\|}\right)_{i\in E} \subset S$ auf der kompakten Menge S einen Häufungspunkt v_∞ besitzt mit einem $n_0 \in \mathbb{N}$, so dass $\left\|\frac{v_i}{\|v_i\|} - v_\infty\right\| < \frac{\epsilon}{2}$ für alle $i \geq n_0$. Dann gilt $|\langle v_i, v_\infty\rangle| = \|v_i\| \cdot \left|\left\langle \frac{v_i}{\|v_i\|}, v_\infty \right\rangle\right| = \|v_i\| \cdot \left|\left\langle \frac{v_i}{\|v_i\|}, v_\infty - \frac{v_i}{\|v_i\|}\right\rangle + \left\langle \frac{v_i}{\|v_i\|}, \frac{v_i}{\|v_i\|}\right\rangle\right| \geq \epsilon\left(1 - \frac{\epsilon}{2}\right) = \frac{\epsilon}{2}$ für alle $i \in E$, so dass die Reihe $\sum_{i\in\mathbb{N}} \langle v_i, v_\infty\rangle$ nicht in eine konvergente Reihe umgeordnet werden kann im Widerspruch zur Voraussetzung der komponentenweise bedingten Konvergenz.

6.12 Divergenzpunkte und absolute Konvergenz

Ein Punkt $x \in S = \{x \in \mathbb{R}^n : \|x\| = 1\}$ heißt **Divergenzpunkt** der Reihe $\sum_{i\in\mathbb{N}} v_i$, wenn die Menge $\mathbb{N}_U := \left\{i \in \mathbb{N} : \frac{v_i}{\|v_i\|} \in U\right\}$ für jede Umgebung $U \subset S$ von x unendlich ist und die Teilfolge $\sum_{i\in\mathbb{N}_U} \|v_i\| = \infty$ divergiert. Die Menge $D \subset S$ aller Divergenzrichtungen einer bedingt konvergenten Reihe ist nach Definition abgeschlossen in S und als Teilmenge der kompakten und insbesondere abgschlossenen **Einheitsspäre** S auch abgeschlossen in $\mathbb{R}^n$. Im Fall $D = \emptyset$ gibt es eine endliche Überdeckung $\mathcal{U}$ offener Mengen $U \subset S$, für die jeweils $\sum_{i\in\mathbb{N}_U} \|v_i\| < \infty$ und folglich $\sum_{i\in\mathbb{N}} \|v_i\| \leq \sum_{U\in\mathcal{U}} \sum_{i\in\mathbb{N}_U} \|v_i\| < \infty$, d.h., **absolute Konvergenz** mit $\Gamma = \mathbb{R}^n$, $\Gamma^\perp = \emptyset$ und $\Sigma = \{v\}$ für $v = \sum_{i\in\mathbb{N}} v_i$.

6.13 Konvexe Hülle der Divergenzpunkte

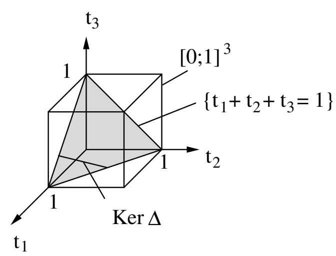
Im Fall $D \neq \emptyset$ enthält die **konvexe Hülle**

$$\mathrm{co}(D) = \left\{\sum_{k=0}^{m} t_k x_k : \sum_{k=0}^{m} t_k = 1,\, t_k \in [0;1],\, x_k \in D,\, 0 \leq k \leq m,\, m \in \mathbb{N}\right\}$$

den Koordinatenursprung, denn ansonsten existiert nach dem Satz von Hahn-Banach ([7, 5.2]) ein $w \in \mathbb{R}^n$ und ein $\epsilon > 0$ mit $\langle x, w\rangle > \epsilon \forall x \in \mathrm{co}(D)$ sowie wegen der Stetigkeit des linearen Funktionals $\langle \ldots, w\rangle : \mathbb{R}^n \to \mathbb{R}$ auch eine Umgebung $U \in \mathcal{U}(x)$ mit $\langle x, w\rangle > \frac{\epsilon}{2} \forall x \in U$, so dass $\sum_{i\in\mathbb{N}} \left\langle v_{p(i)}, w\right\rangle$ für keine Permutation p konvergieren kann im Gegensatz zur Annahme der komponentenweisen bedingten Konvergenz von $\sum_{i\in\mathbb{N}} v_i$.

6.14 Konvexe Hülle der minimalen Divergenzpunkte

Die minimale Teilmenge $D_0 \subset D$ mit $0 \in \mathrm{co}(D_0)$ ist affin unabhängig und besteht daher aus höchstens $n+1$ Punkten, denn im Fall $D_0 = \{x_o, ..., x_{n+1}\}$ ist der **Kern** $\mathrm{Ker}\Delta = \{t = (t_1; ...; t_{n+1}) \in \mathbb{R}^{n+1} : \Delta(t) = 0\}$ der linearen Abbildung $\Delta : \mathbb{R}^{n+1} \to \mathbb{R}^n$ mit $\Delta(t) = \sum_{k=0}^{n+1} t_k x_k \wedge t_0 = 1 - \sum_{k=1}^{n+1} t_k$ für $x_0, ..., x_{n+1} \in D \subset \mathbb{R}^n$ ein mindestens eindimensionaler und höchstns $(n+1)$-dimensionaler Unterraum des $\mathbb{R}^{n+1}$ in der Form $\mathrm{Ker}\Delta = \left\{t \in \mathbb{R}^{n+1} : t = a + \sum_{r=1}^{n+1} s_r b_r,\, s_1, ..., s_{n+1} \in \mathbb{R}\right\}$ mit $a \in \mathbb{R}^{n+1}$ und linear unabhängigen $b_1, ..., b_{n+1} \in \mathbb{R}^{n+1}$ sowie $b_1 \neq 0$, der wegen 3.4 den Definitionsbereich $[0;1]^{n+2}$ der konvexen Hülle und für jedes $0 \leq j \leq n+1$ mit $\langle e_j, b_r\rangle \neq 0$ seinen **Rand** $\left\{(t_0, ..., t_{n+1}) \in [0;1]^{n+2} : \exists 0 \leq j \leq n+1 : t_j = 0\right\}$ schneidet. Dabei kann man sich das Urbild $\Delta^{-1}[\mathrm{co}(D_0)]$ der konvexen Hülle als mindestens eindimensionale und höchstns $(n+1)$-dimensionale Hyperfläche vorstellen, die durch den Schnitt der $(n+1)$-dimensionalen Hyperebene $\left\{t = (t_0; ...; t_{n+1}) \in \mathbb{R}^{n+2} : \sum_{k=0}^{n+1} t_k = 1\right\}$ mit dem Einheitswürfel $[0;1]^{n+2}$ entsteht. An jedem dieser Durchstoßpunkte ist mindestens ein $t_j = 0$ und $0 = \sum_{\substack{0\leq k\leq n+1\\ k\neq j}} t_k x_k \wedge \sum_{\substack{0\leq k\leq n+1\\ k\neq j}} t_k = 1$ erfüllt, d.h., $0 \in \mathrm{co}(D_0 \setminus \{x_j\})$. Die minimale konvexe Hülle enthält damit sogar eine **offene Umgebung** des Ursprungs: $0 \in U \subset \mathrm{co}(D_0)$ für ein $U \in \mathcal{U}(0)$, denn falls der Ursprung auf dem **Rand**



$\left\{\sum_{k=1}^{m} t_k x_k \in \mathrm{co}(D_0) : \exists 1 \leq j \leq n+1 : t_j = 0\right\}$ liegt, könnte wieder der entsprechende Divergenzpunkt x_j entfernt werden, d.h. $0 \in \mathrm{co}\left(D_0 \setminus \{x_j\}\right)$ im Widerspruch zum minimalen Charakter von D_0.

6.15 Lineare Hülle der minimalen Divergenzpunkte

Sei $X_0 = \left\{\sum_{k=1}^{m} t_k x_k : x_1, ..., x_m \in D_0\right\}$ die **lineare Hülle** von $D_0 = \{x_1; ...; x_m\}$ mit der **Projektion** $\mathrm{pr}_0 : \mathbb{R}^n \to X_0$, wobei $\mathrm{pr}_0(v) = \sum_{k=1}^{m} \left\langle v, \frac{x_k}{\|x_k\|}\right\rangle \frac{x_k}{\|x_k\|}$ und $X_1 = X_0^{\perp} = \{y \in \mathbb{R}^n : \langle x, y\rangle = 0 \forall x \in X_0\}$ das **orthogonale Komplement** mit der entsprechenden **Projektion** $\mathrm{pr}_1 : \mathbb{R}^n \to X_1$, wobei $\mathrm{pr}_1(v) = \sum_{k=m+1}^{n} \left\langle v, \frac{x_k}{\|x_k\|}\right\rangle \frac{x_k}{\|x_k\|}$ für die **Basis** $\{x_{m+1}; ...; x_n\}$ von $D_0^{\perp}$. Dann gibt es eine Teilmenge $\Omega \subset \mathbb{N}$ mit $\sum_{i\in\Omega} \|\mathrm{pr}_1(v_i)\| < \infty$ und $\sum_{i\in\Omega\cap\mathbb{N}_U} \|v_i\| = \infty$ für jede Umgebung $U \subset S$ aller $x_k \in D_0$

Beweis: Für jedes $n \in \mathbb{N}$ und $v \in B_n := B_{2^{-n}}(x) \subset S$ gilt $\|\mathrm{pr}_1(v)\| = \|\mathrm{pr}_1(v-x) + \mathrm{pr}_1(x)\| = \|\mathrm{pr}_1(v-x) + 0\| \leq \|v - x\| < 2^{-n}$. Für jedes $x_k \in D_0 \subset D$ ist $\mathbb{N}_{B_n}$ unendlich und $\sum_{i\in\mathbb{N}_{B_n}} \|v_i\| = \infty$. Daher und nach 3.2 gibt es eine endliche Teilmenge $F_n \subset \mathbb{N}_{B_n}$ mit $n < \sum_{i\in F_n} \|v_i\| < n + 1$. Für $O_k = \bigcup_{n\in\mathbb{N}} F_n$ gilt dann $\sum_{i\in O_k} \|\mathrm{pr}_1(v_i)\| \leq \sum_{n\in\mathbb{N}} \sum_{i\in F_n} \|\mathrm{pr}_1(v_i)\| = \sum_{n\in\mathbb{N}} \sum_{i\in F_n} \left\|\mathrm{pr}_1\left(\frac{v_i}{\|v_i\|}\right)\right\| \cdot \|v_i\| \leq \sum_{n\in\mathbb{N}} \frac{n+1}{2^n} < \infty$ und für $\Omega = \bigcup_{k=1}^{m} O_k$ folglich die erste Behauptung. Für eine beliebige Umgebung $U \subset S$ von $x_k \in D_0$ gibt es ein $n \in \mathbb{N}$ mit $B_{2^{-n}}(x_k) \subset U$ und damit $F_m \subset \mathbb{N}_{B_m} \subset \mathbb{N}_U$ für alle $m \geq n$, so dass $\sum_{i\in\Omega\cap\mathbb{N}_U} \|v_i\| \geq \sum_{i\in\bigcup_{m\geq n} F_m} \|v_i\| \geq \sup_{m\geq n} m = \infty$, womit die zweite Behauptung gezeigt ist.

6.16 Näherung der linearen Hülle der minimalen Divergenzpunkte

Es gibt eine positive Konstante C, so dass für $x \in X_0$, $\epsilon > 0$ und jede endliche Teilmenge $F \subset \Omega$ eine weitere endliche Teilmenge $E \subset \Omega \setminus F$ existiert mit $\|x - \sum_{i\in E} v_i\| < \epsilon$ und $\|\sum_{i\in E'} v_i\| \leq C \cdot \max\{\|x\|, \epsilon\}$ für alle $E' \subset E$.

Beweis: Nach 3.5 gibt es ein $\delta < \frac{1}{4}$ mit $B_\delta(0) \subset \mathrm{co}(D_0)$ und $B_\delta(x_i) \cap B_\delta(x_j) = \emptyset \, \forall x_i, x_j \in D_0$. Seien $x \in X_0$, $\epsilon < \delta$ und die endliche Teilmenge $F \subset \Omega$ gegeben. Für $c = \frac{1}{\delta} \max\{\|x\|, \epsilon\}$ folgt $\frac{x}{c} \in B_\delta(0) \subset \mathrm{co}(D_0)$ und damit $x = \sum_{k=1}^{m} c \cdot t_k x_k$ für $x_1, ..., x_m \in D_0$ und $0 \leq t_k \leq 1$ mit $\sum_{k=1}^{m} t_k = 1$. Auf der sphärischen Kreisscheibe $S_k = B_\delta(x_k) \cap S$ gilt zunächst $x, y \in S_k \Rightarrow \langle x, y\rangle = \langle x, x\rangle + \langle x, y - x\rangle \geq 1 - \|x - y\| > 1 - 2\delta \geq \frac{1}{2}$, woraus für den zugehörigen Richtungskegel $\hat{S}_k = \{t \cdot x : x \in S_k, t > 0\}$ die Abschätzung $x, y \in \hat{S}_k \Rightarrow \|x + y\| \geq \left\langle x + y, \frac{x}{\|x\|}\right\rangle = \|x\| + \|y\| \left\langle \frac{y}{\|y\|}, \frac{x}{\|x\|}\right\rangle \geq \|x\| + \frac{1}{2}\|y\|$ folgt. Nach 3.6 sind die Mengen $\Omega_k := \left\{i \in \Omega : \frac{v_i}{\|v_i\|} \in S_k\right\} = \left\{i \in \Omega : v_i \in \hat{S}_k\right\}$ für $1 \leq k \leq m$ unendlich und $\sum_{i\in\Omega_k} \|v_i\| \geq \sum_{i\in\Omega_k\cap\mathbb{N}_U} \|v_i\| = \infty$. Da wegen $(v_i)_{i\in\Omega_k} \subset \hat{S}_k$ auch $\sum_{i\in\Omega_k} v_i \in \hat{S}_k$ und außerdem $\lim_{i\to\infty} \|v_i\| = 0$ gilt, lässt sich für eine gegebene endliche Teilmenge $F \subset \Omega_k$ eine weitere endliche Teilmenge $E_k \subset \Omega_k \setminus F$ finden, so dass $s_k = \sum_{i\in E_k} v_i \in \hat{S}_k$ und $c \cdot t_k - \frac{\epsilon}{2m} < \|s_k\| \leq c \cdot t_k$. Damit folgt $\|s_k - c \cdot t_k x_k\| \leq \left\|\|s_k\| \cdot \frac{s_k}{\|s_k\|} - c \cdot t_k \cdot \frac{s_k}{\|s_k\|}\right\| + \left\|c \cdot t_k \cdot \frac{s_k}{\|s_k\|} - c \cdot t_k x_k\right\| = \left\|\|s_k\| - c \cdot t_k\right\| + c \cdot t_k \left\|\frac{s_k}{\|s_k\|} - x_k\right\| < \frac{\epsilon}{2m} + c \cdot t_k \cdot \frac{\epsilon}{2c}$ und für die endliche Teilmenge $E = \bigcup_{k=1}^{m} E_k \subset \Omega \setminus F$ folgt $\|\sum_{i\in E} v_i - x\| = \left\|\sum_{i\in E} v_i - \sum_{k=1}^{m} c \cdot t_k x_k\right\| \leq \sum_{k=1}^{m} \left\|\sum_{i\in E_k} v_i - c \cdot t_k x_k\right\| < \sum_{k=1}^{m} \left(\frac{\epsilon}{2m} + c \cdot t_k \cdot \frac{\epsilon}{2c}\right) = \frac{\epsilon}{2}\left(1 + \sum_{k=1}^{m} t_k\right) = \epsilon$.

Für die zweite Behauptung sei $E' \subset E$ und $E'_k := E \cap E_k$. Dann ist $\sum_{i\in E'_k} v_i \in \hat{S}_k$ mit $\left\|\sum_{i\in E'_k} v_i\right\| \leq \left\|\sum_{i\in E_k} v_i\right\| \leq c \cdot t_k \leq c$, also $\sum_{i\in E'_k} v_i = t \cdot y$ mit $y \in S_k$ und $0 \leq t \leq c$, so dass $\|\sum_{i\in E'} v_i\| \leq c \cdot \delta \cdot C = C \cdot \max\{\|x\|, \epsilon\}$ mit $C := \frac{1}{\delta} \sup\left\{\left\|\sum_{k=1}^{m} t_k y_k\right\| : 0 \leq t_k \leq 1, y_k \in B_\delta(x_k), x_k \in D_0\right\}$.

6.17 Die Richtungen absoluter Konvergenz

Die Richtungen absoluter Konvergenz sind orthogonal zur linearen Hülle der Divergenzpunkte: $\Gamma = \{y \in \mathbb{R}^n : \sum_{i\in\mathbb{N}} |\langle y, v_i\rangle| < \infty\} = \{y \in X_1 : \sum_{i\in\mathbb{N}} |\langle y, \mathrm{pr}_1(v_i)\rangle| < \infty\} = \Gamma_1$

Beweis: $\subset$: Angenommen, $\exists y \in \Gamma \setminus X_1$, dann existiert insbesondere ein $x_k \in D_0$ mit $\langle y, x_k\rangle \neq 0$ und die Menge $\mathbb{N}_U := \left\{i \in \mathbb{N} : \frac{v_i}{\|v_i\|} \in U\right\}$ für die offene Umgebung $U = \{x \in S : |\langle y, x\rangle| > |\langle y, x_k\rangle|\}$ von y ist unendlich und $\sum_{i\in\mathbb{N}_\mathbb{U}} \|v_i\| = \infty$. Damit folgt aber $|\langle y, v_i\rangle| > \|v_i\| \cdot |\langle y, x_k\rangle| \forall i \in \mathbb{N}_U$ und folglich $\sum_{i\in\mathbb{N}_\mathbb{U}} |\langle y, v_i\rangle| = \infty$ im Widerspruch zur Auswahl von y. $\supset$: Für $y \in X_1$ ist $\langle y, \mathrm{pr}_1(v_i)\rangle = \langle y, v_i\rangle - \langle y, \mathrm{pr}_0(v_i)\rangle = \langle y, v_i\rangle - 0 = \langle y, v_i\rangle$, woraus sich die Behauptung ergibt.

6.18 Satz von Lévy und Steinitz II

Die Menge Σ aller möglichen Summen $\sum_{i\in\mathbb{N}} v_{p(i)}$ zu den Permutationen $p : \mathbb{N} \to \mathbb{N}$ einer Folge $(v_i)_{i\in\mathbb{N}} \subset \mathbb{R}^n$ von Vektoren ist ein affiner Unterraum: $\Sigma = \Sigma + \Gamma^\perp$, wobei der Untervektorraum $\Gamma = \{y \in \mathbb{R}^n : \sum_{i\in\mathbb{N}} |\langle y, v_i\rangle| < \infty\}$ die Richtungen absoluter Konvergenz und $\Gamma^\perp = \{x \in \mathbb{R}^n : \langle y, x\rangle = 0 \forall y \in \Gamma\}$ das orthogonale Komplement zu Γ beschreiben. Dabei gilt $\Sigma \neq \emptyset$ genau dann, wenn $\sum_{i\in\mathbb{N}} v_i$ komponentenweise bedingt konvergent ist. Im Fall absoluter Konvergenz mit $\sum_{i\in\mathbb{N}} \|v_i\| < \infty$ ist $\Sigma = \{v\}$ mit $v = \sum_{i\in\mathbb{N}} v_i$ sowie $\Gamma = \mathbb{R}^n$und folglich $\Gamma^\perp = \emptyset$.

Beweis durch Induktion nach n: Aus der komponentenweise bedingten Konvergenz von $\sum_{i\in\mathbb{N}} v_i$ in $\mathbb{R}^n$ folgt nach 3.8 die komponentenweise bedingte Konvergenz von $\sum_{i\in\Omega} \mathrm{pr}_1(v_i)$ im Hilbertraum $X_1 \simeq \mathbb{R}^m$ mit der Dimension $m < n$, falls keine absolute Konvergenz vorliegt. (vgl. 3.3). Nach einem geeigneten Basiswechsel kann die Induktionsannahme auf $\sum_{i\in\mathbb{N}} \mathrm{pr}_1(v_i)$ angewandt werden, so dass die Menge Σ_1 aller möglichen Summen $\sum_{i\in\Omega} \mathrm{pr}_1\left(v_{p(i)}\right)$ die Struktur $\Sigma_1 = \Sigma_1 + \Gamma_1^\perp \cap X_1 = \Sigma_1 + \Gamma^\perp \cap X_1$ besitzt, denn nach 3.8 gilt $\Gamma_1 = \Gamma = \{y \in \mathbb{R}^n : \sum_{i\in\mathbb{N}} |\langle y, v_i\rangle| < \infty\}$. Man kann nun zeigen, dass für die Menge Σ aller möglichen Summen $\sum_{i\in\mathbb{N}} v_{p(i)}$ gilt $\Sigma = \Sigma_1 + \left(\Gamma^\perp \cap X_1\right) \oplus X_0 = \Sigma_1 + \Gamma^\perp$, wobei die Gleichung $\left(\Gamma^\perp \cap X_1\right) \oplus X_0 = \Gamma^\perp$ mit Hilfe der Zerlegung $x = \mathrm{pr}_0(x) + \mathrm{pr}_1(x)$ sowie der Beziehung $\Gamma \subset X_1$ eingesehen werden kann. Die Inklusion $\Sigma \subset \Sigma_1 + \left(\Gamma^\perp \cap X_1\right) \oplus X_0$ ergibt sich trivialerweise aus der Zerlegung $\mathbb{R}^n = X_0 \oplus X_1$. Für die Umkehrung ist zu zeigen, dass für jedes $x \in X_0$ und $y \in \Sigma_1$ eine Permutation p von $\mathbb{N}$ existiert mit $\sum_{i\in\mathbb{N}} v_{p(i)} = x + y$. Die gewünschte Permutation p wird zweckmäßigerweise durch eine Wohlordnung $\preccurlyeq$ auf $\mathbb{N}$ gemäß $p(i) = \max_\leq [0; i]_\preccurlyeq$ definiert, so dass $p(i) \leq p(j) \Leftrightarrow i \preccurlyeq j$ und $\sum_{i\in\mathbb{N}}^{\preccurlyeq} v_i = x + y$, d.h., für jedes $\epsilon > 0$ existiert ein $k \in \mathbb{N}$, so dass für alle $j \succcurlyeq k$ gilt $\left\|\sum_{i\succcurlyeq j} v_i - (x + y)\right\| < \epsilon$. Nach Induktionsannahme gibt es eine eine Permutation p von $\mathbb{N}$ mit $\sum_{i\in\mathbb{N}} \mathrm{pr}_1\left(v_{p(i)}\right) = y$ und damit wie eben beschrieben eine Wohlordnung $\preccurlyeq_1$ auf $\mathbb{N}$ und insbesondere auf $\Lambda = \mathbb{N} \setminus \Omega$ mit $i \preccurlyeq_1 j \Leftrightarrow p(i) \leq p(j)$. Damit erhält man bereits $\sum_{i\in\Lambda}^{\preccurlyeq_1} \mathrm{pr}_1(v_i) + \sum_{i\in\Omega} \mathrm{pr}_1(v_i) = y$. Auf Ω ist der Grenzwert der **absolut konvergenten** Teilreihe $\sum_{i\in\Omega} \mathrm{pr}_1(v_i) \in X_1$ **unabhängig von der Permutation bzw. Wohlordnung** und erlaubt daher weitere Anpassung der Wohlordnung $\preccurlyeq_1$ auf Ω, so dass die endgültige Wohlordnung $\preccurlyeq$ auf $\mathbb{N} = \Omega \cup \Lambda$ auf Λ mit $\preccurlyeq_1$ übereinstimmt und wie bisher $\sum_{i\in\mathbb{N}}^{\preccurlyeq} \mathrm{pr}_1(v_i) = y$ gilt sowie zusätzlich $\sum_{i\in\mathbb{N}}^{\preccurlyeq} \mathrm{pr}_0(v_i) = x$. Dazu konstruiert man induktiv die absteigenden wohlgeordneten Mengenfolgen $(\Lambda_k; \preccurlyeq)_{k\in\mathbb{N}}$ und $(\Omega_k; \leq)_{k\in\mathbb{N}}$ sowie eine aufsteigende Folge $(F_k)_{k\in\mathbb{N}}$ mit den folgenden Eigenschaften für $k \in \mathbb{N}$:

1. $\Lambda_{k+1} = \Lambda_k \setminus \{\min \Lambda_k\}$
2. $\Omega_{k+1} \subset \Omega_k \setminus \{\min \Omega_k\}$
3. $F_{k+1} = F_k \cup \{\min \Lambda_k\} \cup (\Omega_k \setminus \Omega_{k+1})$
4. $\left\|x - \sum_{i\in F_{k+1}} \mathrm{pr}_0(v_i)\right\| < 2^{-k}$
5. $\forall E \subset F_{k+1} \setminus F_k : \|\sum_{i\in E} v_i\| \leq C \cdot \left(2^{-k} + \|v_{\min \Lambda_k} + v_{\min \Omega_k}\|\right)$.

Nach 3.7 lässt sich eine endliche Menge $F_0 \subset \Omega$ finden mit $\|x - \sum_{i\in F_0} \mathrm{pr}_0(v_i)\| \leq \|x - \sum_{i\in F_0} v_i\| < 1$. Damit definiert man $\Omega_0 = \Omega \setminus F_0$ und $\Lambda_0 = \Lambda = \mathbb{N} \setminus \Omega$. Für breits konstruierte F_k, Ω_k, Λ_k sei $a_k := x -$

$\mathrm{pr}_0\left(v_{\min\Lambda_k}+v_{\min\Omega_k}+\sum_{i\in F_k}v_i\right)\in X_0$ mit $\|a_k\|\leq\left\|x-\sum_{i\in F_k}\mathrm{pr}_0\left(v_i\right)\right\|+\|v_{\min\Lambda_k}+v_{\min\Omega_k}\|<2^{-k}+\|v_{\min\Lambda_k}+v_{\min\Omega_k}\|$ wegen 4. Mit 3.7 lässt sich eine endliche Teilmenge $E_k\subset\Omega_k\setminus F_k\cup\{\min\Omega_k\}$ finden mit $\left\|a_k-\sum_{i\in E_k}v_i\right\|<2^{-k-1}$ und für jede Teilmenge $E\subset E_k$ gilt $\|\sum_{i\in E}v_i\|<C\cdot\max\left\{2^{-k-1};\|a_k\|\right\}\leq C\cdot\left(2^{-k}+\|v_{\min\Lambda_k}+v_{\min\Omega_k}\|\right)$. Definiere nun $F_{k+1}:=F_k\cup E_k\cup\{\min\Lambda_k;\min\Omega_k\}$, $\Omega_{k+1}:=\Omega_k\setminus(E_k\cup\{\min\Omega_k\})$ und $\Lambda_{k+1}=\Lambda_k\setminus\{\min\Lambda_k\}$. Auf $\bigcup_{k\in\mathbb{N}}F_k=\mathbb{N}$ wird nun die angpasste Fortsetzung $\preccurlyeq$ der Wohlordnung $\preccurlyeq_1$ defniert mit $i\preccurlyeq j\Leftrightarrow\exists k\in\mathbb{N}:i\in F_k\wedge j\notin F_k$. Aufgrund der absoluten Konvergenz von $\sum_{i\in\Omega}\mathrm{pr}_1\left(v_i\right)$ undabhängig von der Ordnung gilt nach wie vor $\sum_{i\in\mathbb{N}}^{\preccurlyeq}\mathrm{pr}_1\left(v_i\right)=\sum_{i\in\Lambda}^{\preccurlyeq}\mathrm{pr}_1\left(v_i\right)+\sum_{i\in\Omega}^{\preccurlyeq}\mathrm{pr}_1\left(v_i\right)=y$. Wegen 4. und 5. sowie 3.2 gilt nun aber zusätzlich $\sum_{i\in\mathbb{N}}^{\preccurlyeq}\mathrm{pr}_0\left(v_i\right)=x$, womit $x+y\in\Sigma$ bewiesen ist. Die Behauptung ergibt sich nun aus $\Sigma+\Gamma^\perp=(X_0\oplus\Sigma_1)+\left(X_0\oplus\Gamma_1^\perp\right)=X_0\oplus\left(\Sigma_1+\Gamma_1^\perp\right)=X_0\oplus\Sigma_1=\Sigma$.

6.19 Beispiel

Das erste Beispiel zeigt, dass die Mengen $X_0\subset\Gamma^\perp$ bzw. $\Gamma\subset X_1$ nicht zusammenfallen müssen: Für Reihe $\sum_{i\in\mathbb{N}}v_i$ mit $v_i=(-1)^i\begin{pmatrix}i^{-0,5}\\i^{-1}\end{pmatrix}$ ist $\sum_{i\in\mathbb{N}}|\langle y,v_i\rangle|=\infty$ u.a. für $y=\binom{0}{1}$, aber der einzige Divergenzpunkt ist $x=\lim\limits_{i\to\infty}\frac{v_i}{\|v_i\|}=\binom{1}{0}$. In diesem Fall ist $X_0=\{\binom{x}{0}:x\in\mathbb{R}\}$, $X_1=\{\binom{0}{y}:y\in\mathbb{R}\}$, $\Gamma=\{\binom{0}{0}\}$ und $\Gamma^\perp=\mathbb{R}^2$. Die Menge der möglichen Summen bzw. Grenzwerte ist $\Sigma=\Sigma+\Gamma^\perp=\mathbb{R}^2$. Auch orthogonal zur linearen Hülle X_0 der Divergenzpunkte können also Divergenzrichtungen liegen, d.h., der Vektorraum Γ der Richtungen absoluter Konvergenz ist i.A. eine echte Teilmenge von X_1.

6.20 Beispiel

Das zweite Beispiel zeigt, dass die Mengen $D_0\subset D$ bzw. $\Gamma\subset X_1$ nicht zusammenfallen müssen: Für die Reihe $\sum_{i\in\mathbb{N}}v_i$ in $\mathbb{R}^3$ mit $v_i=\frac{(-i)^i}{i}\begin{pmatrix}1\\1\\0\end{pmatrix}$ sind die Divergenzrichtungen identisch mit dem Einheitskreis in der x-y-Ebene: $D=\left\{\begin{pmatrix}x\\y\\0\end{pmatrix}:x^2+y^2=1\right\}$. Eine minimale Teilmenge, deren konvexe Hülle den Ursprung enthält, ist z.B. $D_0=\left\{\begin{pmatrix}1\\0\\0\end{pmatrix};\begin{pmatrix}-1\\0\\0\end{pmatrix}\right\}$ mit $\mathrm{co}\left(D_0\right)=\left\{\begin{pmatrix}1-2t\\0\\0\end{pmatrix}:0\leq t\leq 1\right\}$ und $X_0=\left\{\begin{pmatrix}x\\0\\0\end{pmatrix}:x\in\mathbb{R}\right\}$. Das orthogonale Komplement ist $X_1=\left\{\begin{pmatrix}0\\y\\z\end{pmatrix}:y,z\in\mathbb{R}\right\}$. Offensichtlich ist aber $\Gamma=\left\{\begin{pmatrix}0\\0\\z\end{pmatrix}:z\in\mathbb{R}\right\}$ und die Menge der möglichen Summen bzw. Grenzwerte ist $\Sigma=\Sigma+\Gamma^\perp=\left\{\begin{pmatrix}x\\y\\0\end{pmatrix}:x,y\in\mathbb{R}\right\}$. In diesem Fall gilt also $X_0=\Gamma^\perp$.

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