Measure Theory

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November 22, 2023

Preface

This text is essentially a working reference and follows the classical expositions of **Bauer** [1], **Forster** [3], **Hewitt/Stromberg** [4], **Lang** [5] and **Rudin** [7] to develop the foundations of the analysis of functions needed for the research on partial differential equations in probability and physics.The necessary results from set theory and topology can be found in [11] and [13]; the corresponding references are given in the text. For reasons of brevity motivations and proofs for simple definitions and propositions are omitted.

The exposition starts with **measure theory** which is the field of mathematics dedicated to the study of the content or weight of a set expressed by its measure. If the set is defined by a **function** on a certain domain its measure can be written as an **integral**. In this case the function turns out to be the **derivative** of the measure, i.e. it is itself a measure for the rate of change of the given measure depending on the change of the domain. Thus measure theory provides one of the basic methods for the study of functions in **analysis**. Since the measure of a set can be interpreted as the probability for the realization of the events represented by its elements measure theory has proved to be a very useful foundation of **probability theory** and **statistics**.

The first section introduces measurable sets, measures and measurable functions in a pronounced analogy to the open sets, metrics and continuous functions in topology. The concept of integration provides the basis for the extension of measures on product spaces. For the sake of clarity the integral is introduced in the generalized **Bochner** variant for functions with values in **Banach spaces** and later specialized to the usual **Lebesgue integral** so as to profit from the full range of possibilities of **differentiation**. The Lebesgue integral and the associated **product measures** on countable products of measure spaces prove to be a very useful concept for the description of sequences of independent random variables and their mean rep. expected values leading to the **strong law of large numbers**. In analysis they constitute the foundation for the **integral transformations** needed for the solution of **partial differential equations**, e.g. **convolutions**, **distributions** and **fourier transforms**. These integral transformations also provide an easy approach to the **central limit theorem** of probability theory. Mean values resp. integrals of functions on subsets are themselves measures and the **Lebesgue-Radon-Nikodym theorem** states that in fact every **positive σ-finite measure** can be represented as an **integral** over a suitable second measure. This result provides the foundation for two central theorems in **functional** resp. **real analysis**: **Positive** resp. **bounded** measures on **locally compact vector spaces** prove to be eqiuvalent to the corresponding functionals. Hence the set of all such measures on such a space is the **dual space** of a locally compact vector space. This is the content of the **Riesz representation theorem**.

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1 Measurable sets

1.1 σ-algebrae

A family $\mathcal{A} \subset \mathcal{P}(X)$ is an **algebra** iff

1. $\emptyset \in \mathcal{A}$.
2. $A, B \in \mathcal{A} \Rightarrow A \cap B; A \cup B; A \setminus B \in \mathcal{A}$

In the case of

3. $X \in \mathcal{A}$
4. $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A} \wedge (n \neq m \Rightarrow A_n \cap A_m = \emptyset) \Leftrightarrow \mathring{\bigcup}_{n\in\mathbb{N}} A_n \in \mathcal{A}$

we have a **σ-algebra**. The pair $(X; \mathcal{A})$ then is a **measurable space**. Every **σ-algebra** is **closed under arbitrary countable unions and intersections** since for $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ we obtain paiwise disjoint $A'_n := A_n \setminus \bigcup_{1\leq k<n} A_k = \bigcap_{1\leq k<n} (A_n \setminus A_k) \in \mathcal{D}$ whence $\bigcup_{n\in\mathbb{N}} A_n = \mathring{\bigcup}_{n\in\mathbb{N}} A'_n \in \mathcal{D}$ and $\bigcap_{n\in\mathbb{N}} A_n = X \setminus \bigcup_{n\in\mathbb{N}} (X \setminus A_n)$.

1.2 Borel σ-algebrae

For an arbitrary $\mathcal{M} \subset \mathcal{P}(X)$ the intersection $\sigma(\mathcal{M})$ of all σ-algebrae containing $\mathcal{M}$ is again a σ-algebra. It is the σ-algebra **induced** by $\mathcal{M}$ and $\mathcal{M}$ is its **basis**. On a **topological space** $(X; \mathcal{O})$ we have the **Borel σ-algebra $\mathcal{B}(X) = \sigma(\mathcal{O})$** induced by the topology $\mathcal{O}$. Owing to 1.1 it contains the **open sets** and their **countable intersections**, i.e. the **G_δ-sets** as well as the **closed sets** and their **countable unions**, i.e. the **F_σ-sets**. The Borel σ-algebra of a **second countable** topological space $\mathcal{O}(\mathcal{E})$ induced by a **countable topogical basis** $\mathcal{E}$ is induced by $\mathcal{E}$ itself, i.e. $\mathcal{B}(X) = \sigma(\mathcal{O}(\mathcal{E})) = \sigma(\mathcal{E})$. In a **Hausdorff space** all **compact** sets are **closed** and hence **Borel measurable**, i.e. measurable with respect to $\mathcal{B}(X)$. For a **locally compact** X which is **countable at infinity** the **Borel σ-algebra** $\mathcal{B}(X) = \sigma(\mathcal{K})$ is induced by the family $\mathcal{K}$ of all compact sets since due to [13, p. 10.6] every closed set is the countable intersection of compact sets. In a **discrete space** X with $\mathcal{B}(X) = \sigma(\mathcal{O}) = \mathcal{O} = \mathcal{P}(X)$ a set is compact iff it is finite and $\sigma(\mathcal{K})$ is the σ-algebra of all sets $A \subset X$ with countable A or $X \setminus A$. Using **Zorn's lemma** ([11, p. 14.2.4]) we can infer that $\sigma(\mathcal{K}) = \mathcal{B}(X)$ iff X itself is countable.

1.3 The trace of a σ-algebra

The **trace σ-algebra** $\mathcal{A} \cap B := \{A \cap B : A \in \mathcal{A}\}$ on a subset $B \subset X$ of a measurable space $(X; \mathcal{A})$ simply consists of the **inter sections of measurable** A in X with B. On account of $(O_1 \cap O_2) \cap B = (O_1 \cap B) \cap (O_2 \cap B)$, $(O_1 \cup O_2) \cap B = (O_1 \cap B) \cup (O_2 \cap B)$, $(O_1 \setminus O_2) \cap B = (O_1 \cap B) \setminus (O_2 \cap B)$ and $(\bigcup_{n\in\mathbb{N}} O_n) = \bigcup_{n\in\mathbb{N}} (O_n \cap B)$ the trace $\sigma(\mathcal{O}) \cap B$ of the **Borel σ-algebra** $\mathcal{B}(X) = \sigma(\mathcal{O})$ on a **topological space** $(X; \mathcal{O})$ is identical with the σ-algebra $\sigma(\mathcal{O} \cap B)$ of the trace $\mathcal{O} \cap B$ of the **topology** $\mathcal{O}$ on B.

1.4 Intervals and figures

The **finite** unions of **pairwise disjoint left-open intervals** $\mathcal{I} = \{]a; b] : a \leq b \in \mathbb{R}\}$ form the **algebra** $\mathcal{F} = \left\{\mathring{\bigcup}_{0\leq k\leq m} I_k : I_k \in \mathcal{I}; 0 \leq k \neq l \leq m \Rightarrow I_k \cap I_l = \emptyset; m \in \mathbb{N}\right\}$ of the **one-dimensional figures** since $\emptyset =]a; a] \in \mathcal{I}$ and for $I, J \in \mathcal{I}$ we have $I \cap J \in \mathcal{I}$, $I \setminus J \in \mathcal{I}$ as well as $I \cup J \in \mathcal{I}$ in the case of $I \cap J \neq \emptyset$ resp. $I \cup J \in \mathcal{F}$ for $I \cap J = \emptyset$. Hence for $F = \mathring{\bigcup}_{0\leq k\leq m} I_k \in \mathcal{F}$ and $G = \mathring{\bigcup}_{0\leq l\leq n} J_l \in \mathcal{F}$ we have $F \cap G = \mathring{\bigcup}_{0\leq k\leq m} \mathring{\bigcup}_{0\leq k\leq m} I_k \cap J_l \in \mathcal{F}$, $F \setminus G = F \setminus (F \cap G) \in \mathcal{F}$ and $F \cup G \in \mathcal{F}$. The **left-open intervals** $]a; b]$ are **G_δ-sets** hence they are Borel-measurable and because of $]a; b[= \bigcup_{k\in\mathbb{N}} \left]a; b - \frac{1}{n}\right]$ they **induce** the Borel σ-algebra on $\mathbb{R}$ as well as the **algebra** of **figures**: $\mathcal{B} = \sigma(\mathcal{F}) = \sigma(\mathcal{I})$. Alternative basis families are the **closed rays** $]-\infty; b] = \bigcup_{n\in\mathbb{N}} \left]a; b - \frac{1}{n}\right]$ since $]a; b] =]-\infty; b] \setminus]-\infty; a]$ as well as the **open rays** $]-\infty; b[$, $]a; \infty[$, $[a; \infty[$ and by analogous arguments the **right-open intervals** $[a; b[$ for $a \leq b \in \mathbb{R}$.

1.5 Dynkin systems

A family $\mathcal{D} \subset \mathcal{P}(X)$ is a **Dynkin system** or **δ-system** iff

1. $\emptyset \in \mathcal{D}$.
2. $A \in \mathcal{D} \Leftrightarrow X \setminus A \in \mathcal{D}$
3. $(A_n)_{n\in\mathbb{N}} \subset \mathcal{D} \wedge (n \neq m \Rightarrow A_n \cap A_m = \emptyset) \Leftrightarrow \mathring{\bigcup}_{n\in\mathbb{N}} A_n \in \mathcal{D}$

1.6 The Dynkin δ-π-theorem

The **Dynkin system** $\delta(\mathcal{E})$ generated by a **π-basis** $\mathcal{E} \subset \mathcal{P}(X)$ being **closed under intersections** coincides with the corresponding **σ-algebra** $\sigma(\mathcal{E})$.

Proof: For every $B \subset A$ we have $A \setminus B = X \setminus ((X \setminus A) \mathring{\cup} B)$ whence $(X \setminus A) \cap D = D \setminus (A \cap D) \in \mathcal{D}$ for every $D \in \delta(\mathcal{E})$ and $A \subset X$. Hence the family $\mathcal{D}_D := \{A \subset X : A \cap D \in \delta(\mathcal{E})\}$ is itself a Dynkin system including $\mathcal{E}$ and consequently $\delta(\mathcal{E})$. Hence $\delta(\mathcal{E})$ is closed under **intersection**. On account of $A \cup B = X \setminus ((X \setminus A) \cap (X \setminus B))$, $A \setminus B = A \cap (X \setminus B)$ resp. 1.5.3 it is a σ-algebra, i.e. $\sigma(\mathcal{E}) \subset \delta(\mathcal{E})$ and since every σ-algebra is a Dynkin system we have $\sigma(\mathcal{E}) = \delta(\mathcal{E})$.

1.7 The monotone class theorem

A class $\mathcal{M} \subset \mathcal{P}(X)$is **monotone** iff it is closed under the formation of monotone unions and intersections, i.e. for every **increasing** sequence $(A_n)_{n\geq 1}$ we have $\bigcup_{n\geq 1} A_n \in \mathcal{M}$ and for every **decreasing** sequence $(A_n)_{n\geq 1}$ we have $\bigcap_{n\geq 1} A_n \in \mathcal{M}$. Every **monotone class** $\mathcal{M}$ including an **algebra** $\mathcal{A} \subset \mathcal{M}$ also includes the **σ-algebra** $\sigma(\mathcal{A}) \subset \mathcal{M}$ generated by $\mathcal{A}$.

Proof: We apply the "**good set principle**" three times in a row: Since **every monotone algebra is a σ-algebra** it suffices to show that the monotone class $m(\mathcal{A})$ generated by $\mathcal{A}$, i.e. the intersection of all monotone classes including $\mathcal{A}$, is an **algebra**: The class $\mathcal{F} = \{A \subset X : X \setminus A \in m(\mathcal{A})\}$ is monotone and includes $\mathcal{A}$ whence follows $m(\mathcal{A}) \subset \mathcal{F}$, i.e. $m(\mathcal{A})$ is closed under complementation. The class $\mathcal{G} = \{A \subset X : A \cup B \in m(\mathcal{A}) \, \forall B \in \mathcal{A}\}$ is monotone and includes $\mathcal{A}$, hence $m(\mathcal{A})$. The class $\mathcal{H} = \{A \subset X : A \cup B \in m(\mathcal{A}) \, \forall B \in m(\mathcal{A})\}$ is monotone and includes $\mathcal{A}$ since $m(\mathcal{A}) \subset \mathcal{G}$. Hence $m(\mathcal{A}) \subset \mathcal{H}$, i.e. $m(\mathcal{A})$ is closed under formation of unions. Due to $A \cap B = X \setminus ((X \setminus A) \cup (X \setminus B))$ we obtain the intersections which completes the proof.

2 Pre-measures

2.1 Pre-measures

The inclusion of big sets like $X = \mathbb{C}$ into the **domain** of a measure makes it necessary to include the corresponding value ∞ into its **range**. Since we expect integrals of vanishing functions on sets of infinite measure to have the value zero we define $\infty \cdot 0 := 0 \cdot \infty := 0$. The corresponding extended ranges are denoted as $\mathbb{R} \cup \{\infty\} = \overline{\mathbb{R}}$ resp. $\mathbb{C} \cup \{\infty\} = \overline{\mathbb{C}}$ resp. $[0; \infty[\cup \{\infty\} = [0; \infty]$. A set function $\mu : \mathcal{A} \to [0; \infty]$ on an **algreba** $\mathcal{A} \subset P(X)$ is **finitely additive** iff $\mu(A \mathring{\cup} B) = \mu(A) + \mu(B)$ for **disjoint** $A, B \in \mathcal{A}$. In the general case with $A \cap B \in A$ follows the **subadditivity** $\mu(A \cup B) \leq \mu(A) + \mu(B)$. If there is an $A \in \mathcal{A}$ with $\mu(A) < \infty$ we have $\mu(\emptyset) = \mu(A \mathring{\cup} \emptyset) - \mu(A) = 0$. Also μ is **monotone**: For $A \subset B$ and $\mu(A) < \infty$ on account of $A \setminus B \in \mathcal{A}$ and $B = A \cup B \setminus A$ we have $\mu(B \setminus A) = \mu(B) - \mu(A)$ and particularly $\mu(A) < \mu(B)$. Note that $\mu(B) = \infty \Rightarrow \mu(B \setminus A) = \infty$ if $\mu(A) < \infty$. In the case of **σ-additivity** with $\mu\left(\mathring{\bigcup}_{n\in\mathbb{N}}A_n\right) = \sum_{n\in\mathbb{N}}\mu(A_n)$ for pairwise disjoint $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ it is a **pre-measure**. The **supremum property** (cf. [13, p. 14.12]) of the **real numbers** permits the extension of the **subadditivity** to **countable unions**: $\mu\left(\bigcup_{n\in\mathbb{N}}A_n\right) \leq \sum_{n\in\mathbb{N}}\mu(A_n)$.

2.2 Characterization of pre-measures

A **finite and finitely additive** set function $\mu : \mathcal{A} \to [0;\infty[$ on an **algebra** $\mathcal{A} \subset \mathcal{P}(X)$ is a **pre-measure** if one of the following equivalent conditions holds.

1. **σ-additiv**ity: For a sequence $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ of **pairwise disjoint** measurable sets with $\mathring{\bigcup}_{n\in\mathbb{N}}A_n \in \mathcal{A}$ we have $\mu\left(\mathring{\bigcup}_{n\in\mathbb{N}}A_n\right) = \sum_{n\in\mathbb{N}}\mu(A_n)$.
2. **Continuity from below**: For an **increasing** sequence of measurable sets $A_0 \subset A_1 \subset ...$ with $\bigcup_{n\in\mathbb{N}}A_n \in \mathcal{A}$ we have $\lim_{n\in\mathbb{N}}\mu(A_n) = \mu(\bigcup_{n\in\mathbb{N}}A_n)$.
3. **Continuity from above**: For a **decreasing** sequence of measurable sets $A_0 \supset A_1 \supset ...$ with $\bigcap_{n\in\mathbb{N}}A_n \in \mathcal{A}$ we have $\lim_{n\in\mathbb{N}}\mu(A_n) = \mu\left(\bigcap_{n\in\mathbb{N}}A_n\right)$.
4. **$\emptyset$-Continuity**: For a **decreasing** sequence of measurable sets $A_0 \supset A_1 \supset ...$ with $\bigcap_{n\in\mathbb{N}}A_n = \emptyset$ we have $\lim_{n\in\mathbb{N}}\mu(A_n) = 0$.

Note: Owing to the **σ-additivity** every set A with **finite pre-measure** $\mu(A) < \infty$ has at most **countably many disjoint subsets** $A_i \subset A$, $i \in I$ of **non-zero pre-measure** $\mu(A_i) > 0\,\forall i \in I$ since every subfamily $I_n = \left\{i \in I : \mu(A_i) \geq \frac{\mu(A)}{n}\right\}$ must be **finite** and $I = \bigcup_{n\geq 1} I_n$.

Proof:

1. $\Rightarrow$ 2. : With $A'_n := A_n \setminus A_{n-1}$ we obtain a **pairwise disjoint** family $(A'_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ with $\mu(A_n) = \mu\left(\bigcup_{1\leq k\leq n}A'_k\right) = \sum_{1\leq k\leq n}\mu(A'_k)$ such that $\lim_{n\in\mathbb{N}}\mu(A_n) = \sum_{n\in\mathbb{N}}\mu(A'_n) = \mu\left(\bigcup_{n\in\mathbb{N}}A'_n\right) = \mu(\bigcup_{n\in\mathbb{N}}A_n)$.

2. $\Rightarrow$ 3. : We apply 2. to the **increasing** sequence $\emptyset = A'_0 \subset A'_1 \subset ...$ of the **complements** $A'_n := A_0 \setminus A_n \in \mathcal{A}$ such that $\lim_{n\in\mathbb{N}}\mu(A_n) = \lim_{n\in\mathbb{N}}\mu(A_0 \setminus A'_n) = \lim_{n\in\mathbb{N}}(\mu(A_0) - \mu(A'_n)) = \mu(A_0) - \lim_{n\in\mathbb{N}}\mu(A'_n)$ $= \mu(A_0) - \mu\left(\bigcup_{n\in\mathbb{N}}A'_n\right) = \mu\left(A_0 \setminus \bigcup_{n\in\mathbb{N}}A'_n\right) = \mu\left(\bigcap_{n\in\mathbb{N}}A_0 \setminus A'_n\right) = \mu\left(\bigcap_{n\in\mathbb{N}}A_n\right)$.

3. $\Rightarrow$ 4. : Obvious.

4. $\Rightarrow$ 1. : With $A'_k := \mathring{\bigcup}_{n>k}A_n$ we obtain a **decreasing** sequence $(A'_k)_{k\in\mathbb{N}}$ with $\bigcap_{k\in\mathbb{N}}A'_k = \emptyset$ and $\mu(A'_k) < \infty$ such that due to 4. we have $0 = \lim_{k\in\mathbb{N}}\mu(A'_k) = \mu\left(\mathring{\bigcup}_{n\in\mathbb{N}}A_n\right) - \lim_{k\in\mathbb{N}}\mu\left(\mathring{\bigcup}_{n\leq k}A_n\right) = \mu\left(\mathring{\bigcup}_{n\in\mathbb{N}}A_n\right) - \sum_{n\in\mathbb{N}}\mu(A_n)$.

2.3 Examples

1. The **Dirac measure** $\delta_x(A) := \begin{cases} 1, x \in A \\ 0, x \notin A \end{cases}$ for $A \subset X$ and $x \in X$ is a **pre-measure** on every **ring** on a set X.
2. The **measure** $\mu(A) := \begin{cases} 0 & \text{for countable } A \\ \infty & \text{else} \end{cases}$ on the **algebra** $\mathcal{P}(X)$ of a **discrete** space X according to 1.2.

3 Measures

3.1 Measures

A pre-measure μ on a **σ-algebra** $\mathcal{A}$ is a **measure** and $(X;\mathcal{A};\mu)$ is a **measure space**. **Probability measures** have the range $[0;1]$ and in that case $(X;\mathcal{A};\mu)$ is a **probability space**.

3.2 Outer measures

A set function $\tilde{\mu}:P(X)\to[0;\infty]$ is an **outer measure** iff for all $A,B,A_n\in\mathcal{A},n\in\mathbb{N}$ the following properties hold:

1. **Homogeneity**: $\tilde{\mu}(\emptyset)=0$
2. **Monotonicity**: $A\subset B\Rightarrow\tilde{\mu}(A)\leq\tilde{\mu}(B)$
3. **Sub-additivity**: $\tilde{\mu}\left(\bigcup_{n\in\mathbb{N}}A_n\right)\leq\sum_{n\in\mathbb{N}}\tilde{\mu}\left(A_n\right)$

A set $A\subset X$ is **$\tilde{\mu}$-measurable** iff for every $Q\subset X$ we have

4. $\tilde{\mu}(Q)=\tilde{\mu}\left(Q\cap A\right)+\tilde{\mu}\left(Q\setminus A\right)$.

3.3 Carathéodory's theorem

For an outer measure $\tilde{\mu}$ on a set X the system $\mathcal{A}$ of all $\tilde{\mu}$-measurable sets $A\subset X$ is a σ-algebra and the restriction $\tilde{\mu}|_{\mathcal{A}}$ is a measure.

Proof: Obviously we have $\emptyset,X\in\mathcal{A}$ and on account of 3.2.4 every $A\in\mathcal{A}$ has a measurable **complement** $X\setminus A\in\mathcal{A}$. For $A,B\in\mathcal{A}$ the **union** $A\cup B\in\mathcal{A}$ is measurable too since by applying 3.2.4 successively we obtain first an equation (I): $\tilde{\mu}(Q)=\tilde{\mu}\left(Q\cap A\right)+\tilde{\mu}\left(Q\setminus A\right)=\tilde{\mu}\left(Q\cap A\cap B\right)+\tilde{\mu}\left(Q\cap A\setminus B\right)+\tilde{\mu}\left(Q\setminus A\cap B\right)+\tilde{\mu}\left(Q\setminus A\setminus B\right)$ and if we substitute Q with $Q\cap(A\cup B)$ in (I) we arrive at another equatiuon (II): $\tilde{\mu}\left(Q\cap(A\cup B)\right)=\tilde{\mu}\left(Q\cap A\cap B\right)+\tilde{\mu}\left(Q\cap A\setminus B\right)+\tilde{\mu}\left(Q\setminus A\cap B\right)$. We can substitute the first three terms in (I) by (II) and hence obtain the measurability of the **union**: $\tilde{\mu}(Q)=\tilde{\mu}\left(Q\cap(A\cup B)\right)+\tilde{\mu}\left(Q\setminus(A\cup B)\right)$. Thuas and because of $A\cap B=X\setminus((X\setminus A)\cup(X\setminus B))$ and $A\setminus B=A\cap(X\setminus B)$ the family $\mathcal{A}$ is an **algebra**.

For a sequence $(A_n)_{n\in\mathbb{N}}\subset\mathcal{A}$ of **pairwise disjoint measurable** sets $A:=\bigcup_{n\in\mathbb{N}}A_n$ equation (II) yields $\tilde{\mu}\left(Q\cap(A_0\cup A_1)\right)=\tilde{\mu}\left(Q\cap A_0\right)+\tilde{\mu}\left(Q\cap A_1\right)$ resp. by **induction** $\tilde{\mu}\left(Q\cap\bigcup\limits_{k=0}^{n}A_k\right)=\sum\limits_{k=0}^{n}\tilde{\mu}\left(Q\cap A_k\right)$. On account of $\bigcup\limits_{k=0}^{n}A_k\in\mathcal{A}$ and 3.2.2 we conclude that (III): $\tilde{\mu}(Q)=\tilde{\mu}\left(Q\cap\bigcup\limits_{k=0}^{n}A_k\right)+\tilde{\mu}\left(Q\setminus\bigcup\limits_{k=0}^{n}A_k\right)$ $\geq\sum\limits_{k=0}^{n}\tilde{\mu}\left(Q\cap A_k\right)+\tilde{\mu}\left(Q\setminus A\right)$. Since this estimate holds for all $n\in\mathbb{N}$ it extends to $n\to\infty$ such that by 3.2.3 we arrive at the measurability criterion 3.2.4 for A. Due to 1.5.3 the family $\mathcal{A}$ is a **Dynkin system** which is **closed under intersection** and in accordance with 1.6 it is a **σ-algebra**. If in (III) we substitute $Q=A$ and observe 3.2.3 we obtain the **σ-additivity** of $\tilde{\mu}$ on $\mathcal{A}$, i.e. $\tilde{\mu}|_{\mathcal{A}}$ is a **measure**.

3.4 The uniqueness theorem

Two measures μ_1 and μ_2 on a σ-Algebra $\sigma\left(\mathcal{E}\right)$ induced by a **π-basis** $\mathcal{E}\subset\mathcal{P}\left(X\right)$ are identical iff they coincide on $\mathcal{E}$ and are **σ-finite** on $\mathcal{E}$, i.e. $\exists\left(E_n\right)_{n\in\mathbb{N}}\subset\mathcal{E}$ with $\bigcup_{n\in\mathbb{N}}E_n=X$ and $\mu_1\left(E_n\right)=\mu_2\left(E_n\right)<\infty$ for all $n\in\mathbb{N}$.

Proof: For $E\in\mathcal{E}$ with $\mu_1\left(E\right)=\mu_2\left(E\right)<\infty$ the family $\mathcal{D}_E:=\{D\in\sigma\left(\mathcal{E}\right):\mu_1\left(E\cap D\right)=\mu_2\left(E\cap D\right)\}$ is a **Dynkin system** since $\emptyset\in\mathcal{D}_E$ and for every $D\in\mathcal{D}_E$ on account of $\mu_1\left(E\cap X\setminus D\right)=\mu_1(E)-\mu_1\left(E\cap D\right)=\mu_2(E)-\mu_2\left(E\cap D\right)=\mu_2\left(E\cap X\setminus D\right)$ we also have $X\setminus D\in\mathcal{D}_E$. Criterion 1.5.3 follows from the σ-additivity of μ_1 and μ_2. Since $\mathcal{E}$ is **closed under intersection** we have $\mathcal{E}\subset\mathcal{D}_E$

and since $\mathcal{D}_E$ is a Dynkin system 1.6 entails $\sigma(\mathcal{E}) = \delta(\mathcal{E}) \subset \mathcal{D}_E \subset \sigma(\mathcal{E})$, i.e.$\mathcal{D}_E = \sigma(\mathcal{E})$ resp. $\mu_1(E \cap A) = \mu_2(E \cap A)$ for all $E \in \mathcal{E}$ and $A \in \sigma(\mathcal{E})$.

As in the proof of 2.2.2 we define a sequence of pairwise disjoint sets $E_n' := E_n \setminus \bigcup_{1 \leq k < n} E_k \in \sigma(\mathcal{E})$ with $\mathring{\bigcup}_{n \in \mathbb{N}} E_n' = X$ such that for $A \in \sigma(\mathcal{E})$ we have $E_n' \cap A \in \sigma(\mathcal{E})$, hence $\mu_1(E_n \cap E_n' \cap A) = \mu_2(E_n \cap E_n' \cap A)$ and the σ-additivity of μ_1 resp. μ_2 yields $\mu_1(A) = \mu_2(A)$.

3.5 Hahn's extension theorem

Every **σ-finite pre-measure** μ on an **algebra** $\mathcal{A}$ can be extended **in a unique way** to a **measure** μ on $\sigma(\mathcal{A})$.

Proof: For every set $Q \subset X$ let $\mathcal{U}(Q) \neq \emptyset$ be the family of sequences $(A_n)_{n \in \mathbb{N}} \subset \mathcal{A}$ with $Q \subset \bigcup_{n \in \mathbb{N}} A_n$. Then $\tilde{\mu}(Q) := \inf \left\{ \sum_{n \in \mathbb{N}} \mu(A_n) : (A_n)_{n \in \mathbb{N}} \in \mathcal{U}(Q) \right\}$ in case of $\mathcal{U}(Q) \neq \emptyset$ and $\tilde{\mu}(Q) := \infty$ else is an **outer measure** since obviously we have $\tilde{\mu}(\emptyset) = 0$ and for $P \subset Q$ follows $\mathcal{U}(P) \supset \mathcal{U}(Q)$ and hence $\tilde{\mu}(P) \leq \tilde{\mu}(Q)$, particularly $\tilde{\mu}(Q) \geq 0 \ \forall Q \subset X$. For every sequence $(Q_n)_{n \in \mathbb{N}} \subset \mathcal{P}(X)$, $\epsilon > 0$ and $n \in \mathbb{N}$ there is a sequence $(A_{nm})_{m \in \mathbb{N}} \subset \mathcal{U}(Q_n) \neq \emptyset$ with $\sum_{m \in \mathbb{N}} \mu(A_{nm}) < \tilde{\mu}(Q_n) + \epsilon \cdot 2^{-n-1}$ and since $(A_{nm})_{n,m \in \mathbb{N}} \subset \mathcal{U}\left(\bigcup_{n \in \mathbb{N}} Q_n\right)$ it follows that $\tilde{\mu}\left(\bigcup_{n \in \mathbb{N}} Q_n\right) \leq \sum_{n,m \in \mathbb{N}} \mu(A_{nm}) < \sum_{n \in \mathbb{N}} \tilde{\mu}(Q_m) + \epsilon$. Since $\epsilon > 0$ is arbitrary condition 3.2.3 is satisfied.

The algebra $\mathcal{A}$ is **$\tilde{\mu}$-measurable** since for every $A \in \mathcal{A}$ and $Q \subset X$ with $(A_n)_{n \in \mathbb{N}} \subset \mathcal{U}(Q)$ we have $(A_n \cap A)_{n \in \mathbb{N}} \subset \mathcal{U}(Q \cap A)$ resp. $(A_n \setminus A)_{n \in \mathbb{N}} \subset \mathcal{U}(Q \setminus A)$ and since $\mu(A_n) = \mu(A_n \cap A) + \mu(A_n \setminus A)$ we obtain $\tilde{\mu}(Q) \geq \tilde{\mu}(Q \cap A) + \tilde{\mu}(Q \setminus A)$ and hence equality on account of 3.2.3. The assertion then follows from 3.3 and 3.4.

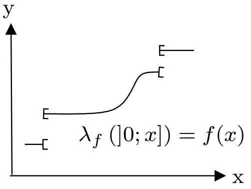
3.6 The approximation property

Every set $Q \in \sigma(\mathcal{A})$ with **finite measure** $\mu(Q) < \infty$ on a **σ-algebra $\sigma(\mathcal{A})$** induced by an **algebra** $\mathcal{A}$ can be approximated **in measure** by a sequence $(C_n)_{n \in \mathbb{N}} \subset \mathcal{A}$ such that $\lim\limits_{n \to \infty} \mu(Q \Delta C_n) = 0$ and particularly $\lim\limits_{n \to \infty} \mu(C_n) = \mu(Q)$.

Proof: As in the proof for 3.5 and since $\mu(Q) < \infty$ for every $\epsilon > 0$ we can find a sequence of w.l.o.g. (cf. proof of 2.2.2) pairwise disjoint sets $(A_k)_{k \in \mathbb{N}} \subset \mathcal{A}$ with $Q \subset \mathring{\bigcup}_{k \in \mathbb{N}} A_k$ and $\mu\left(\mathring{\bigcup}_{k \in \mathbb{N}} A_k\right) - \mu(Q) = \sum_{k \in \mathbb{N}} \mu(A_k) - \mu(Q) < \frac{\epsilon}{2}$. The unions $C_n := \mathring{\bigcup}_{0 \leq k \leq n} A_k$ already constitute the desired sequence since owing to $\mu\left(\mathring{\bigcup}_{k \in \mathbb{N}} A_k\right) < \infty$ we can apply 2.2.2 such that there is an $n_0 \in \mathbb{N}$ with $\mu\left(\mathring{\bigcup}_{n \in \mathbb{N}} A_n\right) - \mu(C_{n_0})$ $< \frac{\epsilon}{2}$ and hence $\mu(Q \Delta C_{n_0}) = \mu(Q \setminus C_{n_0}) + \mu(C_{n_0} \setminus Q) \leq \mu\left(\mathring{\bigcup}_{n \in \mathbb{N}} A_n \setminus C_{n_0}\right) + \mu\left(\mathring{\bigcup}_{n \in \mathbb{N}} A_n \setminus Q\right) < \frac{\epsilon}{2} + \frac{\epsilon}{2}$ $= \epsilon$. The second assertion follows from $\mu(C_n) = \mu(Q) + \mu(C_n \setminus C)$ and $\mu(C_n \setminus C) \leq \mu(Q \Delta C_n)$.

3.7 Distribution functions and Lebesgue-Stieltjes measures

The **real vector space** of **nondecreasing** and **right continuous distribution functions** $f: \mathbb{R} \to \mathbb{R}$ with **existing left limits** (*càdlàg = continue à droite et limite à gauche*) and $f(0) = 0$ by $\mu_f(]a;b]) = f(b) - f(a)$ for every **left-open interval** $]a;b] \in \mathcal{I}$ resp. $f_\mu(x) = \begin{cases} \mu(]0;x]) & : x \geq 0 \\ \mu(]-x;0]) & : x < 0 \end{cases}$ for $x \in \mathbb{R}$ is **isomorphic** to the **real vector space** $\mathcal{M}(\mathcal{B}(\mathbb{R}); \mathbb{R}^+)$ of positive measures on the Borel σ-algebra on the real numbers.



right continuous
left limits

Notes:

1. These measures are sometimes called **Lebesgue-Stieltjes measure**s and in the case of the **identity** $f(x) = x$ we have the **Lebesgue-Borel measure** $\lambda = \mu_{\text{id}}$. Another example is the **Dirac measure** $\delta_x = \mu_{\chi_{[x;\infty[}}$ from 2.3.1 generated by $\chi_{[x;\infty[}$.

2. According to [12, th. 3.1] every distribution function has at most a **countable number of simple discontinuities**.

Proof: The **linearity** of the map $f \to \mu_f$ is obvious. For a given **distribution function** f the set function μ_f defined as above is obviously **finite** and **finitely additive** on the π**-system** of the **left-open intervals** $\mathcal{I} = \{]a;b] : a \leq b \in \mathbb{R}\}$. Hence its extension by $\mu_f(F) = \sum\limits_{k=1}^{m} \mu_f(I_k) = \sum\limits_{k=1}^{m}\sum\limits_{l=1}^{n} \mu_f(I_k \cap J_l) = \sum\limits_{l=1}^{n} \mu_f(J_l)$ for any $F = \mathring{\bigcup}_{0 \leq k \leq m} I_k = \mathring{\bigcup}_{0 \leq l \leq n} J_l \in \mathcal{F}$ to the **algebra** $\mathcal{F}$ of the **one-dimensional figures** from 1.4 is **well defined** and **independent of the representation** of F.

For every **decreasing** sequence of figures $F_0 \supset F_1 \supset \ldots$ with $F_n = \mathring{\bigcup}_{0 \leq k_n \leq l_n}]a_{k_n}; b_{k_n}] \in \mathcal{F}$ and $\bigcap_{n \in \mathbb{N}} F_n = \emptyset$ the decreasing character implies that $\forall m \geq 0 \; \forall n > m \; \forall 0 \leq k_n \leq l_n \; \exists 0 \leq k_m \leq l_m$ with $]a_{k_n}; b_{k_n}] \subset]a_{k_m}; b_{k_m}]$, i.e. from every $m \geq 0$ onwards we are left with at most $l_m + 1$ decreasing sequences $(]a_{k_n}; b_{k_n}])_{n \geq m}$ of intervals. Furthermore the condition $\bigcap_{n \in \mathbb{N}} F_n = \emptyset$ implies that each of these decreasing sequences must terminate in an empty set **after finitely many steps**: $\forall (]a_{k_n}; b_{k_n}])_{n \geq m} \; \exists N \in \mathbb{N}$ with $a_{k_N} = b_{k_N}$ since otherwise due to the **supremum property** [13, th. 14.12] of the real numbers we had limits $a = \sup\limits_{n \to \infty} a_{k_n} \leq \inf\limits_{n \to \infty} b_{k_n} = b$ and consequently $\emptyset \neq [a;b] \subset \bigcap\limits_{n \in \mathbb{N}}]a_{k_n}; b_{k_n}]$. Hence $\lim\limits_{n \to \infty} \mu_f(F_n) = 0$ whence by 2.2.4 μ_f is a **pre-measure** on $\mathcal{F}$. By $\mu_f(]-n;n]) = f(n) - f(-n)$ it is σ**-finite** such that according to **Hahn's extension theorem** 3.5 there is a **uniquely determined** extension to a **measure** μ_f on $\sigma(\mathcal{F}) = \mathcal{B}(\mathbb{R})$ according to 1.4.

Conversely for a given **measure** $\mu \in \mathcal{M}(\mathcal{B}(\mathbb{R}); \mathbb{R}^+)$ the obviously **nondecreasing distribution function** f_μ as defined above must be the **right continuous** for $x \geq 0$ since the **continuity from above** 2.2.3 of μ implies $\lim\limits_{n \to \infty} f_\mu\left(x + \frac{1}{n}\right) = \lim\limits_{n \to \infty} \mu\left(\left]0; x + \frac{1}{n}\right]\right) = \mu\left(\bigcap_{n \geq 1} \left]0; x + \frac{1}{n}\right]\right) = \mu(]0;x]) = f(x)$ and also for $x < 0$ since owing to the **continuity from below** 2.2.4 we have $\lim\limits_{n \to \infty} f_\mu\left(x + \frac{1}{n}\right) = -\lim\limits_{n \to \infty} \mu\left(\left]x + \frac{1}{n}; 0\right]\right) = -\mu\left(\bigcup_{n \geq 1} \left]x + \frac{1}{n}; 0\right]\right) = -\mu(]x;0]) = f(x)$. For every $x \geq 0$ must exist a **left limit** since the **continuity from below** implies $\lim\limits_{n \to \infty} f_\mu\left(x - \frac{1}{n}\right) = \lim\limits_{n \to \infty} \mu\left(\left]0; x - \frac{1}{n}\right]\right) = \mu\left(\bigcup_{n \geq 1} \left]0; x - \frac{1}{n}\right]\right) = \mu(]0;x[) \in \mathbb{R}$ and likewise for $x < 0$ since by the **continuity from above** we have $\lim\limits_{n \to \infty} f_\mu\left(x - \frac{1}{n}\right) = -\lim\limits_{n \to \infty} \mu\left(\left]x - \frac{1}{n}; 0\right]\right) = -\mu\left(\bigcap_{n \geq 1} \left]x - \frac{1}{n}; 0\right]\right) = -\mu(x;0) \in \mathbb{R}$.

3.8 Continuous Lebesgue-Stieltjes measures

Continuous distribution functions $f \in C(\mathbb{R};\mathbb{R})$ imply $\mu_f(\{x\}) = \mu_f\left(\bigcap_{n \geq 1} \left]x - \frac{1}{n}; x + \frac{1}{n}\right]\right) = \lim\limits_{n \to \infty} \mu\left(\left]x - \frac{1}{n}; x + \frac{1}{n}\right]\right) = \lim\limits_{n \to \infty} f\left(x + \frac{1}{n}\right) - f\left(x + \frac{1}{n}\right) = 0$ whence $\mu_f(]a;b]) = \mu_f([a;b]) = \mu_f([a;b[) = \mu_f(]a;b[) = f(b) - f(a)$ for $a \leq b \in \mathbb{R}$. Thus every countable union of single points is a μ_f**-null set**, in particular the **rational numbers**: $\mu_f(\mathbb{Q}) = 0$. The **Cantor set** $T := g\left[\{0;2\}^{\mathbb{N}}\right]$ with $g(x) = \sum_{n \geq 1} \frac{x_n}{3^n}$ for any sequence $x = (x_n)_{n \geq 1}$ with $x_n \in \{0;2\}$ (cf. [13, p. 2.10] is a λ-null set since $T = \bigcap_{n \in \mathbb{N}} T_n$ with $T_0 = [0;1]$ and T_{n+1} is a union of 2^{n+1} disjoint and closed intervals with **length** resp. **measure** $\frac{1}{3^{n+1}}$ obtained by removing the middle third from the 2^n closed intervals T_n with length $\frac{1}{3^n}$ such that $\lambda(T_n) = \frac{2^n}{3^n}$ and $\lambda(T) = \lim\limits_{n \in \mathbb{N}} \lambda(T_n) = 0$ due to the **continuity from above** (2.2.3). The G$_\delta$-set $U = \bigcap_{n \geq 1} U_n$ with **dense open** sets $U_n = \bigcup_{i \geq 1} B_{n^{-1} \cdot 2^{-i-1}}(q_i)$ based on the enumeration $\mathbb{Q} = (q_i)_{i \geq 1}$ includes $\mathbb{Q}$ and hence is dense in $\mathbb{R}$. Again due to 2.2.3 and since $\lambda(U_n) \leq \frac{1}{n}$ it also is a λ-null set: $\lambda(U) = 0$. The **complements** $\mathbb{R} \setminus U_n$ are **closed** and **nowhere dense** in $\mathbb{R}$ but with measure $\lambda(\mathbb{R} \setminus U_n) = \infty$ and $\mathbb{R} \setminus U$ is an example for a set of **first category** with measure $\lambda(\mathbb{R} \setminus U) = \infty$. (cf. [13, th. 16.1])

3.9 Complete measures

A measure μ is **complete** iff every **subset** of a μ-**null set** is **measurable**.

1. A σ-algebra $\mathcal{A}$ can be **completed** to a σ-Algebra $\mathcal{A}_0 = \{A \cup M : A \in \mathcal{A} \wedge M \subset N \in \mathcal{A} : \mu(N) = 0\}$ by simply adding the requested subset of null sets to the given measurable sets: For $A, B \in \mathcal{A}$ resp. $M_A \subset N_A$, $M_B \subset N_B$ and $\mu(N_A) = \mu(N_B) = 0$ we have $(A \cup M_A) \setminus (B \cup M_B) = A \setminus (B \cup M_B) \cup M_A \setminus (B \cup M_B) = (A \setminus B) \cap (A \setminus N_B) \cup (N_B \setminus M_B) \cup M_A \setminus (B \cup M_B) \in \mathcal{A}_0$ since $(A \setminus B) \cap (A \setminus N_B) \in \mathcal{A}$ and $(N_B \setminus M_B) \cup M_A \setminus (B \cup M_B) \subset N_A \cup N_B$ with $\mu(N_A \cup N_B) = 0$. The σ-additivity is obvious.
2. A set E is $\mathcal{A}_0$-measurable iff there are $A, B \in \mathcal{A}$ with $A \subset E \subset B$ and $\mu(B \setminus A) = 0$: One the one hand for any $E = A \cup M$ with $M \subset N \in \mathcal{A}$ and $\mu(N) = 0$ the measurable sets A and $B := A \cup N$ satisfy the criterion. On the other hand for any E and measurable A, B according to the criterion we have $E = A \cup (B \setminus A \cap E)$ with $B \setminus A \cap E \subset B \setminus A$ and hence $E \in \mathcal{A}_0$.
3. The corresponding **extension** $\mu_0 \supset \mu$ with $\mu_0(A \cup N) := \mu(A)$ for $A \in \mathcal{A}$ and $N \subset M : \mu(M) = 0$ obviously is a complete measure. Thus the **Lebesgue-Borel measure** λ on the σ-algebra $\mathcal{B}$ of the **Borel sets** is extended to the **Lebesgue measure** λ_0 on the **completed σ-algebra** $\mathcal{B}_0$ of the **Lebesgue sets**. In the following section the index is usually omitted such that the complete Lebesgue space is still denoted as $(X; \mathcal{B}; \lambda)$.

3.10 Almost everywhere existing properties

In **probability theory** the completion is seldom used since it is not generated by the **open** sets any more and hence restricts the choice of possible **measures** resp. **distributions** without granting any gain in information. In **analysis** it is widely adopted though not always necessarily so since a σ-algebra is a family e.g. larger by far than the topology on $\mathbb{R}$ such that it is not a trivial exercise to find non measurable sets at all. In any case we speak of a property $E(x)$ being satisfied μ**-almost everywhere** (μ-a.e.) iff it is satsfied everywhere with the exception of μ-**null sets**, i.e. iff $\mu(\neg E) = 0$.

3.11 Vitali's theorem on non-measurable sets

There is a set $K \subset \mathbb{R}$ which is **not Lebesgue measurable**.

Proof: The **equivalence relation** defined by $xRY \Leftrightarrow x - y \in \mathbb{Q}$ generates a disjoint cover of $\mathbb{R}$ by equivalence classes with the class $\overline{0} = \mathbb{Q}$ and all other classes represented by irrational numbers. Since $\mathbb{Q}$ is dense in $\mathbb{R}$ every class has representants $x \in [0; 1]$ and the **axiom of choice** [11, p. 14.2.1] permits us to choose **exactly one of those for every equivalence class** and thus define a set $K \subset [0; 1]$ such that we obtain a **disjoint and countable cover** $\mathbb{R} = \mathring{\bigcup}_{q\in\mathbb{Q}} (q + K)$ which due to the **σ-additivity** and the **translation invariance** must satisfy $\infty = \lambda(\mathbb{R}) = \sum_{q\in\mathbb{Q}} \lambda(K)$ and hence $\lambda(K) > 0$. On the other hand we have $\mathring{\bigcup}_{q\in\mathbb{Q}\cap[0;1]} (q + K) \subset [0; 2]$ and due to the **monotonicity** of the measure $\sum_{q\in\mathbb{Q}} \lambda(K) \leq \lambda([0; 2]) = 2$ hence $\lambda(K) = 0$. From this contradiction we must infer that K is **not measurable**.

4 Measurable functions

4.1 Measurable functions

A mapping $f : (X; \mathcal{A}) \to (Y; \mathcal{B})$ between measurable spaces is **measurable** iff every inverse image $f^{-1}(B)$ of a measurable set $B \in \mathcal{B}$ is again measurable in $(X; \mathcal{A})$, i.e. $f^{-1}(B) \in \mathcal{A}$. Since all necessary set operations transfer to inverse images (cf. [11, p. 9.2]) it is sufficient that the inverse images of **basis** sets are measurable in X (cf. [13, p. 3.1]). In analysis the usual basis is the topology $\mathcal{O}$ on Y and the function is **Borel measurable** iff it is measurable with reference to $\mathcal{B} = \sigma(\mathcal{O})$. Hence a function $f : (X; \mathcal{A}) \to (Y; d)$ into a **metric space** is Borel measurable iff $f^{-1}[\mathcal{B}_\epsilon(y)] \in \mathcal{A}$ for every $\epsilon > 0$ and $y \in Y$.

4.2 Real valued Borel measurable functions

According to 2.1 a function $f : X \to \mathbb{R}$ is measurable iff the sets $\{f \geq a\} := f^{-1}[[a; \infty[]$ or the analogously defined $\{f > a\}$, $\{f \leq a\}$ resp. $\{f < a\}$ are measurable in X. In particular for a Borel measurable $f : X \to \mathbb{R}$ the **positive part** $f^+ := \max\{f; 0\}$, the **negative part** $f^- := \min\{f; 0\}$ are Borel measurable. Since $\mathbb{Q}$ is countable and dense in $\mathbb{R}$ the sets $\{f > g\} = \bigcup_{a \in \mathbb{Q}} (\{f > a\} \cap \{a > g\})$ and $\{f \geq g\} = X \setminus \{f < g\}$ are measurable. Hence the **maximum** $\max\{f; g\}$ and the **minimum** $\min\{f; g\}$ are Borel measurable for any for measurable $f, g : X \to \mathbb{R}$. In the expression for the measure μ of the set of all $x \in X$ for which $A(f(x))$ is true we will often omit not only the argument but also the curly brackets: $\mu(A(f)) = \mu(\{A(f)\}) = \mu(\{x \in X : A(f(x))\})$ as e.g. in $\mu(|f| < \epsilon) = \mu(\{|f| < \epsilon\})$.

4.3 The image of a measure space

The image $f(\mathcal{A}) := \{B \subset Y : f^{-1}[B] \in \mathcal{A}\}$ of a σ-algebra $\mathcal{A}$ on X under $f : X \to Y$ is a σ-algebra on Y and the largest σ-algebra such that f is measurable. The **image of the measure** $f \circ \mu : f(\mathcal{A}) \to [0; \infty]$ with $(f \circ \mu)(B) := \mu(f^{-1}[B])$ resp. $(f \circ \mu)(f[B]) := \mu(B)$ is a measure on $f(\mathcal{A})$ and **transitive** with regard to **composition**: $g \circ f \circ \mu : g \circ f(\mathcal{A}) \to [0; \infty]$ obviously is again a measure. E.g. the **Lebesgue measure** λ is **invariant** under the **translation** $T_c(x) = x + c$ with $(T_c \circ f)([a; b[) = \lambda(T_c^{-1}[[a; b[]) = \lambda([a - c; b - c[) = \lambda([a; b[)$ but not under **dilation** $g(x) = mx$ since $(g \circ \lambda)([a; b[) = \lambda(g^{-1}[[a; b[]) = \lambda\left(\left[\frac{a}{m}; \frac{b}{m}\right[\right) = \frac{1}{m}\lambda([a; b[)$.

4.4 The inverse image of a measurable space

The **inverse image** $\sigma(f^{-1}(\mathcal{E})) = f^{-1}(\sigma(\mathcal{E}))$ of the σ-algebra $\sigma(\mathcal{E})$ on Y induced by $\mathcal{E} \subset \mathcal{P}(Y)$ under $f : X \to Y$ is the **smallest** σ-algebra such that f is measurable. The inclusion $\subset$ holds since $f^{-1}(\sigma(\mathcal{E}))$ is a σ-algebra containing $f^{-1}(\mathcal{E})$. The inclusion $\supset$ follows from 4.3 since $f(\sigma(f^{-1}(\mathcal{E})))$ is a σ-algebra on Y including $\mathcal{E}$ and hence $\sigma(\mathcal{E})$.

4.5 Continuous functions

On account of 4.4 a function $f : (X; \mathcal{A}) \to (Y; \sigma(\mathcal{O}_Y))$ into a **topological space** $(Y; \mathcal{O}_Y)$ is **Borel measurable** iff the inverse image of every **open set** in measurable in $(X; \mathcal{A})$: $f^{-1}(\mathcal{O}_Y) \subset \mathcal{A} \Rightarrow f^{-1}(\sigma(\mathcal{O}_Y)) = \sigma(f^{-1}(\mathcal{O}_Y)) \subset \mathcal{A}$. In the case of $\mathcal{A} = \sigma(\mathcal{O}_X)$ also being induced by a topology $\mathcal{O}_X$ on X every **continuous function** is **Borel measurable**. A real function $f : X \to \mathbb{R}$ on a topological space $(X; \mathcal{O})$ is lower resp. upper **semicontinuous** iff $f^{-1}[]a; \infty[] \in \mathcal{O}$ resp. $f^{-1}[]-\infty; b[] \in \mathcal{O}$ for $\forall a, b \in \mathbb{R}$. (cf. [13, p. 3.3]) According to 1.4 resp. 4.1 these functions are again Borel measurable.

4.6 Compositions

The **composition** $h = g \circ f : X \to Z$ is measurable iff $f : X \to Y$ and $g : Y \to Z$ are measurable. Due to [13, 3.1, 4.2.3 and 10.7]

- the **projections** $\pi_j : \prod_{i \in I} X_i \to X_j$ on a **product space** $\prod_{i \in I} X_i$,
- the **metric** $d : X^2 \to [0; \infty[$ on a **metric space** $(X; d)$,
- the **norm** $|| : X \to [0; \infty[$, the **multiple** $\alpha\cdot : X \to X$ for fixed $\alpha \in \mathbb{C}$ and the **addition** $+ : X^2 \to X$ on a **Banach space** $(X; ||)$ (cf. [13, p. 21.9]) ,

- the **multiplication** $\cdot : X^2 \to X$ on a **Banach algebra** $(X;||)$ (cf. [13, p. 18.9]) and
- the **multiple** $x \mapsto \alpha \cdot x$ resp. the **powers** $x \mapsto x^{\alpha}$ for $\alpha \in \mathbb{C}$ as well as in particular the reciprocal $x \mapsto \frac{1}{x}$ on a **field** like $\mathbb{R}$ or $\mathbb{C}$

are **continuous** and hence **Borel measurable**. Hence for Borel measurable $f, g : X \to \mathbb{C}$ the **real part** $\mathrm{Re} f$, **imaginary part** $\mathrm{Im} f$ and **absolute value** $|f|$ are Borel measurable mappings $X \to \mathbb{R}$; likewise the **complex conjugate** $\overline{f}$ as well as $\alpha \cdot f$, f^{α}, $\frac{1}{f}$, $f + g$ and $f \cdot g$ are Borel measurable mappings $X \to \mathbb{C}$.

4.7 Measurable functions into product spaces

A Borel measurable function $f : (X;\mathcal{A}) \to \left(\prod_{i\in I} Y_i; \sigma\left(\bigotimes_{i\in I} \mathcal{O}_i\right)\right)$ has Borel measurable components $f_i := \pi_i \circ f$. Since the **cylinder sets** $\bigcap_{i\in J} \pi_i^{-1}[O_i]$ with O_i open in Y_i and **finite** $J \subset I$ form a basis for the **product topology** $\bigotimes_{i\in I} \mathcal{O}_i$ (cf. [13, p. 4.2]) the converse is true if this basis is **countable** (i.e. the product topology is **first countable**, cf. [13, p. 2.6]) such that the inverse image of every open set in $\prod_{i\in I} Y_i$ is the countable union of inverse images of cylinder sets and hence contained in the σ-algebra $\mathcal{A}$ on X. This condition is satisfied for every **finite product** $\prod\limits_{i=1}^{n} Y_i$ of first countable components Y_i and in particular $\mathbb{C}^n$. Note that the countability condition is not needed for the corresponding statement on **continuous** functions since a **topology** $\mathcal{O}$ on X includes **arbitrary** unions of cylinder sets. Hence $f : X \to \mathbb{C}^n$ is Borel measurable iff every component f_i is Borel measurable.

4.8 Vector spaces of measurable functions

The product Y^2 of two Banach spaces $(Y;||)$ is first countable if Y itself is the **finite product** of first countable spaces, e.g. $\mathbb{C}^n$ or **separable**, e.g. the space $C_c^{\infty}(\mathbb{C})$ of infinitely derivable functions $f : \mathbb{C} \to \mathbb{C}$ with compact support. In these cases the ordered pair $(f, g) : X \to Y^2$ is Borel measurable if each $f, g : X \to Y$ is Borel measurable and so is their sum $f + g$ such that the Borel measurable functions $f : X \to Y$ into **finite dimensional** or **separable** Banach spaces Y themselves form a **vector space**.

4.9 Pointwise limits of measurable functions

The **pointwise** limit $f = \lim\limits_{n\to\infty} f_n$ of a sequence $(f_n)_{n\geq 1}$ of Borel measurable functions $f_n : X \to Y$ from a **measurable space** $(X;\mathcal{A})$ into a **metric space** (Y, d) is again Borel measurable.

Proof: For any **open** $U \subset Y$ and $f(x) \in U$ there is an $m \in \mathbb{N}$ with $f_k(x) \in U$ for all $k \geq m$ and hence $f^{-1}[U] \subset \bigcup\limits_{m=1}^{\infty} \bigcap\limits_{k=m}^{\infty} f_k^{-1}[U] \subset \bigcap\limits_{m=1}^{\infty} \bigcup\limits_{k=m}^{\infty} f_k^{-1}[U]$. On the other hand every **closed** $A \subset Y$ containing infinitely many $f_k(x)$ must contain the limit $f(x)$, i.e. $\bigcap\limits_{m=1}^{\infty} \bigcup\limits_{k=m}^{\infty} f_k^{-1}[A] \subset f^{-1}[A]$. For the open sets $V_n = \left\{x \in U : d(x; X \setminus U) < \frac{1}{n}\right\}$ we have $U = \bigcup\limits_{n=1}^{\infty} V_n = \bigcup\limits_{n=1}^{\infty} \overline{V_n}$ and hence $\bigcup\limits_{n=1}^{\infty} \bigcap\limits_{m=1}^{\infty} \bigcup\limits_{k=m}^{\infty} f_k^{-1}\left[\overline{V_n}\right] \subset \bigcup\limits_{n=1}^{\infty} f^{-1}\left[\overline{V_n}\right] = f^{-1}[U] = \bigcup\limits_{n=1}^{\infty} f^{-1}[V_n] \subset \bigcup\limits_{n=1}^{\infty} \bigcap\limits_{m=1}^{\infty} \bigcup\limits_{k=m}^{\infty} f_k^{-1}[V_n]$ whence follows equality since $V_n \subset \overline{V_n}$.

4.10 Convergence in measure and μ-almost everywhere

A sequence $(f_n)_{n\geq 1}$ of Borel measurable functions $f_n : X \to Y$ from a **measure space** $(X;\mathcal{A};\mu)$ into a **Banach space** $(Y, ||)$ converges to a Borel measurable $f : X \to Y$:

1. **μ-almost everywhere** (**μ-a.e.**) iff one of the following equivalent conditions is satisfied:

   a) $\mu\left(X \setminus \left\{\lim\limits_{n\to\infty} |f_n - f| = 0\right\}\right) = 0$

b) $\lim\limits_{k\to\infty} \mu\left(\sup\limits_{n\geq k} |f_n - f| \geq \epsilon\right) = \lim\limits_{k\to\infty} \mu\left(\bigcup_{n\geq k} \{|f_n - f| \geq \epsilon\}\right) = 0$ for every $\epsilon > 0$
$\overset{*}{\Rightarrow} \mu\left(\bigcap_{k\geq 1}\bigcup_{n\geq k} \{|f_n - f| \geq \epsilon\}\right) = 0$ for every $\epsilon > 0$

c) $\lim\limits_{k\to\infty} \mu\left(\sup\limits_{n\geq k} |f_n - f| \geq \frac{1}{k}\right) = \lim\limits_{k\to\infty} \mu\left(\bigcup_{n\geq k} \left\{|f_n - f| \geq \frac{1}{k}\right\}\right) = 0$
$\overset{*}{\Rightarrow} \mu\left(\bigcap_{k\geq 1}\bigcup_{n\geq k} \left\{|f_n - f| \geq \frac{1}{k}\right\}\right) = 0.$

2. **in measure** μ iff for every $A \in \mathcal{A}$ with $\mu(A) < \infty$ one of the following equivalent conditions is satisfied

   a) $\lim\limits_{n\to\infty} \mu|_A (|f_n - f| \geq \epsilon) = 0$ for every $\epsilon > 0 \Leftrightarrow$

   b) For every $k \in \mathbb{N}$ there is an $n_k \in \mathbb{N}$ such that $\mu|_A\left(|f_{n_k} - f| \geq 2^{-k}\right) < 2^{-k}$.

Notes:

1. The preceding definition is also known as **local convergence in measure** as opposed to the stronger **global convergence in measure** without the restriction to sets with **finite measure** $\mu(A) < \infty$. For an apriori **finite measure** with $\mu(X) < \infty$ the two definitions obviously coincide. In the case of a **probability measure** the convergence in measure is called **stochastic convergence**.
2. The inclusions $\overset{*}{\Rightarrow}$ become **equivalences** if we can presume the **continuity from above** 2.2.3, i.e. $\mu(X) < \infty$ or at least the existence of a $k \in \mathbb{N}$ such that $\mu\left(\bigcup_{n\geq k}\left\{|f_n - f| \geq \frac{1}{k}\right\}\right) < \infty$. Many of the subsequent convergence theorems also depend heavily on 2.2.3 and hence are restricted to **finite measure spaces** resp. to **local convergence in measure**. In partucular for the **Lebesgue measure** λ they **do not extend to global convergence**.
3. Both convergence criterions imply that the limit function f as well as **finally** (i.e. all except for a finite number) all f_n are **μ-a.e. finite.**

4.11 Lebesgue's convergence theorem

A sequence $(f_n)_{n\geq 1}$ of Borel measurable functions $f_n : X \to Y$ from a **measure space** $(X; \mathcal{A}; \mu)$ into a **Banach space** $(Y, ||)$ converging **μ-a.e.** to a Borel measurable function $f : X \to Y$ also converges **in measure** to f.

Proof: For every $A \in \mathcal{A}$ with $\mu(A) < \infty$ and $\epsilon > 0$ we have $\inf\limits_{k\geq 1}\sup\limits_{n\geq k}\mu|_A(\{|f_n - f| \geq \epsilon\}) \overset{2.2.2}{=} \inf\limits_{k\geq 1}\mu|_A\left(\bigcup_{n\geq k}\{|f_n - f| \geq \epsilon\}\right) \overset{2.2.3}{=} \mu|_A\left(\bigcap_{k\geq 1}\bigcup_{n\geq k}\{|f_n - f| \geq \epsilon\}\right) = 0$.

Example: The **Lebesgue measure** λ is **not continuous from above**, e.g. $\lambda\left(\bigcap_{n\in\mathbb{N}} \mathbb{R} \setminus B_n(0)\right) = \lambda(\emptyset) = 0$ but $\inf\limits_{n\in\mathbb{N}} \lambda(\mathbb{R} \setminus B_n(0)) = \infty$ since $\lambda(\mathbb{R} \setminus B_n(0)) = \infty$ for every $n \in \mathbb{N}$. Hence in the case of $f_n(x) = \frac{x^2}{n}$ we observe **pointwise convergence** and particularly **λ-a.e. convergence** as well as **compact convergence** to $f(x) = 0$ hence **local convergence in measure** but not **global convergence in measure** since $\lambda(|x| \geq \epsilon) = \lambda(|f_n - f| \geq \sqrt{n\epsilon}) = \infty$ for every $n \in \mathbb{N}$ and $\epsilon > 0$.

4.12 The Borel-Cantelli lemma

For every sequence $(A_n)_{n\geq 1}$ of measurable sets $A_n \in \mathcal{A}$ on a **measure space** $(X; \mathcal{A}; \mu)$ we have $\sum_{n\geq 1}\mu(A_n) < \infty \Rightarrow \mu\left(\bigcap_{k\geq 1}\bigcup_{n\geq k} A_n\right) = 0$ and in the case of a **probability measure** and **pairwise independent** A_n, i.e. $\mu(A_k \cap A_l) = \mu(A_k) \cdot \mu(A_l)$ for $k \neq l$ the **converse** is also true: $\sum_{n\geq 1}\mu(A_n) = \infty \Rightarrow \mu\left(X \setminus \bigcap_{k\geq 1}\bigcup_{n\geq k} A_n\right) = 0$.

Proof: In the first case for every $\epsilon > 0$ there is a $k_\epsilon \geq 1$ with $\sum_{n \geq k_\epsilon} \mu(A_n) < \epsilon$ such that $\mu\left(\bigcap_{k\geq 1}\bigcup_{n \geq k} A_n\right) \leq \mu\left(\bigcup_{n\geq k_\epsilon} A_n\right) \leq \sum_{n\geq k_\epsilon} \mu(A_n) < \epsilon$ and hence the assertion. In the second case with $\mu(X) = 1$ and the **continuity of the exponential function** we have $\mu\left(\bigcap_{k\geq 1}\bigcup_{n\geq k} A_n\right) =$ $1 - \mu\left(\bigcup_{k\geq 1}\bigcap_{n\geq k} X \setminus A_n\right) \overset{2.2.2}{=} 1 - \sup_{k\geq 1}\mu\left(\bigcap_{n\geq k} X \setminus A_i\right) \overset{2.2.3}{=} 1 - \sup_{k\geq 1}\inf_{n\geq k}\mu\left(\bigcap_{i=k}^{n} X\setminus A_i\right) = 1 - \sup_{k\geq 1}\inf_{n\geq k}\prod_{i=k}^{n}$ $(1-\mu(A_i)) \geq 1 - \sup_{k\geq 1}\inf_{n\geq k}\prod_{i=k}^{n}\exp(-\mu(A_i)) = 1 - \sup_{k\geq 1}\inf_{n\geq k}\exp\left(-\sum_{n\geq i\geq k}\mu(A_i)\right) = 1.$

4.13 Completeness and μ-a.e. convergent subsequence for convergence in measure

For a sequence $(f_n)_{n\geq 1}$ of Borel measurable functions $f_n : X \to Y$ from a **measure space** $(X; \mathcal{A}; \mu)$ into a **Banach space** $(Y, ||)$ the following statements are equivalent::

1. $(f_n)_{n\geq 1}$ is a **Cauchy sequence in measure,** i.e. $\lim_{k\geq 1}\sup_{n\geq k}\mu|_A(|f_n - f_k| > \epsilon) = 0$ for every $A \in \mathcal{A}$ with $\mu(A) < \infty$ and $\epsilon > 0$.
2. $(f_n)_{n\geq 1}$ converges **in measure** to a Borel measurable function $f : X \to Y$.
3. **Riesz convergence theorem**: Every **subsequence** of $(f_n)_{n\geq 1}$ has another **subsequence converging μ-a.e.** to the same Borel measurable function $f : X \to Y$

Proof: Let $A \in \mathcal{A}$ with $\mu(A) < \infty$.

1. ⇒ 2. : Due to the hypothesis for every $k \geq 1$ there is an $n_k \geq 1$ with $\mu|_A\left(|f_n - f_{n_k}| > 2^{-k}\right) < 2^{-k}$ for all $n \geq n_k$. Hence we have a partial sequence $(f_{n_k})_{k\geq 1}$with w.l.o.g. $n_{k+1} > n_k$ and $B_k = \left\{|f_{n_{k+1}} - f_{n_k}| > 2^{-k}\right\}$ such that $\sum_{k\geq 1}\mu|_A(B_k) < \infty$. According to 4.12 we obtain $\mu|_A\left(\bigcap_{m\geq 1}\bigcup_{k\geq m}(B_k)\right)$ $= \mu(X \setminus B) = 0$ for $B = \bigcup_{m\geq 1}\bigcap_{k\geq m}(X\setminus B_k)$. Hence for every $x \in B$ there is an $m \geq 1$ such that $\sup_{k\geq m}|f_{n_k}(x) - f_{n_m}(x)| \leq \sum_{k\geq m}|f_{n_{k+1}}(x) - f_{n_k}(x)| \leq \sum_{k\geq m} 2^{-k} = 2^{-m+1}$. Thus we have a **$\mu$-a.e. Cauchy sequence** $(f_{n_k})_{k\geq 1}$ which due to the **completeness** of Y and according to 4.7 converges μ-a.e. to a **measurable** $f : B \to Y$. Due to $\mu(A) < \infty$ we can apply 4.11 to find for every $\epsilon > 0$ an $m_\epsilon \geq 1$ such that $\mu|_A(|f_{n_m} - f| > \frac{\epsilon}{2}) < \frac{\epsilon}{2}$ for every $m \geq m_\epsilon$. Hence for every $n \geq n_m$ with $m \geq \max(m_\epsilon; k)$ and $2^{-k} < \frac{\epsilon}{2}$ we obtain $\mu|_A(|f_n - f| > \epsilon) \leq \mu|_A(\{|f_n - f_{n_m}| > \frac{\epsilon}{2}\} \cup \{|f_{n_m} - f| > \frac{\epsilon}{2}\}) \leq$ $\mu|_A(|f_m - f_{n_m}| > \frac{\epsilon}{2}) + \mu|_A(|f_{n_m} - f| > \frac{\epsilon}{2}) < \epsilon$. This converse-triangle-inequality argument will be repeatedly used in the subsequent proofs.

2. ⇒ 3. : Due to 4.10.2 b) for every $k \geq 1$ there is an $n_k \geq 1$ such that $\mu(B_k) < 2^{-k}$ for $B_k = \left\{|f_{n_k} - f| \geq \frac{1}{k}\right\}$ whence $\mu|_A\left(\bigcup_{k\geq m} B_k\right) \leq 2^{-m+1}$ due to the **subadditivity** 2.2.1 and $\mu|_A\left(\bigcap_{m\geq 1}\bigcup_{k\geq m} B_k\right) =$ 0 due to the **continuity from above** 2.2.3. Both properties require $\mu(A) < \infty$. The assertion then follows from 4.10.1 c).

3. ⇒ 1. : Suppose there is an $\epsilon > 0$ such that $\forall n_k \geq 1\ \exists n_{k+1} \geq n_k$ with $\mu|_A(|f_{n_{k+1}} - f_{n_k}| > \epsilon)$ $> \epsilon$. As above we get $\mu|_A(f_{n_k} - f > \frac{\epsilon}{2}) + \mu|_A(|f_{n_{k+1}} - f| > \frac{\epsilon}{2}) \geq \mu|_A(|f_{n_k} - f_{n_{k+1}}| > \epsilon) > \epsilon$, i.e. either $\mu|_A(|f_{n_k} - f| > \frac{\epsilon}{2}) \geq \frac{\epsilon}{2}$ or $\mu|_A(|f_{n_{k+1}} - f| > \frac{\epsilon}{2}) \geq \frac{\epsilon}{2}$. For each $k \in \mathbb{N}$ we choose the f_{n_k} with respectively larger probability $\mu(\ldots)$ of deviation and thus obtain a subsequence $\left(f'_{n_k}\right)_{k\geq 1}$ with $\mu|_A\left(\left|f'_{n_k} - f\right| > \frac{\epsilon}{2}\right) \geq \frac{\epsilon}{2}$ for all $k \geq 1$ such that no part of this subsequence can possibly converge in measure to f and according to 4.11 with $\mu(A) < \infty$ this behaviour transfers to μ-a.e. convergence.

4.14 Completeness of μ-a.e. convergence

A sequence $(f_n)_{n\geq 1}$ of Borel measurable functions $f_n : X \to Y$ from a **measure space** $(X; \mathcal{A}; \mu)$ into a **Banach space** $(Y, ||)$ **converges** μ**-a.e.** to a Borel measurable $f : X \to Y$ iff $\lim\limits_{k\to\infty} \mu|_A \left(\sup\limits_{n\geq k} |f_k - f_n| > \epsilon\right) = 0$ for every $\epsilon > 0$.

Proof:

$\Rightarrow$: Applying the converse-triangle-inequality argument to suprema we obtain

$$\mu|_A \left(\sup_{n\geq k} |f_k - f_n| > \epsilon\right) \leq \mu|_A \left(|f_k - f| > \frac{\epsilon}{2}\right) + \mu|_A \left(\sup_{n\geq k} |f - f_n| > \frac{\epsilon}{2}\right)$$

The assertion follows from the **convergence in measure** due to 4.11 presuming $\mu(A) < \infty$ resp. the μ**-a.e. convergence** due to 4.10.1 b).

$\Leftarrow$: Due to the **continuity from below** 2.2.2 we obtain

$$\sup_{n\geq k} \mu|_A (|f_k - f_n| > \epsilon) \leq \mu|_A \left(\bigcup_{n\geq k} |f_k - f_n| > \epsilon\right) = \mu|_A \left(\sup_{n\geq k} |f_k - f_n| > \epsilon\right),$$

i.e. $(f_n)_{n\geq 1}$ **converges in measure** to f . Using again the converse-triangle-inequality we get

$$\mu|_A \left(\sup_{n\geq k} |f - f_n| > \epsilon\right) \leq \mu|_A \left(|f - f_k| > \frac{\epsilon}{2}\right) + \mu|_A \left(\sup_{n\geq k} |f_k - f_n| > \frac{\epsilon}{2}\right)$$

and hence the μ**-a.e. convergence** to f due to 4.10.1 b).

4.15 Egorov's convergence theorem

For every sequence $(f_n)_{n\geq 1}$ of Borel measurable functions $f_n : X \to Y$ from a **finite** measure space $(X; \mathcal{A}; \mu)$ into a **Banach space** $(Y, ||)$ **converging** μ**-a.e.** to a Borel measurable $f : X \to Y$ and every $\epsilon > 0$ there is a set $A_\epsilon \in \mathcal{A}$ with $\mu(A_\epsilon) < \epsilon$ such that $(f_n)_{n\geq 1}$ **uniformly converges** to f on $X \setminus A_\epsilon$.

Proof: Follows directly from 4.10.1 b) since for $\epsilon > 0$ there is a $k_\epsilon \geq 1$ such that we have $\mu(A_\epsilon) < \epsilon$ for $A_\epsilon := \bigcup_{n\geq k_\epsilon} \left\{|f_n(x) - f(x)| \geq \frac{1}{n}\right\}$ and $(f_n)_{n\geq 1}$ obviously converges **uniformly** to f on $X \setminus A_\epsilon$.

4.16 Examples

1. The **function** sequence $(f_n)_{n\geq 1}$ with $f_n = \chi_{\left[\frac{j}{2^k}; \frac{j+1}{2^k}\right]}$ for $n = 2^k + j$, $0 \leq j < 2^k$ and $k \geq 1$ on $\left([0;1]; \mathcal{B}_{[0;1]}; \lambda_{[0;1]}\right)$ converges **globally in measure** λ to $f = 0$ but the **point** sequences $(f_n(x))_{n\geq 1}$ converge for **no** $x \in [0;1]$ hence $(f_n)_{n\geq 1}$ converges **not** λ**-a.e.**
2. The **function** sequence $(f_n)_{n\geq 1}$ with $f_n = \chi_{[n;n+1]}$ for $n \geq 1$ on $(\mathbb{R}; \mathcal{B}; \lambda)$ **converges for every** $x \in \mathbb{R}$ **hence** λ**-a.e.** to $f = 0$ and hence **locally in measure but not globally so** since for $\epsilon < 1$ there is no $k \geq 1$ such that $\lambda\left(\bigcup_{n\geq k} \{|f_n - f| \geq \epsilon\}\right) < \infty$: The **continuity from above** 2.2.3 resp. theorem 4.12 do not apply.

5 Integration

Throughout this section and if not specified otherwise any function from X to Y is **Borel measurable** from a measure space $(X; \mathcal{A}; \mu)$ with **positive measure** $\mu : A \to [0; \infty]$ into a Banach space $(Y, ||)$ over a **field** K.

5.1 Step functions

The **characteristic functions** $\chi_A : X \to \{0; 1\}$ for a measurable **support** $A \in \mathcal{A}$ with $\chi_A(x) = \begin{cases} 1, x \in A \\ 0, x \notin A \end{cases}$ are the most simple measurable functions on a measurable space $(X; \mathcal{A})$. They are identical with the **Dirac measure** δ_x from 2.3.1 albeit with interchanged roles for x and A. The family $\mathcal{S}(X; Y)$ denotes the **step functions** of the form $\sum\limits_{i=0}^{m} y_i \chi_{A_i}$ with $m \in \mathbb{N}$ such that $\bigcup\limits_{i=0}^{m} A_i = X$ with values $y_i \in Y$ and $\mu(A_i) < \infty$ for $1 \leq i \leq m$ but **vanishing** outside of these sets, i.e. $\alpha_0 = 0$. The step functions form a **vector space** of Borel measurable functions and according to 4.9 their **closure** $\overline{\mathcal{S}(X; Y)}$ with regard to pointwise convergence includes Borel measurable maps with **separable range** and **vanishing outside of a countable union of sets with finite measure**. Countable unions of sets with finite measure are called **σ-finite** with the most prominent example represented by $\mathbb{C}^n$ which is also separable. The following theorem shows that under these two conditions $\overline{\mathcal{S}(X; Y)}$ already contains **all** Borel measurable functions modulo null sets, i.e. $\mathcal{S}(X; Y)$ is **dense** in the **quotient space** of the Borel measurable functions with regard to the **equivalence relation** $f \sim g \Leftrightarrow f = g$ μ-a.e.

5.2 Limits of step functions

For every Borel measurable function $f : X \to Y$ from a **σ-finite measure space** $(X; \mathcal{A}; \mu)$ into a **separable Banach space** $(Y, ||)$ there is a sequence $(\varphi_n)_{n \in \mathbb{N}} \subset \mathcal{S}(X; Y)$ of step functions converging μ-a.e. to f. Also for every set A of **finite measure** $\mu(A) < \infty$ and $\epsilon > 0$ there is a set $Z_\epsilon \subset X$ with measure $\mu(Z_\epsilon) < \epsilon$ such that $(\varphi_n)_{n \in \mathbb{N}}$ converges **uniformly** on $A \setminus Z_\epsilon$.

Note: With 4.9 we obtain a necessary and sufficient condition for measurability: A function $f : X \to Y$ from a **σ-finite measure space** $(X; \mathcal{A}; \mu)$ into a **separable Banach space** $(Y, ||)$ is **Borel measurable** iff there is a sequence $(\varphi_n)_{n \in \mathbb{N}} \subset \mathcal{S}(X; Y)$ of step functions converging μ-a.e. to f.

Proof: The image $f[A]$ of a set with finite measure $\mu(A) < \infty$ includes a dense subset $(y_l)_{l \geq 1}$ such that for every $n \geq 1$ we have $f[A] \subset \bigcup\limits_{l=1}^{\infty} B_{1/n}(y_l)$ resp. $A \subset \bigcup\limits_{l=1}^{\infty} C_{l,n}$ with $C_{l,n} = f^{-1}\left[B_{1/n}(y_l)\right]$ and consequently there is an $L_n \in \mathbb{N}$ with $\mu\left(A \setminus \bigcup\limits_{l=1}^{L_n} C_{l,n}\right) < 2^{-n}$.

Then the step functions $\varphi_n = \sum\limits_{l=1}^{L_n} y_l \chi_{D_l}$ with $D_l = C_{l,n} \setminus \bigcup\limits_{i=1}^{l-1} C_{i,n}$ converge to f

- uniformly on every $A \cap \bigcup\limits_{l=1}^{L_n} C_{l,n}$ with $\mu(Z_n) < 2^{-n}$ for $Z_n = A \setminus \bigcup\limits_{l=1}^{L_n} C_{l,n}$ and $n \geq 1$
- pointwise on $A \cap \bigcup\limits_{n=1}^{\infty} \bigcup\limits_{l=1}^{L_n} C_{l,n}$ with $\mu(Z) = 0$ for $Z = A \setminus \bigcup\limits_{n=1}^{\infty} \bigcup\limits_{l=1}^{L_n} C_{l,n}$.

For $X = \bigcup\limits_{k=1}^{\infty} A_k$ with w.l.o.g. pairwise disjoint A_k and $\mu(A_k) < \infty$ for every $k \geq 1$ there is a sequence $(\varphi_{k;j})_{j \geq 1}$ of step functions converging to f

- uniformly on $A_k \setminus Z_{k,n}$ with $\mu(Z_{k,n}) \leq 2^{-n}$.
- pointwise on $A_k \setminus Z_k$ with $\mu(Z_k) = 0$.

Then for every set $A \subset \bigcup\limits_{k=1}^{m} A_k$ with $m \geq 1$ the step functions $\psi_n(x) = \begin{cases} \varphi_{k;n}(x) & \text{if } x \in A_k; 1 \leq k \leq n \\ 0 & \text{if } x \in X \backslash \bigcup\limits_{k=1}^{n} A_k \end{cases}$ converges to f

- uniformly on $A \backslash \bigcup\limits_{k=1}^{\infty} Z_{k,n+m}$ with $\mu\left(\bigcup\limits_{k=1}^{\infty} Z_{k,n+m}\right) < 2^{-m}$ and
- pointwise on $X \backslash \bigcup\limits_{k=1}^{\infty} Z_k$ with $\mu\left(\bigcup\limits_{k=1}^{\infty} Z_k\right) = 0$.

5.3 The integral for step functions

For any **step function** $\varphi = \sum_{0 \leq i \leq m} y_i \chi_{A_i}$ with $y_i \in Y$ and $A_i \in \mathcal{A}$ the **integral** is defined by $\int \varphi d\mu := \sum_{0 \leq i \leq m} y_i \mu(A_i)$. **Uniqueness** and **linearity** $\int (\alpha\varphi + \beta\psi)\, d\mu = \alpha \int \varphi d\mu + \beta \int \psi d\mu$ for $\alpha, \beta \in K$ are obvious if we consider representations with **common** and **pairwise disjoint supports** $A_i \cap B_j$ for two elementary functions f und g as in 5.1 and observe the **additivity** 2.2.1 of the measure. Also we define integrals on **measurable subsets** as $\int_A \varphi d\mu := \int \varphi|_A d\mu$. On account of $\varphi|_{A \mathring{\cup} B} = \varphi|_A + \varphi|_B$ we have $\int_{A \mathring{\cup} B} \varphi d\mu = \int_A \varphi d\mu + \int_B \varphi d\mu$. For **positive** integrands φ with $\varphi[X] \subset [0; \infty[$ we have **monotonicity** in the form $\varphi < \psi \Rightarrow \int \varphi d\mu < \int \psi d\mu$. In general Banach spaces we still have $\left|\int_A \varphi d\mu\right| \leq \int_A |\varphi|\, d\mu \leq \|\varphi\|_\infty \mu(A)$ with the **supremum norm** $\|\varphi\|_\infty = \sup\limits_{x \in X} |\varphi(x)|$. The expression $\|\varphi\|_1 := \int |\varphi|\, d\mu$ defines the $\mathcal{L}^1$**- seminorm** (c.f. [13, p. 1.3]) on $\mathcal{S}(X;Y)$ with obvious **linearity** $\|\alpha\varphi + \beta\psi\|_1 = |\alpha| \cdot \|\varphi\|_1 + |\beta| \cdot \|\psi\|_1$ and the **triangle inequality** $\|\varphi + \psi\|_1 \leq \|\varphi\|_1 + \|\psi\|_1$. The latter follows from an application of the triangle inequality $|y_\varphi + y_\psi|_K \leq |x_\varphi|_K + |y_\psi|_K$ on the field K to representations with **common** as well as **pairwise disjoint supports** $A_i \cap B_j$ and invoking the **monotonicity of the integral** for the positive integrand $|\varphi|$. A sequence $(\varphi_n)_{n \in \mathbb{N}}$ of step functions **converges in mean** or with respect to $\mathcal{L}^1$ to a step function φ iff $\lim\limits_{n \to \infty} \|\varphi_n - \varphi\|_1 = 0$.

5.4 Convergence of step functions

For any $\mathcal{L}^1$- Cauchy sequence $(\varphi_n)_{n \in \mathbb{N}}$ of **step functions** $\varphi_n : X \to Y$ there exists a subsequence $(\varphi_{n_k})_{k \in \mathbb{N}}$ and for every $\epsilon > 0$ a set $Z_\epsilon \subset X$ with measure $\mu(Z_\epsilon) < \epsilon$ such that $(\varphi_{n_k})_{k \in \mathbb{N}}$ converges **absolutely** and **uniformly** on $X \setminus Z_\epsilon$ as well as μ**-a.e.** on X.

Proof: For every $k \geq 1$ there is an $n_k \geq n_{k-1} \in \mathbb{N}$ such that $\|\varphi_n - \varphi_{n_k}\|_1 \leq \frac{1}{2^{2k}}$ for every $n \geq n_k$. Then for $Y_k = \left\{|\psi_{k+1} - \psi_k| \geq \frac{1}{2^k}\right\}$ with $\psi_k := \varphi_{n_k}$ we have $\frac{1}{2^k}\mu(Y_k) = \int_{Y_n} \frac{1}{2^k} d\mu \leq \int_X |\psi_{k+1} - \psi_k|\, d\mu \leq \frac{1}{2^{2k}}$ whence $\mu(Y_k) \leq \frac{1}{2^k}$. Hence $\mu(Z_m) \leq \frac{1}{2^{m-1}}$ for $Z_m = \bigcup\limits_{k=m}^{\infty} Y_k$ and $|\psi_{k+1}(x) - \psi_k(x)| < \frac{1}{2^k}$ for every $x \in X \setminus Z_m$ resp. $k \geq m$ such that $\sum\limits_{k=m}^{\infty} (\psi_{k+1} - \psi_k)$ converges **absolutely** and **uniformly** on $X \setminus Z_m$. Hence $(\varphi_{n_k})_{k \geq m}$ converges **absolutely** and **uniformly** on $X \setminus Z_m$ resp. **pointwise** on $X \backslash \bigcap\limits_{m=1}^{\infty} Z_m$. Due to the **continuity from above** 2.2.3 we have $\mu\left(\bigcap\limits_{m=1}^{\infty} Z_m\right) = 0$.

5.5 The Bochner integral

The **Bochner integral** $\int f d\mu := \lim\limits_{n \to \infty} \int \varphi_n d\mu < \infty$ is **well defined** and **finite** for every function $f : X \to Y$ with an **approximating sequence** $(\varphi_n)_{n \in \mathbb{N}}$, i.e. an $\mathcal{L}^1$**-Cauchy** sequence of **step functions** converging μ**-a.e.** to f. The **vector space** $\mathcal{B}(X;Y)$ of these **integrable functions** is the **Bochner space** whereas $\mathcal{L}^1(X;Y) = \{f : X \to Y : \|f\|_1 < \infty\} \subset \mathcal{B}(X;Y)$ of **Lebesgue integrable** functions is called the **Lebesgue space**. Hence the integral is a **linear functional** $I : \mathcal{B} \to K$. According to 4.9 every integrable $f : X \to Y$ from a σ**-finite measure space** $(X; \mathcal{A}; \mu)$ into a **separable Banach space** $(Y, ||)$ is measurable.

In order to prove that the definition is independent of the approximating sequence we show: For two $\mathcal{L}^1$**- Cauchy** sequences $(\varphi_n)_{n\in\mathbb{N}}$ and $(\psi_n)_{n\in\mathbb{N}}$ of **step function**s converging μ**-a.e**. to the same function $f : X \to Y$ we have $\lim\limits_{n\to\infty} \int \varphi_n d\mu = \lim\limits_{n\to\infty} \int \psi_n d\mu < \infty$ as well as $\lim\limits_{n\to\infty} \|\varphi_n - \psi_n\|_1 = 0$.

Proof: The existence of the limits is a consequence of the **completeness** of Y since $\left|\int (\varphi_n - \varphi_m)\, d\mu\right| \leq \|\varphi_n - \varphi_m\|_1$ such that $\left(\int \varphi_n d\mu\right)_{n\in\mathbb{N}}$ and likewise $\left(\int \psi_n d\mu\right)_{n\in\mathbb{N}}$ are again Cauchy sequences in Y. The differences $\gamma_n = \varphi_n - \psi_n$ also are $\mathcal{L}^1$-Cauchy and converge μ-a.e. to 0 such that for every $\epsilon > 0$ there is an $N \in \mathbb{N}$with $\|\gamma_m - \gamma_n\|_1 < \epsilon$ for all $m, n \geq N$. According to definition 5.1 there is a set A with $\mu(A) < \infty$ and $X \setminus A \subset \{\gamma_N = 0\}$ such that $\int_{X\setminus A} |\gamma_n|\, d\mu = \int_{X\setminus A} |\gamma_n - \gamma_N|\, d\mu \leq \|\gamma_n - \gamma_N\|_1 < \epsilon$. By the preceding lemma 5.4 there exists a subset $Z \subset A$ with $\mu(Z) < \frac{\epsilon}{1+\|f_N\|_\infty}$ and a subsequence converging to 0 uniformly on $A \setminus Z$ such that there is an $M \geq N$ with $\int_{A\setminus Z} |\gamma_n|\, d\mu < \epsilon$ for all $n \geq M$. Finally for $n \geq N$ we have $\int_Z |\gamma_n|\, d\mu \leq \int_Z |\gamma_n - \gamma_N|\, d\mu + \int_Z |\gamma_N|\, d\mu \leq \|\gamma_n - \gamma_N\|_1 + \|f_N\|_\infty \cdot \mu(Z) < 2\epsilon$. In sum we arrive at $\int_{X\setminus A} |\gamma_n|\, d\mu + \int_{A\setminus Z} |\gamma_n|\, d\mu + \int_Z |\gamma_n|\, d\mu < 4\epsilon$ which proves the assertion.

5.6 μ-a.e. properties of integrable functions

1. Due to 5.1 integrable functions with approximating sequence $(\varphi_n)_{n\in\mathbb{N}}$ **vanish outside of the** σ**-finite set** $\bigcup_{n\in\mathbb{N}} \{\varphi_n \neq 0\}$.
2. According to 5.4 the integrable functions are μ**-a.e. finite** and **bounded** outside of a set of **finite measure**: Since the φ_n converge uniformly outside of a set Z_ϵ with $\mu(Z_\epsilon) < \epsilon$ for any $c \geq 0$ there is an $n \in \mathbb{N}$ such that $\{|f| \geq c\} \setminus Z_e \subset \{|\varphi_n| \geq \frac{c}{2}\}$ and hence $\mu(|f| \geq c) < \mu\left(|\varphi_n| \geq \frac{c}{2}\right) < \infty$.
3. In the case of **positive integrands** we have $\int f d\mu = 0 \Rightarrow f = 0$ μ-a.e. since for $A_n = \{f > 0\}$ the estimate $\frac{1}{n}\mu(A_n) \leq \int_{A_n} f d\mu \leq \int f d\mu = 0$ yields $\mu(f > 0) = \mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = 0$ on account of the **continuity form above** 2.2.3. In particular for positive integrable $f, g \in \mathcal{L}^1(X; \mathbb{R})$ with $f \leq g$ we have $\int f d\mu = \int g d\mu \Rightarrow f = g$ μ-a.e.

5.7 Special cases

For every integrable f with approximating sequence $(\varphi_n)_{n\in\mathbb{N}}$ the **restriction** $f|_A$ on any measurable subset is again integrable with the approximating sequence $(\varphi_n|_A)_{n\in\mathbb{N}}$. Hence we can define the **integral on measurable subsets** $\int_A f d\mu := \int f|_A d\mu$ with **additivity** extending to domains by $\int_{A\mathring{\cup}B} f d\mu = \int f|_{A\mathring{\cup}B} d\mu = \int (f|_A + f|_B)\, d\mu = \int_A f d\mu + \int_B f d\mu$. Likewise the **components** of functions in **finite dimensional Banach spaces** can be integrated separately since for every **continuous** $g : Y \to Z$ into another Banach space Z we have an approximating sequence $(g \circ \varphi_n)_{n\in\mathbb{N}}$ for $g \circ f$ with $\lim\limits_{n\to\infty} (g \circ \varphi_n) = g \circ \lim\limits_{n\to\infty} \varphi_n$ and in the case of **continuous and linear** g we even have $\int g \circ f d\mu = g \circ \int f d\mu$. For $Y = Y_1 \times Y_2$ and the continuous as well as linear **projections** $g = \pi_1 : Y \to Y_1$ resp. $g = \pi_2 : Y \to Y_2$ we obtain $\int (f_1, f_2)\, d\mu = \left(\int f_1 d\mu, \int f_2 d\mu\right)$. In particular f is integrable iff each of its **components** is integrable or in the case of $Y = \mathbb{C}$ iff $\mathrm{Re}f$ and $\mathrm{Im}f$ are integrable with $\int (\mathrm{Re}f + i\mathrm{Im}f)\, d\mu = \int \mathrm{Re}f d\mu + i \int \mathrm{Im}f d\mu$. For $Z = \mathbb{R}$ and the **continuous but not linear Banach norm** $g = |\,|$ we see that for every $f \in \mathcal{B}(X; Y)$ its Banach norm $|f| \in \mathcal{L}^1(X; \mathbb{R})$ is also integrable with approximating sequence $(|\varphi_n|)_{n\in\mathbb{N}}$. Note that in particular $(|\varphi_n|)_{n\in\mathbb{N}}$ is $\mathcal{L}_1$**-Cauchy** since $\||\varphi_n| - |\varphi_m|\|_1 \leq \|\varphi_n - \varphi_m\|_1$. The **converse statement** $|f| \in \mathcal{L}^1(X; \mathbb{R}) \Rightarrow f \in \mathcal{L}^1(X; Y)$ is only true for σ**-finite** $(X; \mathcal{A}; \mu)$ and **separable** $(Y, |\,|)$. (cf. 5.16). The **well ordering** of the real numbers provides the space $\mathcal{L}^1(X; \mathbb{R})$ with additional properties: For $f, g \in \mathcal{L}^1(X; \mathbb{R})$ we have $\sup\{f; g\} = \frac{1}{2}(f + g + |f - g|) \in \mathcal{L}^1(X; \mathbb{R})$ and $\inf\{f; g\} = \frac{1}{2}(f + g - |f - g|) \in \mathcal{L}^1(X; \mathbb{R})$. $f = f^+ - f^- \in \mathcal{L}^1(X; \mathbb{R})$ iff its **positive part** $f^+ = \sup\{f; 0\} \in \mathcal{L}^1(X; \mathbb{R})$ and its **negative part** $f^- = \inf\{f; 0\} \in \mathcal{L}^1(X; \mathbb{R})$. Also for real valued functions the integral is **monotone**, i.e. $f \leq g \Rightarrow \int f d\mu \leq \int g d\mu$ which for positive integrands $f \geq 0$ extends to the domain in the form $A \subset B \Rightarrow \int_A f d\mu \leq \int_B f d\mu$.

5.8 The integral transformation formula

For every **Borel measurable** $T : X \to Y$ from a **measure space** $(X; \mathcal{A}; \mu)$ into a into a **separable Banach space** $(Y, ||_Y)$ and every **Borel measurable** $f : Y \to Z$ into a further **separable Banach space** $(Z, ||_Z)$ the **composition** $f \circ T : X \to Z$ is μ-integrable iff f is $(T \circ \mu)$-integrable and in that case we have $\int f d(T \circ \mu) = \int (f \circ T)\, d\mu$.

Proof: For an approximating sequence $(\varphi_n)_{n\in\mathbb{N}} \in \mathcal{S}(Y; Z)$ of f with $\varphi_n = \sum\limits_{i=1}^{k_n} z_{n;i}\chi_{B_{n;i}}$we have

$$(T \circ \mu)\text{-a.e. } \lim_{n\to\infty} \varphi_n = f$$

$$\Leftrightarrow \mu\left(T^{-1}\left(\lim_{n\to\infty} \varphi_n \neq f\right)\right) = 0$$

$$\Leftrightarrow \mu\left(T^{-1}\left(\lim_{n\to\infty}\left|\sum_{i=1}^{k_n} z_{n;i}\chi_{B_{n;i}} - f\right| > \epsilon\right)\right) = 0\,\forall\epsilon > 0$$

$$\overset{4.10.1.c)}{\Leftrightarrow} \mu\left(T^{-1}\left(\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty}\bigcup_{i=1}^{k_n}\{y \in B_{n;i} : |z_{n;i} - f(y)| \geq \epsilon\}\right)\right) = 0\,\forall\epsilon > 0$$

$$\Leftrightarrow \mu\left(\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty}\bigcup_{i=1}^{k_n}\{x \in T^{-1}[B_{n;i}] : |z_{n;i} - f(T(x))| \geq \epsilon\}\right) = 0\,\forall\epsilon > 0$$

$$\Leftrightarrow \mu\left(\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty}\left|\sum_{i=1}^{k_n} z_{n;i}\chi_{T^{-1}[B_{n;i}]} - f \circ T\right| > \epsilon\right) = 0\,\forall\epsilon > 0$$

$$\Leftrightarrow \mu\left(\lim_{n\to\infty} \varphi_n \circ T \neq f \circ T\right) = 0$$

$$\Leftrightarrow \mu\text{-a.e. } \lim_{n\to\infty} \varphi_n \circ T = f \circ T$$

and also

$$0 = \lim_{n\to\infty}\sup_{m\geq n}\int |\varphi_n - \varphi_m|\, d(T \circ \mu)$$

$$= \lim_{n\to\infty}\sup_{m\geq n}\int\left|\sum_{i=1}^{k_n} z_{n;i}\chi_{B_{n;i}} - \sum_{j=1}^{k_m} z_{m;j}\chi_{B_{m;j}}\right| d(T \circ \mu)$$

$$= \lim_{n\to\infty}\sup_{m\geq n}\sum_{i=1}^{k_n}\sum_{j=1}^{k_m} |z_{n;i} - z_{m;j}|\,\mu\left(T^{-1}[B_{n;i} \cap B_{m;j}]\right)$$

$$= \lim_{n\to\infty}\sup_{m\geq n}\sum_{i=1}^{k_n}\sum_{j=1}^{k_m} |z_{n;i} - z_{m;j}|\,\mu\left(T^{-1}[B_{n;i}] \cap T^{-1}[B_{m;j}]\right)$$

$$= \lim_{n\to\infty}\sup_{m\geq n}\int\left|\sum_{i=1}^{k_n} z_{n;i}\chi_{T^{-1}[B_{n;i}]} - \sum_{j=1}^{k_m} z_{m;j}\chi_{T^{-1}[B_{m;j}]}\right| d\mu$$

$$= \lim_{n\to\infty}\sup_{m\geq n}\int |\varphi_n \circ T - \varphi_m \circ T|\, d\mu$$

whence $(\varphi_n \circ T)$ is an approximating sequence of $f \circ T$. Hence

$$\begin{aligned}\int f d\left(T\circ\mu\right) &= \lim_{n\to\infty}\int\varphi_n d\left(T\circ\mu\right)\\ &= \lim_{n\to\infty}\int\sum_{i=1}^{k_n} z_{n;i}\chi_{B_{n;i}} d\left(T\circ\mu\right)\\ &= \lim_{n\to\infty}\sum_{i=1}^{k_n} z_{n;i}\mu\left(T^{-1}\left[B_{n;i}\right]\right)\\ &= \lim_{n\to\infty}\int\sum_{i=1}^{k_n} z_{n;i}\chi_{T^{-1}\left[B_{n;i}\right]} d\mu\\ &= \lim_{n\to\infty}\int\left(\varphi_n\circ T\right)d\mu\\ &= \int\left(f\circ T\right)d\mu.\end{aligned}$$

5.9 The seminorm for Lebesgue integrable functions

According to 5.7 for every $f\in\mathcal{L}^1\left(X;Y\right)$ the integral $\|f\|_1 := \int|f|\,d\mu = \lim\limits_{n\to\infty}\|\varphi_n\|_1$ is well defined and a **pseudonorm** on $\mathcal{L}^1\left(X;Y\right)$: For $f,g\in\mathcal{L}^1\left(X;Y\right)$ with approximating sequences $\left(\varphi_n\right)_{n\in\mathbb{N}},\left(\psi_n\right)_{n\in\mathbb{N}}\in\mathcal{S}\left(X;Y\right)$ we have $|f+g|\in\mathcal{L}^1\left(X;Y\right)$ with approximating sequence $\left(|\varphi_n+\psi_n|\right)_{n\in\mathbb{N}}$ and by **continuity of the addition** we obtain $\|f+g\|_1 = \lim\limits_{n\to\infty}\|\varphi_n+\psi_n\|_1 \leq \lim\limits_{n\to\infty}\left(\|\varphi_n\|_1+\|\psi_n\|_1\right) = \lim\limits_{n\to\infty}\|\varphi_n\|_1 + \lim\limits_{n\to\infty}\|\psi_n\|_1 = \|f\|_1+\|g\|_1$, i.e. the **triangle inequality.** Likewise the **continuity of the absolute value** extends the **continuity of the integral** from $\mathcal{S}\left(X;Y\right)$ to $\mathcal{L}^1\left(X;Y\right)$: $\left|\int f d\mu\right| = \left|\lim\limits_{n\to\infty}\int\varphi_n d\mu\right| = \lim\limits_{n\to\infty}\left|\int\varphi_n d\mu\right| \leq \lim\limits_{n\to\infty}\int|\varphi_n|\,d\mu = \int|f|\,d\mu = \|f\|_1$.

5.10 Completeness of $\mathcal{L}^1$

The space $\left(\mathcal{L}^1\left(X;Y\right);\|\|_1\right)$ of Lebesgue integrable functions is **complete**.

Proof: For an $\mathcal{L}^1$-Cauchy sequence $\left(f_n\right)_{n\in\mathbb{N}}\subset\mathcal{L}^1\left(X;Y\right)$ there is a $\varphi_n\in\mathcal{S}\left(X;Y\right)$ with $\|f_n-\varphi_n\|_1 < \frac{1}{n}$. Hence for every $\epsilon>0$ there is an $N\in\mathbb{N}$ such that for every $n,m\geq N$ we have $\|f_n-f_m\|_1<\frac{\epsilon}{3}$ and consequently $\|\varphi_n-\varphi_m\|_1 \leq \|\varphi_n-f_n\|_1+\|f_n-f_m\|_1+\|f_m-\varphi_m\|_1 \leq \frac{1}{n}+\frac{\epsilon}{3}+\frac{1}{m}\leq\epsilon$ for $n,m\geq\max\left\{N;\frac{3}{\epsilon}\right\}$, i.e. $\left(\varphi_n\right)_{n\in\mathbb{N}}$ is $\mathcal{L}^1$-Cauchy. Due to the 5.4 a subsequence $\left(\varphi_{n_k}\right)_{k\in\mathbb{N}}$ converges μ-a.e. to an $f\in\mathcal{L}^1\left(X;Y\right)$, whence 5.5 yields $\int f d\mu = \lim\limits_{k\to\infty}\int\varphi_{n_k}d\mu$ and furthermore $\|f\|_1 = \lim\limits_{l\to\infty}\|\varphi_{n_l}\|_1$ and particularly $\|f-\varphi_{n_k}\|_1 = \lim\limits_{l\to\infty}\|\varphi_{n_l}-\varphi_{n_k}\|_1$ for every $k\in\mathbb{N}$ with 5.7. Since $\left(\varphi_k\right)_{k\in\mathbb{N}}$ is $\mathcal{L}^1$-Cauchy for every $\epsilon>0$ there is an $k\in\mathbb{N}$ with $n_k\geq\frac{3}{\epsilon}$ such that on the one hand $\|\varphi_{n_l}-\varphi_{n_k}\|_1<\frac{\epsilon}{3}$ and on the other hand $\left|\|f-\varphi_{n_k}\|_1-\|\varphi_{n_l}-\varphi_{n_k}\|_1\right|<\frac{\epsilon}{3}$ for every $l\geq k$ whence $\|f-f_{n_k}\|_1\leq\|f-\varphi_{n_k}\|_1+\|\varphi_{n_k}-f_{n_k}\|_1\leq\|\varphi_{n_l}-\varphi_{n_k}\|_1+\frac{\epsilon}{3}+\frac{1}{n_k}\leq\epsilon$. Hence $\left(\varphi_{n_k}\right)_{k\in\mathbb{N}}$ is $\mathcal{L}^1$-convergent to f and due to its $\mathcal{L}^1$-Cauchy property the convergence extends to the complete sequence $\left(\varphi_n\right)_{n\in\mathbb{N}}$.

5.11 Convergence in mean and μ-a.e

For any $\mathcal{L}^1$-Cauchy sequence $\left(f_n\right)_{n\in\mathbb{N}}\subset\mathcal{L}^1\left(X;Y\right)$ of **Lebesgue integrable functions** $f_n:X\to Y$ there exists a subsequence $\left(f_{n_k}\right)_{k\in\mathbb{N}}$ and for every $\epsilon>0$ a set $Z_\epsilon\subset X$ with measure $\mu\left(Z_\epsilon\right)<\epsilon$ such that $\left(f_{n_k}\right)_{k\in\mathbb{N}}$ converges **absolutely** and **uniformly** on $X\setminus Z_\epsilon$ as well as **μ-a.e.** and **in mean** on X to an integrable $f\in\mathcal{L}^1\left(X;Y\right)$.

Proof: According to the preceding theorem $\left(f_n\right)_{n\in\mathbb{N}}$ converges **in mean** to an $f\in\mathcal{L}^1\left(X;Y\right)$ such that for every $k\geq1$ there is an $n_k\geq n_{k-1}\in\mathbb{N}$ with $\|f-f_{n_k}\|_1\leq\frac{1}{2^{2k}}$. Then for $Y_k=\left\{|f-f_{n_k}|\geq\frac{1}{2^k}\right\}$ we have $\frac{1}{2^k}\mu\left(Y_k\right)=\int_{Y_n}\frac{1}{2^k}d\mu\leq\int_X|f-f_{n_k}|\,d\mu\leq\frac{1}{2^{2k}}$ whence $\mu\left(Y_k\right)\leq\frac{1}{2^k}$. Hence $\mu\left(Z_m\right)\leq\frac{1}{2^{m-1}}$ for

$Z_m = \bigcup\limits_{k=m}^{\infty} Y_k$ and $|f(x) - f_{n_k}(x)| < \frac{1}{2^k}$ for every $x \in X \setminus Z_m$ resp. $k \geq m$, i.e. $(f_{n_k})_{k \geq m}$ converges to f **absolutely** and **uniformly** on $X \setminus Z_m$ as well as **pointwise** on $X \setminus \bigcap\limits_{m=1}^{\infty} Z_m$ with $\mu\left(\bigcap\limits_{m=1}^{\infty} Z_m\right) = 0$.

5.12 The norm for Lebesgue integrable functions

Lebesgue integrable functions with common approximatig sequences are μ-a.e. equal and partition $\mathcal{L}^1(X;Y)$ into **equivalence classes** (c.f. 5.1). The corresponding **quotient space** is equally called a **Lebesgue space** and denoted as $L^1(X;Y)$. On this quotient space $\|f\|_1$ is **positive definite** and hence a **norm**, since for $\|f\|_1 = 0$ the null sequence $(0)_{n \in \mathbb{N}}$converges in mean to f and due to the preceding paragraph it also converges μ-a.e. to f whence μ-a.e. $f = 0$. Note that $\left(L^1(X;Y); \|\|_1\right)$ is a **Banach space**, but there is no topology on $\mathcal{L}^1(X;Y)$ corresponding to **μ-a.e. convergence**. (cf. [6])

5.13 Levi's monotone convergence theorem

For every **monotone** sequence $(f_n)_{n \in \mathbb{N}} \in L^1(X;\mathbb{R})$ of **real valued** $f_n : X \to \mathbb{R}$ we have $\int \lim\limits_{n \in \mathbb{N}} f_n d\mu = \lim\limits_{n \in \mathbb{N}} \int f_n d\mu$. In the case of $\lim\limits_{n \in \mathbb{N}} \left|\int f_n d\mu\right| < \infty$ the sequence converges **both in mean and μ-a.e.** to $f = \lim\limits_{n \in \mathbb{N}} f_n \in L^1(X;\mathbb{R})$.

Proof: Due to the **monotonicity of the integral** 5.7 in the case of an **increasing sequence** we have $\sup\limits_{n \in \mathbb{N}} \int f_n d\mu \leq \int \sup\limits_{n \in \mathbb{N}} f_n d\mu$ which proves the assertion in the case of $\sup\limits_{n \in \mathbb{N}} \int f_n d\mu = \infty$. For $\sup\limits_{n \in \mathbb{N}} \int f_n d\mu < \infty$ and $n \geq m$ we have $\|f_n - f_m\|_1 = \int (f_n - f_m) d\mu = \int f_n d\mu - \int f_m d\mu$ whence follows that $(f_n)_{n \in \mathbb{N}}$ is L^1-Cauchy. According to 5.11 a subsequence converges μ-a.e. and in mean to an $f = \lim\limits_{n \in \mathbb{N}} f_n \in L^1(X;\mathbb{R})$ and due to the increasing character this must be true for the complete sequence. Finally for every $\epsilon > 0$ there is an $n \in \mathbb{N}$ with $\int (f - f_n) d\mu < \epsilon$ and hence $\int f d\mu = \int (f - f_n) d\mu + \int f_n d\mu = \epsilon + \int f_n d\mu$ which proves $\int \sup\limits_{n \in \mathbb{N}} f_n d\mu = \sup\limits_{n \in \mathbb{N}} \int f_n d\mu$. In the case of a decreasing sequence apply the proof to $(-f_n)_{n \in \mathbb{N}}$.

5.14 Fatou's lemma

For every sequence $(f_n)_{n \in \mathbb{N}} \in L^1\left(X;\mathbb{R}_0^+\right)$ of **positive Borel measurable** functions with $\lim\limits_{k \to \infty} \inf\limits_{k \leq n} \int f_n d\mu < \infty$ we have $f = \lim\limits_{k \to \infty} \inf\limits_{k \leq n} f_n \in L^1\left(X;\mathbb{R}_0^+\right)$ with $\int \lim\limits_{k \to \infty} \inf\limits_{k \leq n} f_n d\mu \leq \lim\limits_{k \to \infty} \inf\limits_{k \leq n} \int f_n d\mu$.

Proof: For every $k \in \mathbb{N}$ the **decreasing** sequence $\left(\inf\limits_{k \leq n \leq m} f_n\right)_{m \in \mathbb{N}}$ converges μ-a.e. to $\inf\limits_{k \leq n} f_n$ such that due to the preceding theorem we have $\int \inf\limits_{k \leq n} f_n d\mu = \lim\limits_{m \to \infty} \int \inf\limits_{k \leq n \leq m} f_n d\mu \leq \lim\limits_{m \to \infty} \inf\limits_{k \leq n \leq m} \int f_n d\mu = \inf\limits_{k \leq n} \int f_n d\mu \leq \lim\limits_{k \to \infty} \inf\limits_{k \leq n} \int f_n d\mu$. Now we apply the monotone convergence theorem a second time to the **increasing** sequence $\left(\inf\limits_{k \leq n} f_n\right)_{k \in \mathbb{N}}$ and obtain $\int \lim\limits_{k \to \infty} \inf\limits_{k \leq n} f_n d\mu = \lim\limits_{k \to \infty} \int \inf\limits_{k \leq n} f_n d\mu \leq \lim\limits_{k \to \infty} \inf\limits_{k \leq n} \int f_n d\mu$.

5.15 Lebesgue's dominated convergence theorem

A sequence $(f_n)_{n \in \mathbb{N}} \subset L^1(X;Y)$ **converging μ-a. e.** to some f **converges in mean** to f with $f \in L^1(X;Y)$ iff there is an **Lebesgue integrable majorant** $g \in L^1(X;\mathbb{R}_0^+)$ such that for every $n \in \mathbb{N}$ and μ-a.e. we have $|f_n| \leq g$.

Proof: For every $K \in \mathbb{N}$ the **increasing** sequence $\left(\sup\limits_{k\leq m;n\leq l} |f_n - f_m|\right)_{l>k}$ is μ-a.e. bounded by $|f_n - f_m| \leq 2g$ and hence has bounded integrals $\int \left(\sup\limits_{k\leq m;n\leq l} |f_n - f_m|\right) d\mu \leq 2\int g d\mu$. According to the **monotone convergence theorem** 5.13 we conclude $\int \left(\sup\limits_{k\leq m;n} |f_n - f_m|\right) d\mu \leq 2\int g d\mu$ for every $k \in \mathbb{N}$. Hence we can apply the monotone convergence theorem a second time to the **decreasing** sequence $\left(\sup\limits_{k\leq m;n} |f_n - f_m|\right)_{k\geq 1}$ converging μ-a.e. to 0 to obtain $\lim\limits_{k\to\infty} \int \left(\sup\limits_{k\leq m;n} |f_n - f_m|\right) d\mu = 0$. Hence $(f_n)_{n\in\mathbb{N}}$ is L^1-Cauchy and due to the **completeness** 5.10 of $L^1(X;Y)$ it converges **in mean** to an $f^{\#} \in L^1(X;Y)$ coninciding μ-a.e. with f according to 5.11.

5.16 The absolute value of integrable functions

A Borel measurable function $f : X \to Y$ from a **σ-finite measure space** $(X;\mathcal{A};\mu)$ into a **separable Banach space** $(Y,||)$ is **integrable** with $\int f d\mu \leq \int |f|\, d\mu \leq \int g d\mu$ if there is a $g \in L^1(X;\mathbb{R})$ with μ-a.e. $|f| \leq g$. In particular f is integrable if its **absolute value** $|f|$ is integrable. The inequality is a trivial consequence of the **continuity of the integral** according to 5.9. The converse is true for the subset of the **Lebesgue-integrable** functions but neither for the **Bochner integral** nor for the **improper Riemann integral** which is not included in the Lebesgue integral (cf. 5.26). E.g. $f(x) = \frac{\sin(x)}{x}$ is integrable with **Bochner** and **Riemann** but not with **Lebesgue**.

Proof: According to 5.2 there is a sequence $(\varphi_n)_{n\in\mathbb{N}} \subset S(X;Y)$ of step functions converging μ-a.e. to f. Due to 5.5 the function g is Borel measurable. Hence the sets $\{|\varphi_n| \leq 2g\}$ are measurable and by $\psi_n(x) = \begin{cases} \varphi_n(x) & \text{if } |\varphi_n(x)| \leq 2g(x) \\ 0 & \text{if } |\varphi_n(x)| > 2g(x) \end{cases}$ we have a sequence of **integrable** step functions **bounded** by g and **converging μ-a.e.** to f . Due to 5.15 the convergence is also in mean and f is integrable.

5.17 Dominated convergence for series

For a sequence $(f_n)_{n\in\mathbb{N}}$ of functions $f_n : X \to Y$ from a **σ-finite measure space** $(X;\mathcal{A};\mu)$ into a **separable Banach space** $(Y,||)$ with $\sum\limits_{n\in\mathbb{N}} \int |f_n|\, d\mu < \infty$ the series $\sum\limits_{n\in\mathbb{N}} f_n := f$ converges **μ-a.e.** as well as **in mean:** $\sum\limits_{n\in\mathbb{N}} \int f_n d\mu = \int f d\mu$.

Proof: Since $|\int f d\mu| \overset{5.8}{\leq} \int \left|\sum\limits_{n\in\mathbb{N}} f_n\right| d\mu \overset{5.15}{\leq} \int \sum\limits_{n\in\mathbb{N}} |f_n|\, d\mu \overset{5.12}{=} \sum\limits_{n\in\mathbb{N}} \int |f_n|\, d\mu < \infty$ the limit $f \in L^1(X;Y)$ is **Lebesgue integrable** and also μ-a.e. **finite** resp**. convergent** due to **monotone convergence** 5.13. The convergence **in mean** follows by 5.15 with the **majorant** $g := \sum\limits_{n\in\mathbb{N}} |f_n|$.

5.18 Sequences with bounded norms

For the **μ-a. e.** limit $f : X \to Y$ of a sequence $(f_n)_{n\in\mathbb{N}} \subset L^1(X;Y)$ of **Lebesgue integrable** functions from a **σ-finite measure space** $(X;\mathcal{A};\mu)$ into a **separable Banach space** $(Y,||)$ with **bounded norms** $\|f_n\|_1 \leq C$ for some $C \geq 0$ and every $n \in \mathbb{N}$ we have $\|f\|_1 \leq C$ and in particular $f \in L^1(X;Y)$.

Proof: The f_n are measurable due to 5.5 and so is f according to 4.9. Because of $\lim\limits_{n\to\infty} |f_n| = |f|$ we can apply first **Fatou's lemma** 5.14 to obtain $\|f\|_1 \leq C$ and then 5.16 to infer $f \in L^1(X;Y)$.

Note: Due to the **missing bound** for the absolute values $|f_n|$ **we can not assert convergence in mean**. E.g. for the sequence $(\varphi_n)_{n\geq 1}$ with $\varphi_n = n \cdot \chi_{[0;\frac{1}{n}]}$ we have $\lim\limits_{n\to\infty} |\varphi_n| = 0$ but $\lim\limits_{n\to\infty} \|\varphi_n\|_1 = 1$.

5.19 Products of Lebesgue integrable and bounded functions

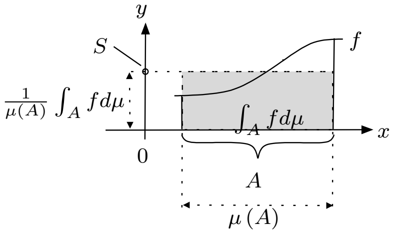
For **σ-finite measure space** $(X;\mathcal{A};\mu)$ and a **separable Banach space** $(Y,||)$ the product fg of an **(Lebesgue) integrable** $f : X \to Y$ and a **bounded measurable** $g : X \to K$ into the **normed, complete and separable field** K is **(Lebesgue) integrable**.

Proof: Due to 5.2, 5.5 and 5.10 there are sequences $(\varphi_n)_{n\in\mathbb{N}} \subset \mathcal{S}(X;Y)$ and $(\psi_n)_{n\in\mathbb{N}} \subset \mathcal{S}(X;K)$ of step functions with $(\varphi_n)_{n\in\mathbb{N}}$ converging both in **mean** and μ-a.e. to f and $(\psi_n)_{n\in\mathbb{N}}$ converging μ-a.e. to g. Then $(\varphi_n\cdot\psi_n)_{n\in\mathbb{N}} \subset \mathcal{S}(X;Y)$ is a L^1-**Cauchy** sequence converging μ-a.e. and according to 5.4 also **in** mean to fg which is hence integrable with $|fg| \le |f|\cdot\|g\|_\infty < \infty$. In the case of $f \in L^1(X;Y)$ we have $|f| \in L^1(X;Y)$ whence $|f|\cdot\|g\|_\infty \in L^1(X;Y)$ and hence $|fg| \in L^1(X;Y)$ due to 5.16.

5.20 The mean value theorem for integration

For every **integrable** $f \in \mathcal{B}(X;Y)$ from a **σ-finite measure space** $(X;\mathcal{A};\mu)$ into a **separable Banach space** $(Y,||)$ with the mean value $\frac{1}{\mu(A)}\int_A fd\mu \in S$ for some **closed** subset $S \subset Y$ and every $A \in \mathcal{A}$ with $0 < \mu(A) < \infty$ we have $\mu(f \notin S) = 0$.

Proof: In the case of $\mu(X) < \infty$ for any closed disk $\overline{B}_r(z) \subset Y \setminus S$ with $\mu(A) > 0$ for $A = f^{-1}\left[\overline{B}_r(z)\right]$ we have $\left|\frac{1}{\mu(A)}\int_A fd\mu - z\right| = \left|\frac{1}{\mu(A)}\int_A (f-z)\,d\mu\right| \le \frac{1}{\mu(A)}\int_A |f-z|\,d\mu \le r$ contrary to $\frac{1}{\mu(A)}\int_A fd\mu \in S$. Therefore we must assume $\mu\left(f \in \overline{B}_r(z)\right) = 0$ and since $Y \setminus S$ is a countable union of such disks the assertion follows from the σ-additivity of μ. Hence if we assume the hypothesis for every $A \cap X_n$ with $A \in \mathcal{A}$, $\mu(X_n) < \infty$ and $X = \bigcup_{n\in\mathbb{N}} X_n$ we obtain $f(x) \in S$ for every $x \in X_n \setminus Z_n$ with $\mu(Z_n) = 0$ and hence for $X \setminus \bigcup_{n\in\mathbb{N}} Z_n$ with $\mu\left(\bigcup_{n\in\mathbb{N}} Z_n\right) = 0$.



The following theorem asserts that step functions on arbitrary measurable sets can be approximated by step functions on an algebra of sets with **finite measures**, e.g. the algebra $\mathcal{F}$ of **figures** in $\mathbb{R}^n$. This step is necessary to identify the **Lebesgue integral** as special case of the **Bochner integral**. The theorem will be prepared by two lemmata:

5.21 L^1-limits of sets of finite measure

For every **algebra** $\mathcal{F} \subset \mathcal{A}$ of sets of **finite measure** in a measure space $(X;\mathcal{A};\mu)$ and every $F \in \mathcal{F}$ we consider the **vector space** $\mathcal{S}(\mathcal{F}_F;\mathbb{R})$ of step functions on the **trace algebra** $\mathcal{F}_F$ of the form $\sum_{i=0}^{m} y_i\chi_{F_i}$ with $m \in \mathbb{N}$ on sets $F_0 = X \setminus F$ resp. $F_i \in \mathcal{F}_F$ with $\bigcup_{i=1}^{m} F_i = F$ and with values $y_0 = 0$ resp. $y_i \in \mathbb{R}$ for $1 \le i \le m$. Then for every $F \in \mathcal{F}$ the family $\mathcal{N}_F = \left\{A \in \mathcal{A}_F : \chi_A \in \overline{\left(\mathcal{S}(\mathcal{F}_F;\mathbb{R});\|\|_1\right)}\right\} \subset \mathcal{A}_F$ is a σ-algebra on the set F.

Proof: Note that every $A \in \mathcal{N}_F$ must be of finite measure but not necessarily be an element of the algebra $\mathcal{F}_F$. Since for $\varphi,\psi \in \mathcal{S}(\mathcal{F}_F;\mathbb{R})$ we obviously have $\sup\{\varphi;\psi\}, \inf\{\varphi;\psi\} \in \mathcal{S}(\mathcal{F}_F;\mathbb{R})$ the closure $\overline{\left(\mathcal{S}(\mathcal{F}_F;\mathbb{R});\|\|_1\right)}$ is again a **vector space closed** with respect to **sup** and **inf**. $\mathcal{N}_F$ is an **algebra** since obviously $\emptyset \in \mathcal{N}_F$ and for every $A, B \in \mathcal{N}_F$ the characteristic functions $\chi_{A\cup B} = \sup\{\chi_A;\chi_B\}$, $\chi_{A\cap B} = \inf\{\chi_A;\chi_B\}$ as well as $\chi_{A\setminus B} = \chi_A - \chi_B$ are all in $\overline{\left(\mathcal{S}(\mathcal{F}_F;\mathbb{R});\|\|_1\right)}$and consequently their supports $A\cup B$, $A\cap B$ resp. $A\setminus B$ are in $\mathcal{N}_F$. It is a **σ-algebra** since $F \in \mathcal{N}_F$ and for every paiwise disjoint sequence $(A_n)_{n\in\mathbb{N}} \subset \mathcal{N}_F$ with union $A = \mathring{\bigcup}_{n\in\mathbb{N}} A_n$ and every $\epsilon > 0$ due to the **continuity from below** 2.2.2 we have an $N \in \mathbb{N}$ with $\mu\left(\bigcup_{k>N} A_k\right) < \epsilon$ and approximating step functions $(\varphi_n)_{n\in\mathbb{N}} \subset \mathcal{S}(\mathcal{F}_F;\mathbb{R})$ such that $\|\chi_{A_n} - \varphi_n\|_1 < \frac{\epsilon}{2^n}$ for $n \in \mathbb{N}$ whence

$$\begin{aligned}\left\|\chi_A - \sum_{k=0}^{n}\varphi_n\right\|_1 &\le \left\|\chi_A - \chi_{\bigcup_{0\le k\le n} A_k}\right\|_1 + \left\|\chi_{\bigcup_{0\le k\le n} A_k} - \sum_{k=0}^{n}\varphi_n\right\|_1\\ &= \left\|\chi_{\bigcup_{k>N} A_k}\right\|_1 + \left\|\sum_{k=0}^{n}\chi_{A_k} - \sum_{k=0}^{n}\varphi_n\right\|_1\\ &\le \mu\left(\bigcup_{k>N} A_k\right) + \sum_{k=0}^{n}\left\|\chi_{A_n} - \varphi_n\right\|_1\\ &< 2\epsilon.\end{aligned}$$

5.22 Coverings of L^1-limits of sets of finite measure

For an **algebra** $\mathcal{F}\subset\mathcal{A}$ of sets with **finite measure** in a measure space $(X;\mathcal{A};\mu)$ and $F_{n\in\mathbb{N}}\in\mathcal{F}$ with $X=\bigcup_{n\in\mathbb{N}}F_n$ with σ-algebrae $\mathcal{N}_n\subset\mathcal{A}_{F_n}$ according to 5.21 the family $\mathcal{N}=\{A\subset X : A\cap F_n\in\mathcal{N}_n\forall n\in\mathbb{N}\}$ is a σ-algebra on X.

Proof: For every $A\in\mathcal{N}$ we have $X\setminus A\cap F_n\in\mathcal{N}_n$ whence $X\setminus A\in\mathcal{N}$. For every $A,B\in\mathcal{N}$ we have $(A\cap B)\cap F_n=(A\cap F_n)\cap(B\cap F_n)\in\mathcal{N}_n$whence $A\cap B\in\mathcal{N}$. Finally for $(A_m)_{m\in\mathbb{N}}\subset\mathcal{N}$ the equality $\left(\bigcup_{m\in\mathbb{N}}A_m\right)\cap F_n=\mathring{\bigcup}_{m\in\mathbb{N}}(A_m\cap F_n)$ shows that $\bigcup_{m\in\mathbb{N}}A_m\in\mathcal{N}$.

5.23 L^1-limits of step functions

For every **algebra** $\mathcal{F}\subset\mathcal{A}$ of sets with **finite measure generating** $\mathcal{A}=\sigma(\mathcal{F})$ on a **σ-finite measure space** $(X;A;\mu)$ we have $\overline{(\mathcal{S}(\mathcal{F};Y);\|\|_1)}=\mathcal{B}(X;Y)$.

Proof:

According to the hypothesis there is a sequence $(F_n)_{n\geq1}\subset\mathcal{F}$ of w.l.o.g. pairwise disjoint sets with finite measure $\mu(F_n)<\infty$ and $\mathring{\bigcup}_{n\geq1}F_n=X$. By lemma 5.21 $\mathcal{N}_{F_n}\subset\mathcal{A}_F$ is a σ-algebra and by lemma 5.22 the family $\mathcal{N}$ is a σ-algebra containing $\mathcal{F}$ and hence $\mathcal{A}=\sigma(\mathcal{F})$ such that for every measurable set $A\in\mathcal{A}$ with finite measure $\mu(A)<\infty$ we have $A\cap F_n\in\mathcal{N}_{F_n}$, i.e. for every $\epsilon>0$ there is a $\varphi_n\in\mathcal{S}(\mathcal{F}_{F_n};\mathbb{R})$ with $\|\chi_{A\cap F_n}-\varphi_n\|_1<\frac{\epsilon}{2^n}$. Due to the **continuity from above** 2.2.3 there is an $N\in\mathbb{N}$ such that $\left\|\chi_A-\sum\limits_{n=1}^{N}\chi_{A\cap F_n}\right\|_1=\mu\left(A-\bigcup\limits_{n=1}^{N}(A\cap F_n)\right)<\epsilon$ whence $\left\|\chi_A-\sum\limits_{n=1}^{N}\varphi_n\right\|_1\leq\left\|\chi_A-\sum\limits_{n=1}^{N}\chi_{A\cap F_n}\right\|_1+\left\|\sum\limits_{n=1}^{N}\chi_{A\cap F_n}-\sum\limits_{n=1}^{N}\varphi_n\right\|_1<2\epsilon$. Thus for every step map $\psi=\sum\limits_{i=0}^{m}y_i\chi_{A_i}\in\mathcal{S}(\mathcal{A};Y)$ with $m\in\mathbb{N}$ such that $\bigcup\limits_{i=0}^{m}A_i=X$ with values $y_i\in Y$ and $\mu(A_i)<\infty$ for $1\leq i\leq m$ and $\alpha_0=0$ there are step maps $\varphi_i=\sum\limits_{n=1}^{N}\varphi_{i,n}\in\mathcal{S}(\mathcal{F};\mathbb{R})$ with $\|\chi_{A_i}-\varphi_i\|_1<\frac{\epsilon}{m\cdot|y_i|}$ such that $\left\|\sum\limits_{i=0}^{m}y_i\chi_{A_i}-\sum\limits_{i=0}^{m}y_i\varphi_i\right\|_1=\|\psi-\varphi\|_1<\epsilon$ with $\varphi=\sum\limits_{i=0}^{m}y_i\varphi_i\in\mathcal{S}(\mathcal{F};Y)$. The assertion now follows from the definition 5.5 of integrable functions since $\overline{(\mathcal{S}(\mathcal{A};Y);\|\|_1)}=\mathcal{B}(X;Y)$.

5.24 Uniqueness of integrable functions

For every **algebra** $\mathcal{F}\subset\mathcal{A}$ of sets with **finite measure generating** $\mathcal{A}=\sigma(\mathcal{F})$ on a **σ-finite measure space** $(X;A;\mu)$ and every **integrable** $f\in\mathcal{B}(X;Y)$ into a **separable Banach space** $(Y,||)$ the following propositions hold:

1. If $\int_F fd\mu=0$ for every $F\in\mathcal{F}$ then $f=0$ μ-a.e.
2. If $\int f\varphi d\mu=0$ for every $\varphi\in\mathcal{S}(\mathcal{F};\mathbb{R})$ then $f=0$ μ-a.e.
3. If $\int_F fd\mu\leq c\cdot\mu(F)$ for some $c\geq0$ and every $F\in\mathcal{F}$ with $\mu(F)>0$ then $|f|\leq c$ μ-a.e.

Proof: According to 5.5 and 5.23 f or every measurable set $A \in \mathcal{A}$ with finite measure $\mu(A) < \infty$ there exists a sequence $(\varphi_n)_{n\in\mathbb{N}} \subset \mathcal{S}(\mathcal{F}; Y)$ converging in mean as well as μ-a.e. to χ_A. Taking $\sup\{\varphi_n; 0\}$ resp. $\inf\{\varphi_n; 1\}$ we can w.l.o.g. assume $0 \leq \varphi_n \leq 1$. Then we have $|\varphi_n f| \leq |f|$ for every $n \in \mathbb{N}$ and $(\varphi_n f)_{n\in\mathbb{N}}$ converges μ-a.e. to $\chi_A f$. By **dominated convergence** 5.15 and since $\int \varphi_n f d\mu = 0$ we conclude $\int \chi_A f d\mu = 0$. Now every measurable set is a countable union of w.l.o.g. pairwise disjoint sets of finite measure such that a second instance of **dominated convergence** yields $\int \chi_A f d\mu = 0$ for every **measurable** $A \in \mathcal{A}$. **Proposition 1.** now follows from the **mean value** theorem for integrals 5.20 applied to $S = \{0\}$. **Proposition 2.** is obtained from 1. by taking $\varphi = \chi_F$. Finally we derive **Proposition 3.** from 5.20 applied to $S_n = \overline{B}_{c+1/n}(0)$ for $n \geq 1$ and considering $\{|f| \leq c\} = \bigcap_{n\in\mathbb{N}} \{f \in S_n\}$.

5.25 Characterization of integrable functions

A function $f : X \to Y$ from a **σ-finite measure space** $(X; A; \mu)$ into a **separable Banach space** $(Y, ||)$ is **integrable** iff there is an **increasing** sequence $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ with $\bigcup_{n\in\mathbb{N}} A_n = X$ and $\lim\limits_{n\to\infty} \int_{A_n} f d\mu \in Y$ exists. In that case we have $\int f d\mu = \lim\limits_{n\to\infty} \int_{A_n} f d\mu$.

Proof: $\Rightarrow$: Take $A_n = X$ for $n \in \mathbb{N}$. $\Leftarrow$: Due to the hypothesis for every $m \geq 1$ there is an $n(m) \in \mathbb{N}$ such that $\left|S - \int_{A_{n(m)}} f d\mu\right| < \frac{1}{2m}$ with $S = \lim\limits_{n\to\infty} \int_{A_n} f d\mu$. Also due to 5.4 there is a $\varphi_{n(m)} \in S(X; Y)$ with $\int \left|f \cdot \chi_{A_{n(m)}} - \varphi_{n(m)}\right| d\mu < \frac{1}{2m}$ and $\left|f(x) - \varphi_{n(m)}(x)\right| < \frac{1}{m}$ for every $x \in A_{n(m)} \setminus Z_{n(m)}$ with $\mu\left(Z_{n(m)}\right) < \frac{1}{m}$. Hence we have $\left|S - \int \varphi_{n(m)} d\mu\right| \leq \left|S - \int_{A_{n(m)}} f d\mu\right| + \left|\int_{A_{n(m)}} f d\mu - \int \varphi_{n(m)} d\mu\right| \leq \frac{1}{2m} + \int \left|f \cdot \chi_{A_{n(m)}} - \varphi_{n(m)}\right| d\mu < \frac{1}{m}$. Furthermore $\lim\limits_{m\to\infty} \left(\varphi_{n(m)}(x)\right) = f(x)$ for every $x \in \bigcup_{m\geq 1} \left(A_{n(m)} \setminus Z_{n(m)}\right) = \bigcup_{m\geq 1} A_{n(m)} \setminus \bigcap_{m\geq 1} Z_{n(m)} = X \setminus \bigcap_{m\geq 1} Z_{n(m)}$ with $\mu\left(\bigcap_{m\geq 1} Z_{n(m)}\right) = 0$. Hence $\left(\varphi_{n(m)}\right)_{m\geq 1} \subset \mathcal{S}(X; Y)$ is an approximating sequence for f and we have $S = \lim\limits_{n\to\infty} \int_{A_n} f d\mu = \lim\limits_{m\to\infty} \int \varphi_{n(m)} d\mu = \int f d\mu$.

5.26 Comparison with the Riemann integral

1. Every **Riemann integrable** function $f : [a; b] \to \mathbb{R}$ is **integrable** and the two integrals are equal: $\int_a^b f(x)dx = \int_{[a;b]} f d\lambda$.
2. $f : \mathbb{R} \to \mathbb{R}$ is **integrable** on $\mathbb{R}$ iff the **improper Riemann integral** exists and in this case the two integrals again coincide: $\lim\limits_{n\in\mathbb{N}} \int_{-n}^{n} f(x)dx = \int_{\mathbb{R}} f d\lambda$.

Proofs:

1. For every partition $z_n := (a = a_0 \leq a_1 \leq ... \leq a_n = b)$ of the interval $[a; b]$ we can compare the **lower Darboux sum** $L_{z_n} := \sum\limits_{i=0}^{n} \overline{\gamma}_i (a_i - a_{i-1}) \leq \int_{[a;b]} l_{z_n} d\lambda$ with $\overline{\gamma}_i := \inf f[[a_{i-1}; a_i]]$ resp. the **upper Darboux sum** $U_{z_n} := \sum\limits_{i=0}^{n} \overline{\Gamma}_i (a_i - a_{i-1}) \geq \int_{[a;b]} u_{z_n} d\lambda$ with $\overline{\Gamma}_i := \sup f[[a_{i-1}; a_i]]$ to the **integrals** of the **corresponding step functions** $l_{z_n} := \sum\limits_{i=0}^{n} \gamma_i \chi_{[a_{i-1}; a_i[}$ with $\gamma_i := \inf f[[a_{i-1}; a_i[] \geq \overline{\gamma}_i$ resp. $u_{z_n} := \sum\limits_{i=0}^{n} \Gamma_i \chi_{[a_{i-1}; a_i[}$ with $\Gamma_i := \sup f[[a_{i-1}; a_i[] \leq \overline{\Gamma}_i$. According to the hypothesis there are sequences $(z_n)_{n\in\mathbb{N}}$ of partitions such that z_{n+1} is a refinement of z_n such that due to the **monotonicity of the integral** 5.7 we obtain $\int_a^b f(x)dx = \lim\limits_{n\to\infty} L_{z_n} \leq \lim\limits_{n\to\infty} \int_{[a;b]} l_{z_n} d\lambda \leq \lim\limits_{n\to\infty} \int_{[a;b]} u_{z_n} d\lambda \leq \lim\limits_{n\to\infty} U_{z_n} = \int_a^b f(x)dx$ whence $\int_a^b f(x)dx = \lim\limits_{n\to\infty} \int_{[a;b]} l_{z_n} d\lambda = \lim\limits_{n\to\infty} \int_{[a;b]} u_{z_n} d\lambda$. Since $(u_{z_n})_{n\in\mathbb{N}}$ decreases, $(l_{z_n})_{n\in\mathbb{N}}$ increases, $(u_{z_n} - l_{z_n})_{n\in\mathbb{N}}$ is a decreasing sequence bounded below by 0 such that due to the **completeness** of the **real numbers** there must be a limit $\lim\limits_{n\in\mathbb{N}} (u_{z_n} - l_{z_n}) \geq 0$. According to 4.9 this limit function is **measurable** and from 5.14 follows

$0 \leq \int \lim\limits_{n \in \mathbb{N}} (u_{z_n} - l_{z_n}) \leq \liminf\limits_{n \in \mathbb{N}} (U_{z_n} - L_{z_n}) = 0$ whence λ-a.e. $\lim\limits_{n \in \mathbb{N}} (u_{z_n} - l_{z_n}) = 0$ due to 5.6.3. Since λ-a.e. $l_{z_n} \leq f \leq u_{z_n}$ we infer that λ-a.e. $\lim\limits_{n \in \mathbb{N}} l_{z_n} = f$. By **dominated convergence** 5.15 with majorant u_{z_0} we obtain $\int_a^b f(x)dx = \lim\limits_{n \in \mathbb{N}} \int_{[a;b]} l_{z_n} d\lambda = \int_{[a;b]} \left(\lim\limits_{n \in \mathbb{N}} l_{z_n} \right) d\lambda = \int_{[a;b]} f d\lambda$.

2. Follows directly from the preceding theorem 5.25.

Note: In essential, 5.13, 5.14 and 5.15 assert the **continuity of the Bochner and Lebesgue integral**s regarding **pointwise** esp. **μ-a.e. convergence** whereas the **Riemann integral** is only continuous with reference to **uniform convergence** (cf.[8, Th 7.16]).

The classical definition of the **Lebesgue** integral is restricted to **positive** functions such that the Lebesgue integral of real functions requires **separate computing of positive and negative parts** entailing the failure of this method in the case of certain integrands with **alternating** signs like e.g. $\int \frac{\sin(x)}{x} dx = \lim\limits_{n \to \infty} \int_{-n}^{n} \frac{\sin(x)}{x} dx = \pi$. (cf. [9, p. 3.7.1]). Theorem 5.25 does not work with the Lebesgue integral.

6 Lebesgue spaces

6.1 Convex functions

A real function $f : \mathbb{R} \to \mathbb{R}$ is **convex** on the open interval $]a;b[$ iff $f(s) \leq f(r) + (s-r) \cdot \frac{f(t)-f(r)}{t-r} = f(t) - (t-s) \cdot \frac{f(t)-f(r)}{t-r}$ resp. $\frac{f(t)-f(s)}{t-s} \geq \frac{f(t)-f(r)}{t-r} \geq \frac{f(s)-f(r)}{s-r}$ for every $a < r < s < t < b$. Every convex function is **continuous** and in particular **Borel-measurable** since for $s \in]a;b[$ and w.l.o.g. $\min\{1; b-s\} > \epsilon > 0$ we have $|f(r) - f(s)| < |r-s| \cdot \frac{|f(s+\epsilon)-f(s)|}{\epsilon} < \epsilon$ for every $|r-s| < \delta := \frac{\epsilon^2}{\max\{1; |f(s+\epsilon)-f(s)|\}}$.

6.2 Jensen's inequality

For every integrable $g : A \to]a;b[\subset \mathbb{R}$ with $A \subset X$ and $\mu(A) < \infty$ on a measure space $(X; \mathcal{A}, \mu)$ and every convex $f :]a;b[\to \mathbb{R}$ we have $f\left(\frac{1}{\mu(A)} \int_A g d\mu\right) \leq \frac{1}{\mu(A)} \int_A (f \circ g) \, d\mu$.

Proof: For $s := \frac{1}{\mu(A)} \int_A g d\mu$ we have $a < s < b$ and due to 6.1 also $\beta := \sup\limits_{a<r<s} \frac{f(s)-f(r)}{s-r} \leq \frac{f(t)-f(s)}{t-s}$ for all $s < t < b$, hence $f(s) + \beta(t-s) \leq f(t)$ resp. $f(s) + \beta(g(x) - s) \leq f(g(x))$. All summands of this inequality are integrable over A such that on account of the monotonicity of the integral we can infer $\mu(A) \cdot f(s) \leq \int_A (f \circ g) \, d\mu$ and hence the assertion.

6.3 Applications

Choosing $A = \{p_1; ...; p_n\} \subset [0; \infty[$ and $\mu(\{p_i\}) = \alpha_i$ with $\mu(A) = \sum\limits_{i=1}^{n} \alpha_i = 1$ as well as $g(p_i) = \ln(x_i)$ and $f(x) = \exp(x)$ Jensen's inequality yields the following very useful special cases:

1. $x_1^{\alpha_1} \cdot ... \cdot x_n^{\alpha_n} \leq \alpha_1 x_1 + ... + \alpha_n x_n$
2. $(x_1 \cdot ... \cdot x_n)^{\frac{1}{n}} \leq \frac{1}{n}(x_1 + ... + x_n)$ (**geometric** and **arithmetic mean** for $\alpha_i := \frac{1}{n}$)
3. $F \cdot G \leq \frac{1}{p} F^p + \frac{1}{q} G^q$ for $\frac{1}{p} + \frac{1}{q} = 1$ with **equality iff** $F^p = G^q$ for $\alpha_1 = \frac{1}{p}; \alpha_2 = \frac{1}{q}; x_1 = F^p; x_2 = G^q$.

6.4 Hölder and Minkowski inequalities

For any positive Borel measurable $f, g : X \to Y$ from a measure space $(X; \mathcal{A}, \mu)$ into a Banach space $(Y; |\,|)$ and $\frac{1}{p} + \frac{1}{q} = 1$ resp. $p + q = p \cdot q$ with $\|f\|_p := \left(\int |f|^p d\mu\right)^{\frac{1}{p}}$ we have

1. $\|fg\| \leq \|f\|_p \cdot \|g\|_q$ (**Hölder** resp. **Schwarz** for $p = q = 2$) with equality iff μ-a.e. $\frac{f(x)}{\|f\|_p} = \frac{g(x)}{\|g\|_q}$.
2. $\|f+g\| \leq \|f\|_p + \|g\|_p$ (**Minkowski**) with equality iff μ-a.e. $\frac{f(x)}{\|f\|_p} = \frac{g(x)}{\|g\|_p} = \frac{f(x)+g(x)}{\|f+g\|_p}$.

Proof: The integrand is measurable on account of 4.6. For one of the integrals disappearing 5.12 tells us that the integrands $f \cdot g, f+g, f$ and g will disappear μ-a.e. too such that we have equality in this case. Therefore we can assume all integrals > 0 in the following proof.

1. With $F := \frac{|f|}{\|f\|_p}$ resp. $G := \frac{|g|}{\|g\|_q}$ in 6.3.3 an integration yields $\int (F \cdot G)\, d\mu \leq \frac{1}{p} + \frac{1}{q} = 1$ and hence the assertion. In particular $f \cdot g$ is integrable if f^p and g^q are integrable.
2. Applying 1. twice to $(f+g)^p = f \cdot (f+g)^{p-1} + g \cdot (f+g)^{p-1}$ and observing $q(p-1) = p$ we obtain $\|f+g\|_p^p \leq \|f\|_p \cdot \left\|(f+g)^{p-1}\right\|_q + \|g\|_p \cdot \left\|(f+g)^{p-1}\right\|_q = \left(\|f\|_p + \|g\|_p\right) \cdot \|f+g\|_p^{\frac{q}{p}}$. Substituting $p - \frac{p}{q} = 1$ yields the assertion. The convexity of t^p provides the inequality $\left(\frac{f+g}{2}\right)^p \leq \frac{f^p+g^p}{2}$, i.e. the integrability of f^p and g^p entails the integrability of $(f+g)^p$.

6.5 L^p-spaces

For $1 \leq p < \infty$ and any $f : X \to Y$ from a measure space $(X; \mathcal{A}; \mu)$ into a Banach space $(Y; |\,|)$ the expressions $\|f\|_p := (\int |f|^p d\mu)^{\frac{1}{p}}$ resp. $\|f\|_\infty := \inf \{0 < \alpha < \infty : \mu(|f| > \alpha) = 0\}$ define a **seminorm** (cf. [13, p. 21.1]) on the **vector space** $\mathcal{L}^p(\mu) := \left\{f : X \to Y : \|f\|_p :< \infty\right\}$. The **absolute homogeneity** follows from the **linearity** 5.5 whereas the **triangle inequation** is provided by the **Hölder inequality** 6.4.2. $\mathcal{L}^1(\mu)$ contains the **Lebesgue integrable** functions and $\mathcal{L}^\infty(\mu)$ is the set of all μ-a.e. **bounded** and measurable functions furnished with the **supremum norm** $\|\|_\infty$. Analogously to 5.1 resp. 5.12 the contraction to the **quotient space** $L^p(\mu) := \mathcal{L}^p / \sim$ defined by the **equivalence relation** $f \sim g \Leftrightarrow \mu(f \neq g) = 0$ makes $\|\|_p$ a **norm**. Convergence with respect to $\|\|_p$ is called in the p**-th mean**. On account of 5.6 all $f \in \mathcal{L}^p$ are μ-a.e. finite for $1 \leq p \leq \infty$.

6.6 Relations between L^p-spaces

For $1 \leq p, q \leq \infty$ we have

1. For μ **bounded above** , i.e. $\mu(A) < \alpha \forall A \in \mathcal{A}$ we have $p < q \Rightarrow L^p \supset L^q$.
2. For μ **bounded below**, i.e. $\mu(A) > \alpha \forall A \in \mathcal{A}$ we have $p < q \Rightarrow L^p \subset L^q$.

Note : The Lebesgue measure $\mu = \lambda^n$ satisfies none of the above requested conditions such that $L^p(\lambda^n)$ cannot be linearly ordered by inclusion. E.g. owing to 5.26.2 on the one hand for $g_n(x) := \min\{1; |x|^{-n}\}$ we have $g_n \in L^p \Leftrightarrow n > \frac{1}{p}$ but in the other hand fro $h_n(x) := \max\{1; |x|^{-n}\}$ the relation $g_n \in L^p \Leftrightarrow n < \frac{1}{p}$ holds.

Proof:

1. With $p = \frac{r}{s} \geq 1$, $f = h^s$ and $g = 1$ Hölder 6.4.1 yields $\int |h|^s d\mu \leq (\int |h|^r d\mu)^{\frac{s}{r}} \cdot (\int 1 d\mu)^{\frac{s-s}{r}}$ resp. $\|h\|_s = (\int |h|^s d\mu)^{\frac{1}{s}} \leq (\int |h|^r d\mu)^{\frac{1}{r}} \cdot (\mu(X))^{\frac{1}{s}-\frac{1}{r}} = \|h\|_r \cdot (\mu(X))^{\frac{1}{s}-\frac{1}{r}}$ and hence the assertion.
2. On account of **Zorn's lemma** ([13, p. 14.2.4]) the set $\{|f| \geq 1\}$ possesses a **maximal cover** of measurable sets referring to **inclusion** resp. refinement and since $\mathcal{A}$ is closed under intersection this mus be a **partition**. Due to $\int |f|^p d\mu < \infty$ we have $\mu(f \geq 1) < \infty$ and since μ is **bounded below** this maximal partition consists of $n := \frac{\mu(f \geq 1)}{\alpha} + 1$ sets $(A_i)_{1 \leq i \leq n}$ with $\mu(A_i) > \alpha$. Owing to 5.3 for every $\epsilon > 0$ there is an elementary function $e = \sum_{i=1}^{n} \alpha_i \chi_{A_i} \leq f$ with $\int_{\{|f| \geq 1\}} e d\mu = \sum_{i=1}^{n} \alpha_i \mu(A_i) \geq \int_{\{|f| \geq 1\}} |f|^p d\mu - \epsilon \cdot \alpha$. Hence on the one hand for every $x \in A_i$ with $1 \leq i \leq n$ we have $|f|^p(x) \geq \alpha_i \Leftrightarrow |f|^q(x) \geq \alpha_i^{\frac{q}{p}}$ and on the other hand for every

$1 \leq i \leq n$ there is an $x_i \in A_i$ with $\alpha_i \geq |f^p(x_i)| - \epsilon \Leftrightarrow \alpha_i^{\frac{q}{p}} \geq (|f^p(x_i)| - \epsilon)^{\frac{q}{p}} \geq |f^q(xxi)| - \epsilon \cdot \frac{q}{p} \cdot (|f^p(x_i)| - \epsilon)^{\frac{q}{p}-1} \geq |f^q(x_i)| - \epsilon \cdot \frac{q}{p} \cdot |f^{q-p}(x_i)|$ since the tangent $t(x+\epsilon) = x^{\frac{q}{p}} + \epsilon \cdot \frac{q}{p} \cdot x^{\frac{q}{p}-1}$ on the convex function $g(x) = x^{\frac{q}{p}}$ always runs below the curve, i.e. $g(x+\epsilon) = (x+\epsilon)^{\frac{q}{p}}$. Thus follows $\int_{\{|f|\geq 1\}} |f|^q d\mu < \sum\limits_{i=1}^{n} \left(\alpha_i^{\frac{q}{p}} + \epsilon \cdot \frac{q}{p} \cdot |f^{q-p}(x_i)\right) \chi_{A_i} < \infty$ and also on the whole set $\int |f|^q d\mu = \int_{\{|f|<1\}} |f|^q d\mu + \int_{\{|f|\geq 1\}} |f|^q d\mu \leq \int_{\{|f|<1\}} |f|^p d\mu + \int_{\{|f|\geq 1\}} |f|^q d\mu < \infty$.

6.7 Completeness

Every L^p-**Cauchy sequence**$(f_n)_{n\in\mathbb{N}} \subset L^p(\mu)$ with $1 \leq p \leq \infty$ converges in the p-**th mean** to a $f \in L^p(\mu)$. Hence $L^p(\mu)$ is a **Banach space**.

Proof: For a Cauchy sequence $(f_n)_{n\in\mathbb{N}} \subset L^p(\mu)$ with $p < \infty$ exists a partial sequence $(f_{n_i})_{i\in\mathbb{N}}$ with $\left\|f_{n_{i+1}} - f_{n_i}\right\|_p < \frac{1}{2^{i+1}}$ which entails $\left\|\sum\limits_{i=0}^{k} \left|f_{n_{i+1}} - f_{n_i}\right|\right\|_p \leq 1$ due to 6.4.2, hence $\left\|\sum\limits_{i=0}^{\infty} \left|f_{n_{i+1}} - f_{n_i}\right|\right\|_p \leq 1$ owing to 5.13 and finally μ-**a.e.** $g := \sum\limits_{i=0}^{\infty} \left|f_{n_{i+1}} - f_{n_i}\right| < \infty$ according to 5.6.2. Since Y is **complete** the sequence $(f_{n_i})_{i\in\mathbb{N}} = \sum\limits_{k=1}^{i} (f_{n_k} - f_{n_{k+1}})$ μ-**a.e. converges** to an $f = \lim\limits_{i\to\infty} f_{n_i} = \sum\limits_{i=0}^{\infty} (f_{n_{i+1}} - f_{n_i})$ with $|f| < g$. On account of the **completeness** of μ (cf. 3.10) we can define $f(x) = 0$ on the remaining null set $\{|f| = \infty\}$. According to the hypothesis for every $\epsilon > 0$ there is a $j \in \mathbb{N}$ with $\left\|f_m - f_{n_j}\right\|_p < \epsilon$ for all $m \geq n_j$ whence **Fatou's lemma** 5.14 yields $\left(\liminf\limits_{m\geq n_j} \left|f_m - f_{n_j}\right|\right)^p = \liminf\limits_{m\geq n_j} \left|f_m - f_{n_j}\right|^p \in L_1(X;\mathbb{R})$ with $\int \left(\liminf\limits_{m\geq n_j} \left|f_m - f_{n_j}\right|^p\right) d\mu \leq \liminf\limits_{m\geq n_j} \int \left|f_m - f_{n_j}\right|^p d\mu < \epsilon^p$ Since μ-a.e. $f = \lim\limits_{i\to\infty} f_{n_i}$ we have μ-a.e $\liminf\limits_{m\geq n_j} \left|f_m - f_{n_j}\right| = \left|f - f_{n_j}\right|$ so that $\left\|f - f_{n_j}\right\|_p = \left\|\liminf\limits_{m\geq n_j} \left|f_m - f_{n_j}\right|\right\|_p < \epsilon$, i.e. the subsequence $(f_{n_i})_{i\in\mathbb{N}}$ and hence the entire Cauchy sequence $(f_n)_{n\in\mathbb{N}}$ (cf. [13, p. 14.1.2]) converges in the p-th mean to f. On account of $\|f\|_p \leq \|f - f_n\|_p + \|f_n\|_p < \infty$ we have $f \subset L^p(\mu)$.

For $p = \infty$ let $A := \bigcup_{m,n\in\mathbb{N}} (\{|f_m - f_n| > \|f_m - f_n\|_\infty\} \cup \{|f_m| > \|f_m\|_\infty\})$. Then we have $\mu(A) = 0$ and $(f_n)_{n\in\mathbb{N}}$ is a **Cauchy sequence** on $X \setminus A$ referring to the **supremum norm.** Due to the **completeness** of Y it converges uniformly and in particular with reference to $\|\|_\infty$ to a bounded function $|f| < \lim\limits_{n\to\infty} \|f_n\|_\infty$. Again we define $f(x) = 0$ for $x \in A$ and finally obtain $f \subset L^\infty(\mu)$.

6.8 Special cases

1. The sequence $(f_n)_{n\in\mathbb{N}} \subset L^p(\lambda)$ with $f_n := \chi_{A_n}$ for $A_n := \left[\frac{n}{2^k}; \frac{n+1}{2^k}\right]$ with $k(n) = \min\left\{k : n < 2^k\right\}$ shows that in general the μ-a.e. convergence cannot be extended to the entire sequence:: $\lim\limits_{n\to\infty} \|f_n\|_p = \lim\limits_{n\to\infty} \left(\lambda\left(\left[\frac{n}{2^k}; \frac{n+1}{2^k}\right]\right)\right)^{\frac{1}{p}} = \lim\limits_{n\to\infty} 2^{-\frac{k(n)}{p}} = \lim\limits_{n\to\infty} \frac{1}{n^{1/p}} = 0$ but for every $x \in \left[\frac{1}{2}; 1\right]$ and $k \geq 1$ there is an $n \in \mathbb{N}$ with $x \in \left[\frac{n}{2^k}; \frac{n+1}{2^k}\right]$ such that $(f_n)_{n\in\mathbb{N}}$ does not converge for any $x \in \left[\frac{1}{2}; 1\right]$ whereas the partial sequence $(f_{2^k})_{k\in\mathbb{N}}$ converges for every $x \neq \frac{1}{2}$..
2. $L^2(\mu)$ is a **Hilbert spac**e with the **inner product** $\langle f, g\rangle := \int f\overline{g} d\mu$ and the **norm** $\|f\| := \langle f, g\rangle^{\frac{1}{2}} := \left(\int f\overline{f} d\mu\right)^{\frac{1}{2}} = \left(\int |f|^2 d\mu\right)^{\frac{1}{2}}$.

6.9 Convergence in the p-th mean, in measure and μ-a.e

Every sequence $(f_n)_{n\in\mathbb{N}} \subset L^p(\mu)$ with $1 \leq p < \infty$ converging **in the p-th mean** to an $f \in L^p(\mu)$ converges in measure to f. Also there exists a **subsequence** $(f_{n_k})_{k\in\mathbb{N}}$ converging **μ-a.e.** to f and for every $\epsilon > 0$ there is a set $Z_\epsilon \subset X$ with measure $\mu(Z_\epsilon) < \epsilon$ such that $(f_{n_k})_{k\in\mathbb{N}}$ converges **absolutely** and **uniformly** on $X \setminus Z_\epsilon$.

Proof: The **convergence in measure** follows at once from $\epsilon{\cdot}\mu(|f-f_n| \geq \epsilon) = \epsilon^p{\cdot}\mu(|f-f_n|^p \geq \epsilon^p) \leq \int |f-f_n|^p \, d\mu$ According to the hypothesis for every $k \geq 1$ there is an $n_k \geq n_{k-1} \in \mathbb{N}$ such that $\|f-f_{n_k}\|_p \leq \frac{1}{2^{2k}}$. Then for $Y_k = \left\{|f-f_{n_k}|^p \geq \frac{1}{2^k}\right\}$ we have $\frac{1}{2^k}\mu(Y_k) = \int_{Y_n} \frac{1}{2^k} d\mu \leq \int_X |f-f_{n_k}|^p \, d\mu \leq \frac{1}{2^{2k}}$ whence $\mu(Y_k) \leq \frac{1}{2^k}$. Hence $\mu(Z_m) \leq \frac{1}{2^{m-1}}$ for $Z_m = \bigcup\limits_{k=m}^{\infty} Y_k$ and $|f(x) - f_{n_k}(x)|^p < \frac{1}{2^k}$ for every $x \in X \setminus Z_m$ resp. $k \geq m$ such that $(f_{n_k})_{k\geq m}$ converges to f **absolutely** and **uniformly** on on $X \setminus Z_m$ as well as **pointwise** on $X \setminus \bigcap\limits_{m=1}^{\infty} Z_m$ with $\mu\left(\bigcap\limits_{m=1}^{\infty} Z_m\right) = 0$.

6.10 Lebesgue's dominated convergence theorem for L^p-spaces

A sequence $(f_n)_{n\in\mathbb{N}} \subset L^p(X;Y)$ **converging μ-a. e.** to some f **converges in the p-th mean** to $f \in L^p(X;Y)$ iff there is an **integrable majorant** $g \in L^p(X;\mathbb{R}_0^+)$ such that for every $n \in \mathbb{N}$ and μ-a.e. we have $|f_n| \leq g$.

Proof: For every $K \in \mathbb{N}$ the **increasing** sequence $\left(\sup\limits_{k\leq m;n\leq l} |f_n - f_m|^p\right)_{l>k}$ is bounded by $|f_n - f_m|^p \leq 2^p g^p$ and hence has bounded integrals $\int \left(\sup\limits_{k\leq m;n\leq l} |f_n - f_m|^p\right) d\mu \leq 2^p \int g^p d\mu = 2^p \|g\|_p^p$. According to the **monotone convergence theorem** 5.13 we conclude $\int \left(\sup\limits_{k\leq m;n} |f_n - f_m|^p\right) d\mu \leq 2^p \int g^p d\mu$ for every $k \in \mathbb{N}$. Hence we can apply the monotone convergence theorem a second time to the **decreasing** sequence $\left(\sup\limits_{k\leq m;n} |f_n - f_m|^p\right)_{k\geq 1}$ converging μ-a.e. to 0 to obtain $\lim\limits_{k\to\infty} \int \left(\sup\limits_{k\leq m;n} |f_n - f_m|^p\right) d\mu = 0$. Hence $(f_n)_{n\in\mathbb{N}}$ is L^p-Cauchy and due to the **completeness** 6.7 of $L^p(X;Y)$ it converges **in the p-th mean** to an $f^{\#} \in L^p(X;Y)$ coninciding μ-a.e. with f according to 5.11.

Note: The proofs of the preceding two theorems is completely analogous to those of the corresponding statements 5.11 resp. 5.14 for L^1 with the small but essential difference that the generalized theorems 6.9 resp. 6.10 require the completeness 6.7 of L^p which like the dominated convergence for L^1 is based in the completeness5.10 of L^1. Alas the proof of this latter property depends on an **elementary approximation by step functions and cannot be duplicated for L^p.**

6.11 L^p- and uniform limits of step functions

1. For $1 \leq p < \infty$ we have $\overline{\left(\mathcal{S}(\mathcal{A};Y);\|\|_p\right)} = L^p(X;Y)$
2. For a **finite** measure space $(X;A;\mu)$ and a **finite dimensional Banach** space $(K^n;|\,|)$ we have $\overline{(\mathcal{S}(\mathcal{A};K^n);\|\|_\infty)} = L^\infty(X;K^n)$.

Proof:

1. According to 5.5 for every $f \in L^p(\mu)$ resp. $f^p \in L^1(\mu)$ and $p < \infty$ there is a sequence $(\varphi_n)_{n\in\mathbb{N}} \subset \mathcal{S}(\mathcal{A};Y)$ of **step functions** converging μ-a.e. to f. The truncated version $\psi_n(x) = \begin{cases} |\varphi_n(x)| & \text{for } |\varphi_n(x)| \leq 2|f(x)| \\ 0 & \text{else} \end{cases}$ still converges μ-a-e- to f and satisfies the hypothesis for 6.10 with the majorant $2|f| \in L^p(X;\mathbb{R}_0^+)$ which yields the convergence in tne p-th mean and hence the assertion.

2. For$|f| \leq N$; $M \geq 1$ and $\mathbf{k} = (k_1; ...; k_n) \in K_M = [-NM; NM[^n \subset \mathbb{Z}^n$ we define $\varphi_M = \sum_{\boldsymbol{k} \in K_M} \frac{\boldsymbol{k}}{M} \chi_{A_{\boldsymbol{k},M}} \in \mathcal{S}(\mathcal{A}; K^n)$ with $A_{\boldsymbol{k},M} = f^{-1}\left[\prod_{i=1}^{n} \left[\frac{k_i}{M}; \frac{k_i+1}{M}\right[\right] \in \mathcal{A}$ and $\mu(A_{\boldsymbol{k},M})$ such that $\|f - \varphi_M\|_\infty \leq \frac{\sqrt{n}}{M}$.

6.12 Continuity of the integral measure

For an **integrable** $f \in L^p(X; Y)$ and every $\epsilon > 0$ there is a $\delta > 0$ such that for every $E \in \mathcal{A}$ with $\mu(E) < \delta$ we have $\int_E |f| \, d\mu < \epsilon$.

Proof: The sequence $(\varphi_n)_{n\geq 1}$ with $\varphi_n(x) = \left\{ \begin{smallmatrix} |f(x)|, & \text{for } |f(x)| \leq n \\ n & \text{else} \end{smallmatrix} \right.$ satisfies the conditions for **monotone convergence** 5.13 such that $\lim\limits_{n\to\infty} \int \varphi_n d\mu = \int |f| \, d\mu$. Hence for $\epsilon > 0$ there is an $n_0 \geq 1$ such that $\int (|f| - \varphi_n) \, d\mu < \frac{\epsilon}{2}$. Since for $\delta = \frac{\epsilon}{2n}$ and every $E \in \mathcal{A}$ with $\mu(E) < \delta$ we have $\int_E \varphi_n d\mu \leq n \cdot \mu(E) = \frac{\epsilon}{2}$ it follows that $\int_E |f| \, d\mu \leq \int_E (|f| - \varphi_n) \, d\mu + \int_E \varphi_n d\mu \leq \epsilon$.

6.13 Vitali's convergence theorem

A sequence $(f_n)_{n\geq 1} \subset L^p(\mu)$ **converging** μ**-a.e.** for $1 \leq p < \infty$ to some f **also converges in the** p**-th mean** to $f \in L^p(\mu)$ **iff** for every $\epsilon > 0$

1. there is an $A_\epsilon \in \mathcal{A}$ with $\mu(A_\epsilon) < \infty$ and $\int_{X \setminus A_\epsilon} |f_n|^p \, d\mu < \epsilon$ for all $n \geq 1$.
2. there is a $\delta > 0$ such that for every $E \in \mathcal{A}$ with $\mu(E) < \delta$ we have $\int_E |f_n|^p \, d\mu < \epsilon$ for all $n \geq 1$.

Proof:

$\Rightarrow$: 1.: Due to the hypothesis for $\epsilon > 0$ there is an $n_0 \geq 1$ such that $\int |f_n - f|^p \, d\mu < \epsilon$ for all $n \geq n_0$. Owing to 5.13 with $|f|^p = \sup\limits_{m\geq 1} |f|^p \cdot \chi_{\{|f|^p > \frac{1}{m}\}}$ and $f \in L^p(\mu)$ there is an $m_0 \geq 1$ with $\int |f|^p \cdot \chi_{\{|f|^p \leq \frac{1}{m}\}} d\mu = \int |f|^p \, d\mu - \int |f|^p \cdot \chi_{\{|f|^p > \frac{1}{m}\}} d\mu < \epsilon$ and $\mu\left(|f|^p \leq \frac{1}{m}\right) \leq \mu\left(|f|^p \leq \frac{1}{m}\right) \leq \int |f|^p \, d\mu < \infty$ for all $m \geq m_0$. For those f_n with $1 \leq n \leq n_0$ we use the same reasoning as above to find an $m_1 \geq m_0$ such that the sets $B_\epsilon = \left\{|f|^p > \frac{1}{m_1}\right\} \in \mathcal{A}$ resp. $C_\epsilon = \left\{\max\limits_{1\leq n<n_0} |f_n|^p > \frac{1}{m_1}\right\} \in \mathcal{A}$ with $\mu(X \setminus B_\epsilon), \mu(X \setminus C_\epsilon) < \infty$ satisfy $\int_{X\setminus B_\epsilon} |f|^p \, d\mu < \epsilon$ resp. $\int_{X\setminus C_\epsilon} |f_n|^p \, d\mu < \epsilon$ for all $1 \leq n < n_0$. For $A_\epsilon = B_\epsilon \cup C_\epsilon$ **Minkowski's inequality** 6.4.2 yields $\left(\int_{X\setminus A_\epsilon} |f_n|^p \, d\mu\right)^{\frac{1}{p}} \leq \left(\int_{X\setminus A_\epsilon} |f_n - f|^p \, d\mu\right)^{\frac{1}{p}} + \left(\int_{X\setminus A_\epsilon} |f|^p \, d\mu\right)^{\frac{1}{p}} < \epsilon^{\frac{1}{p}} + \epsilon^{\frac{1}{p}}$ resp. $\int_{X\setminus A_\epsilon} |f_n|^p \, d\mu < 2^p \epsilon$ for all $n \geq 1$.

2.: For a given $\epsilon > 0$ choose $n_0 \geq 1$ as in 1. such that $\int |f_n - f|^p \, d\mu < \epsilon$ for all $n \geq n_0$. According to the preceding lemma 6.12 there is a $\delta > 0$ such that for all $E \in \mathcal{A}$ with $\mu(E) < \delta$ we have $\int_E |f|^p \, d\mu < \epsilon$ resp. $\int_E |f_n|^p \, d\mu < \epsilon$ for all $1 \leq n < n_0$. As in 1. **Minkowski's inequality** 6.4.2 yields the desired estimate $\int_E |f_n|^p \, d\mu < 2^p \epsilon$ for the remaining $n \geq n_0$.

$\Leftarrow$: According to 1. for $\epsilon > 0$ there is an $A_\epsilon \in \mathcal{A}$ with $\mu(A_\epsilon) < \infty$ such that $\int_{X\setminus A_\epsilon} |f_n|^p \, d\mu < \epsilon$ for all $n \geq 1$ so thath with **Fatou** 5.14 we obtain $\int_{X\setminus A_\epsilon} |f|^p \, d\mu \leq \liminf\limits_{n\geq 1} \int_{X\setminus A_\epsilon} |f_n|^p \, d\mu < \epsilon$. As **Minkowski** 6.4.2 gives $\left(\int_{X\setminus A_\epsilon} |f - f_n|^p \, d\mu\right)^{\frac{1}{p}} \leq \left(\int_{X\setminus A_\epsilon} |f_n|^p \, d\mu\right)^{\frac{1}{p}} + \left(\int_{X\setminus A_\epsilon} |f|^p \, d\mu\right)^{\frac{1}{p}} < 2\epsilon^{\frac{1}{p}}$. According to 2. resp. **Egorov** 4.15 for every $\delta > 0$ there is a $B_\delta \in \mathcal{A}$ as well as an $n_0 \geq 1$ with $\mu(B_\delta) < \delta$ such that $|f(x) - f_n(x)|^p < \epsilon$ for every $x \in A_\epsilon \setminus B_\delta$ and hence $\left(\int_{A_\epsilon \setminus B_\delta} |f - f_n|^p \, d\mu\right)^{\frac{1}{p}} < \epsilon^{\frac{1}{p}}$ for every $n \geq n_0$. On the set B_δ we follow the reasoning for $X \setminus A_\epsilon$ from above to find $\int_{B_\delta} |f|^p \, d\mu \leq \liminf\limits_{n\geq 1} \int_{B_\delta} |f_n|^p \, d\mu < \epsilon$ with **Fatou** and finally $\left(\int_{B_\delta} |f - f_n|^p \, d\mu\right)^{\frac{1}{p}} \leq \left(\int_{B_\delta} |f_n|^p \, d\mu\right)^{\frac{1}{p}} + \left(\int_{B_\delta} |f|^p \, d\mu\right)^{\frac{1}{p}} < 2\epsilon^{\frac{1}{p}}$. Combining our results over $X \setminus A_\epsilon$, $A_\epsilon \setminus B_\delta$ and B_δ we obtain $(\int_X |f - f_n|^p \, d\mu)^{\frac{1}{p}} < 5\epsilon^{\frac{1}{p}}$ for $n \geq n_0$.

7 Product spaces

7.1 The initial σ-algebra

The **initial σ-algebra** $\sigma\left(f_i : i \in I\right) := \sigma\left(\bigcup_{i\in I} f_i^{-1}\left(\mathcal{A}_i\right)\right)$ on a set X referring to the functions $f_i : X \to \left(Y_i; \mathcal{A}_i\right)$ with $i \in I$ is the smallest σ-algebra on X such that all f_i are **measurable**. This concept is closely related to that of the **initial topology**, cf. [**13**].

7.2 The trace of a measure space

The **trace σ-algebra** $\mathcal{A}_B = \sigma\left(i\right)$ on a subset $B \subset X$ of a measure space $\left(X; \mathcal{A}; \mu\right)$ ist the **initial σ-algebra** with reference to the **canonical injection** $i : B \to X$. On account of $i^{-1}\left[A\right] = A \cap B$ the measurable sets in B simply are the **intersections of the measurable sets in** A in X with B. The **trace of the measure** μ is its **restriction** $\mu|_B$.

7.3 The product-σ-algebra

The **product-σ-algebra** $\mathcal{A}_I = \bigotimes_{i\in I} \mathcal{A}_i = \sigma\left(\pi_i : i \in I\right)$ on the product $X_I = \prod_{i\in I} X_i$ of the measurable spaces $\left(X_i; \mathcal{A}_i\right)_{i\in I}$ is the **initial** σ-Algebra with reference to the **projections** $\pi_i : X_I \to X_i$. A mapping $f : Y \to X_I$ is measurable iff the inverse images $f^{-1}\left[\pi_i^{-1}\left[A_i\right]\right] = \left(\pi_i \circ f\right)^{-1}\left[A_i\right]$ of measurable sets in X_i are measurable in $\left(Y; \mathcal{A}\right)$. Hence f is measurable iff every **component** $\pi_i \circ f : \left(Y; \mathcal{A}\right) \to \left(X_i; \mathcal{A}_i\right)$ is measurable. Due to 4.4 the product σ-algebra induced by the families $\mathcal{E}_i \subset \mathcal{P}\left(X_i\right)$ with $i \in I$ is $\bigotimes_{i\in I} \sigma\left(\mathcal{E}_i\right) = \sigma\left(\bigcup_{i\in I} \pi_i^{-1}\left(\sigma\left(\mathcal{E}_i\right)\right)\right) = \sigma\left(\bigcup_{i\in I} \sigma\left(\pi_i^{-1}\left(\mathcal{E}_i\right)\right)\right) = \sigma\left(\bigcup_{i\in I} \pi_i^{-1}\left(\mathcal{E}_i\right)\right)$.

7.4 Measurable rectangles and cylinder sets

1. The family $\mathcal{S}_I = \left\{\bigcap_{j\in J} \pi_j^{-1}\left[A_j\right] = \prod_{j\in J} A_i \times \prod_{i\in I\setminus J} X_i : A_j \in \mathcal{A}_j,\ j \in J \subset I \wedge J\,\text{finite}\right\}$ of **measurable rectangles is closed under intersections** and a **basis** for the product-σ-algebra $\mathcal{A}_I = \sigma\left(\mathcal{S}_I\right)$.
2. For $J \subset K \subset I$ the **projections** $\pi_K^J : \left(X_J; \mathcal{A}_J\right) \to \left(X_K; \mathcal{A}_K\right)$ are measurable and for $J \cap K = \emptyset$ we have $\mathcal{A}_{J\cup K} = \mathcal{A}_J \otimes \mathcal{A}_K$.
3. The **algebra** $\mathcal{Z}_I = \left\{\pi_J^{-1}\left[A_J\right] = A_J \times \prod_{i\in I\setminus J} X_i : A_J \in \mathcal{A}_J, J \subset I \wedge J\,\text{finite}\right\}$ of **cylinder sets** also is a **π-basis** for the product-σ-algebra: $\mathcal{A}_I = \sigma\left(\mathcal{Z}_I\right)$. The cylinder sets $\mathcal{Z}_J = \sigma\left(\mathcal{S}_J\right)$ themselves are **σ-algebrae** with $\mathcal{Z}_J \subset \mathcal{Z}_K$ for $J \subset K$.
4. The family $\mathcal{A}_Z = \left\{\pi_J^{-1}\left[A_J\right] = A_J \times \prod_{i\in I\setminus J} X_i : A_J \in \mathcal{A}_J, J \subset I \wedge J\,\text{countable}\right\}$ of **countable cylinder sets** is a **σ-algebra** and **identical with** the **product-σ-algebra**: $\mathcal{A}_I = \mathcal{A}_Z$. Every measurable set A of a product-σ-algebra may depend from a **countable set of coordinates** in contrast to the **product topology** whose open sets are defined by **finitely many coordinates** (cf. [13, p. 4.2]).

Proof:

1. $\mathcal{S}_I$ is **closed under intersection** since for **finite** $J, K \subset I$ and $A_j \in \mathcal{A}_j$ with $j \in J$ resp. $B_k \in \mathcal{A}_k$ with $k \in K$ we have $\left(\bigcap_{j\in J} \pi_j^{-1}\left[A_j\right]\right) \cap \left(\bigcap_{k\in K} \pi_k^{-1}\left[B_k\right]\right) = \left(\bigcap_{j\in J\setminus K} \pi_j^{-1}\left[A_j\right]\right) \cap \left(\bigcap_{l\in J\cap K} \pi_l^{-1}\left[A_l \cap B_l\right]\right) \cap \left(\bigcap_{k\in K\setminus J} \pi_k^{-1}\left[B_k\right]\right) \in \mathcal{S}_I$ with $A_l \cap B_l \in \mathcal{A}_l$ for $l \in J \cap K$. Due to $\left\{\pi_i^{-1}\left[A_i\right] : A \in \mathcal{A}_i, i \in I\right\} \subset \mathcal{S}_I$ we have $\mathcal{A}_I = \sigma\left(\left\{\pi_i^{-1}\left[A_i\right] : A_i \in \mathcal{A}_i, i \in I\right\}\right) \subset \sigma\left(\mathcal{S}_I\right)$ and on account of $\mathcal{S}_I \subset \mathcal{A}_I$ the converse follows: $\sigma\left(\mathcal{S}_I\right) \subset \mathcal{A}_I$.

2. The **projections** are measurable since with $\bigcap_{k\in K}\left(\pi_k^K\right)^{-1}[A_k]\in\mathcal{S}_K$ for $A_k\in\mathcal{A}_k$ and $k\in K$ we have $\left(\pi_K^J\right)^{-1}\left(\bigcap_{k\in K}\left(\pi_k^K\right)^{-1}[A_k]\right)=\bigcap_{k\in K}\left(\pi_K^J\right)^{-1}\left(\left(\pi_k^K\right)^{-1}[A_k]\right)=\bigcap_{k\in K}\left(\pi_k^J\right)^{-1}[A_k]\in\mathcal{A}_J$ and hence with 1. follows the assertion. The measurability of $\pi_J^{J\cup K}$ resp. $\pi_K^{J\cup K}$ entails $\mathcal{A}_{J\cup K}\supset\mathcal{A}_J\otimes\mathcal{A}_K$ and from 1. resp. $\mathcal{S}_{J\cup K}\subset\mathcal{A}_J\otimes\mathcal{A}_K$ follows the converse $\mathcal{A}_{J\cup K}=\sigma\left(\mathcal{S}_{J\cup K}\right)\subset\mathcal{A}_J\otimes\mathcal{A}_K$.

3. $\mathcal{Z}_I$ is an **algebra** since obviously $\emptyset,X\in\mathcal{A}_Z$ and for $\pi_J^{-1}[A_J],\pi_K^{-1}[A_K]\in\mathcal{Z}_I$ with $A_J\in\mathcal{A}_J,B_K\in\mathcal{A}_K$ and **finite** $J,K\subset I$ owing to 2. we have $\left(\pi_J^{J\cup K}\right)^{-1}[A_J]$, $\left(\pi_K^{J\cup K}\right)^{-1}[B_K]\in\mathcal{A}_{J\cup K}$. Hence the **intersection** $\left(\pi_J^{-1}[A_J]\right)\cap\left(\pi_K^{-1}[B_K]\right)=\pi_{J\cup K}^{-1}\left(\left(\left(\pi_J^{J\cup K}\right)^{-1}[A_J]\right)\cap\left(\pi_K^{J\cup K}\right)^{-1}[B_K]\right)$ $\in\mathcal{Z}_I$ and likewise the **union** are contained in $\mathcal{Z}_I$. Concerning the **complements** we consult e.g. [**Vorwerg2022a**] to obtain $X_I\setminus\pi_J^{-1}[A_J]=\left(\pi_J^{-1}[X_J]\right)\setminus\left(\pi_J^{-1}[A_J]\right)=\pi_J^{-1}[X_J\setminus A_J]\in\mathcal{Z}_I$ since $X_J\setminus A_J\in\mathcal{A}_J$. On the one hand we have $\sigma\left(\mathcal{Z}_I\right)\subset\mathcal{A}_I$ since according to 2. we have $\mathcal{Z}_I\subset\mathcal{A}_I$. On the other hand 1. yields $\mathcal{A}_I=\sigma\left(\mathcal{S}_I\right)\subset\sigma\left(\mathcal{Z}_I\right)$ since $\mathcal{S}_I\subset\mathcal{Z}_I$. Again on account of 2. the families $\mathcal{Z}_J=\pi_J^{-1}\left(\mathcal{A}_J\right)$ are **σ-algebrae** whereas the linear order by inclusion on the family of cylinder sets follows from $\left(\pi_J^K\right)^{-1}\left(\mathcal{A}_J\right)\subset\mathcal{A}_K$ by application of π_K^{-1}. **Note**: The properties of a σ-algebra as well as the linear ordering by inclusion obviously extend to arbitrary index sets, notable countable ones, as shown below:

4. The family $\mathcal{A}_Z$ is again an **algebra** since the reasoning from 3. can be transferred to countable index sets. It is a **σ-algebra** since $\bigcup_{n\in\mathbb{N}}\pi_{J_n}^{-1}[A_{J_n}]=\pi_J^{-1}\left(\bigcup_{n\in\mathbb{N}}\left(\left(\pi_{J_n}^J\right)^{-1}[A_{J_n}]\right)\right)\in\mathcal{A}_Z$ with $\left(\pi_{J_n}^J\right)^{-1}[A_{J_n}]\in\mathcal{A}_J$ and countable $J=\bigcup_{n\in\mathbb{N}}J_n$. In particular we have $\mathcal{A}_Z\subset\sigma\left(\mathcal{Z}_I\right)=\mathcal{A}_I$. Conversely from $\mathcal{A}_Z\supset\mathcal{Z}_I$ and 3. follows the inclusion $\mathcal{A}_Z\supset\sigma\left(\mathcal{Z}_I\right)=\mathcal{A}_I$.

7.5 The product of Borel σ-algebrae and the Borel σ-algebra of a product

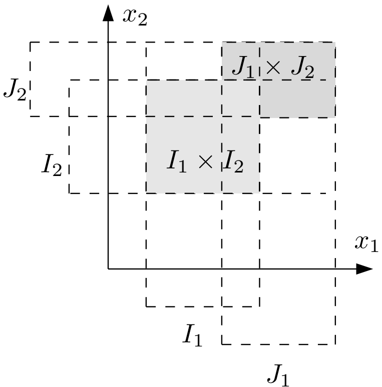
The **product** $\mathcal{B}_I:=\bigotimes_{i\in I}\sigma\left(\mathcal{O}_i\right)$ of the **Borel σ-algebrae** $\mathcal{B}_i$ of the topological spaces $(X_i;\mathcal{O}_i)_{i\in I}$ is the smallest σ-Algebra on $X=\prod_{i\in I}X_i$ or **initial σ-algebra** such that all **projections** $\pi_i:(X;\mathcal{B}_I)\to(X_i\mathcal{B}_i)$ are **measurable**. The π_i are **continuous** with reference to the **product topology** $\mathcal{O}=\bigotimes_{i\in I}\mathcal{O}_i$ (cf. [13, p. 4.2]) and hence due to 4.4 **measurable** with regard to the Borel σ-algebra $\mathcal{B}=\sigma\left(\bigotimes_{i\in I}\mathcal{O}_i\right)$, i.e. $\mathcal{B}_I=\bigotimes_{i\in I}\sigma\left(\mathcal{O}_i\right)\subset\sigma\left(\bigotimes_{i\in I}\mathcal{O}_i\right)=\mathcal{B}$. For **countable** I and **second countable** $\mathcal{O}_i$ the converse inclusion is also true since with **countable bases** $\mathcal{E}_i$ of $\mathcal{O}_i$ the basis $\mathcal{E}=\left\{\pi_i^{-1}\left(E_i\right):\ E_i\in\mathcal{E}_i,\ i\in I\right\}$ of the **topology** is again countable and hence also generates the **Borel σ-algebra** $\mathcal{B}=\sigma\left(\mathcal{O}\left(\mathcal{E}\right)\right)=\sigma\left(\mathcal{E}\right)$ due to 1.2 such that from $\mathcal{E}\subset\mathcal{B}_I$ follows $\mathcal{B}=\sigma\left(\mathcal{E}\right)\subset\mathcal{B}_I$. Especially on **polish spaces** the two σ-algebrae coincide: $\mathcal{B}=\mathcal{B}_I$. For **Hausdorff** components according to [13, p. 7.10] the **separation axiom** T_2 extends to the product space and owing to **Tychonoff's theorem** (cf. [13, p. 9.9]) any **product** of **compact** sets is again **compact** and hence **Borel measurable** due to 1.2.

7.6 Finite products of σ-algebrae

If every basis $\mathcal{E}_j$ for $1\leq j\leq m$ includes a **countable cover** $\left(E_{jn}\right)_{n\in\mathbb{N}}\subset\mathcal{E}_j$ with $\bigcup_{n\in\mathbb{N}}E_{jn}=X_j$ the **product** $\bigotimes\limits_{j=1}^{m}\sigma\left(\mathcal{E}_j\right)$ is generated by the **intersections** $\bigcap\limits_{j=1}^{m}\pi_j^{-1}[E_j]=\prod\limits_{j=1}^{m}\mathcal{E}_j$ for all possible $E_j\in\mathcal{E}_j$: $\bigotimes\limits_{j=1}^{m}\sigma\left(\mathcal{E}_j\right)=\sigma\left(\prod\limits_{j=1}^{m}\mathcal{E}_j\right)$. Due to 7.4.1 on the one hand we have $\sigma\left(\prod\limits_{j=1}^{m}\mathcal{E}_j\right)\subset\bigotimes\limits_{j=1}^{m}\sigma\left(\mathcal{E}_j\right)$ and on the other hand $\pi_i^{-1}[E_i]=\bigcup_{n\in\mathbb{N}}\left(\prod\limits_{j=1}^{m}\pi_j^{-1}[E_{jn}]\cap\pi_i^{-1}[E_i]\right)\in\sigma\left(\left\{\bigcap\limits_{j=1}^{m}\pi_j^{-1}[E_j]:E_j\in\mathcal{E}_j\right\}\right)=\sigma\left(\prod\limits_{j=1}^{m}\mathcal{E}_i\right)$ whence $\bigotimes\limits_{j=1}^{m}\sigma\left(\mathcal{E}_j\right)\subset\sigma\left(\prod\limits_{j=1}^{m}\mathcal{E}_j\right)$ on account of 4.1.

7.7 Finite products of Borel σ-algebrae

Analogously to the one dimensional case dealt with in 1.4 and according to 7.5 the **n-dimensional intervals** $\mathcal{I}^n := \left\{ \prod\limits_{i=1}^{n} \left]a_i; b_i\right] : a_i \leq b_i \in \mathbb{R} \right\} \subset \mathbb{R}^n$ are G_δ and hence $\mathcal{B}^n$-measurable. Also on account of $\left(\prod\limits_{i=1}^{n} I_i \right) \cap \left(\prod\limits_{i=1}^{n} J_i \right) = \prod\limits_{i=1}^{n} \left(I_i \cap J_i \right)$ with **intervals** $I_i; J_i \subset \mathbb{R}$ they are **closed under intersection**. Their **finite unions** form the **algebra** $\mathcal{F}^n$ of the **n-dimensional figures**: For $F = \bigcup\limits_{k=1}^{p} \prod\limits_{i=1}^{n} I_{k,i}; G = \bigcup\limits_{l=1}^{q} \prod\limits_{i=1}^{n} J_{l,i} \in \mathcal{F}^n$ we obviously have $F \cup G \in \mathcal{F}^n$; $F \cap G = \bigcup\limits_{k=1}^{p} \bigcup\limits_{l=1}^{q} \prod\limits_{i=1}^{n} \left(I_{k,i} \cap J_{l,i} \right) \in \mathcal{F}^n$ and $F \setminus G = \bigcup\limits_{k=1}^{p} \bigcup\limits_{l=1}^{q} \prod\limits_{i=1}^{n} \left(I_{k,i} \setminus J_{l,i} \right) \in \mathcal{F}^n$. On account of $\prod\limits_{i=1}^{n} \left]a_i; b_i\right[= \bigcup_{k \in \mathbb{N}} \prod\limits_{i=1}^{n} \left]a_i; b_i - \frac{1}{k}\right]$ both the intervals and the figures generate the Borel σ-algebra: $\mathcal{B}^n := \bigotimes\limits_{i=1}^{n} \mathcal{B}_i = \sigma \left(\mathcal{I}^n \right) = \sigma \left(\mathcal{F}^n \right) = \sigma \left(\left\{ \prod\limits_{i=1}^{n} \left]a_i; \infty\right[: a_i \in \mathbb{R} \right\} \right)$ due to 1.1.2.



8 Product measure

8.1 Measurable cuts

For two measure spaces $(X_i; \mathcal{A}_i; \mu_i)$ with $i \in \{1; 2\}$, every $A \subset \mathcal{A}_1 \otimes \mathcal{A}_2$ and $x_1 \in X_1, x_2 \in X_2$ the **cuts** $A_{x_1} := \{x_2 \in X_2 : (x_1; x_2) \in A\}$ resp. A_{x_2} are measurable with respect to $\mathcal{A}_2$ resp. $\mathcal{A}_1$.

Proof: Due to $(X \setminus Q)_{x_1} = X_2 \setminus Q_{x_1}$ and $\left(\bigcup_{n \in \mathbb{N}} Q_n \right)_{x_1} = \bigcup_{n \in \mathbb{N}} \left(Q_n \right)_{x_1}$ the family of all sets $Q \subset X_1 \times X_2$ with measurable cuts $Q_{x_1} \in \mathcal{A}_2$ is a **σ-algebra** containing all **measurable rectangles** $A_1 \times A_2$ with $A_1 \in \mathcal{A}_1$ resp. $A_2 \in \mathcal{A}_2$ since $\left(A_1 \times A_2 \right)_{x_1} = \left\{ \begin{smallmatrix} A_2, & x_1 \in A_1 \\ \emptyset, & x_1 \notin A_1 \end{smallmatrix} \right.$. Hence according to 7.4.3 it includes the σ-algebra $\mathcal{A}_1 \otimes \mathcal{A}_2$ generated by these sets.

8.2 Measurable measures of cuts

For two **σ-finite** measure spaces $(X_i; \mathcal{A}_i; \mu_i)$ with $i \in \{1; 2\}$ and every $A \in \mathcal{A}_1 \otimes \mathcal{A}_2$ the mappings $s_{1A} : X_2 \to [0; \infty]$ with $s_{1A}(x_2) = \mu_1(A_{x_2})$ resp. $s_{2A} : X_1 \to [0; \infty]$ with $s_{2A}(x_1) = \mu_2(A_{x_1})$ are measurable.

Proof: Preliminarily so as to have access to **complements** we confine ourselves to $s_{1nA}(x_2) := \mu_1|_{A_n}(A_{x_2})$ with the restriction $\mu_1|_{A_n}$ on one of the μ_1-finite sets A_{1n} from the w.l.o.g. **increasing** cover $\bigcup_{n \in \mathbb{N}} A_{1n} = X_1$. The family $\mathcal{D}$ of subsets $D \subset X_1 \times X_2$ with a measurable s_{1nD} is a **Dynkin system** since the constant function $s_{1n\emptyset} = 0$ is measurable, for every measurable s_{1nA} the **complement** function $s_{1n(X_1 \times X_2) \setminus A}(x_2) = \mu_1|_{A_n}\left(\left((X_1 \times X_2) \setminus A \right)_{x_2} \right) = \mu_1|_{A_n}\left((X_1 \times X_2)_{x_2} \setminus A_{x_2} \right) = \mu_1|_{A_n}\left((X_1 \times X_2)_{x_2} \right) - \mu_1|_{A_n}(A_{x_2}) = \mu_1|_{A_n}(X_1) - s_{1nA}(x_2)$ is measurable and so is the **summation** function $s_{1n\mathring{U}D_m} = \sum_{m \in \mathbb{N}} s_{1nD_m}$ with $\left(s_{1nD_m} \right)_{m \in \mathbb{N}}$ for pairwise disjoint sets $\left(D_m \right)_{m \in \mathbb{N}}$owing to 4.9. Furthermore $s_{1n(A_1 \times A_2)}(x_2) = \mu_1|_{A_n}\left((A_1 \times A_2)_{x_2} \right) = \mu_1|_{A_n}(A_1) \cdot \chi_{A_2}(x_2)$ is measurable for every **measurable rectangle** $A_1 \times A_2$ with $A_1 \in \mathcal{A}_1$ resp. $A_2 \in \mathcal{A}_2$. Hence the system $\mathcal{A}_1 \times \mathcal{A}_2$ of measurable rectangles is included in $\mathcal{D}$ and since it is **closed under intersection** we can apply the **Dynkin δ-π-theorem** 1.6 resp. 7.4.3 to obtain $\sigma(\mathcal{A}_1 \times \mathcal{A}_2) = \mathcal{A}_1 \otimes \mathcal{A}_2 \subset \mathcal{D}$. According to the **continuity from below** 2.2.2 and 4.9 the measurability of the s_{1nA} extends to $\sup\limits_{n \in \mathbb{N}} s_{1nA}(x_2) = \sup\limits_{n \in \mathbb{N}} \mu_1|_{A_n}(A_{x_2}) = \sup\limits_{n \in \mathbb{N}} \mu_1(A_n \cap A_{x_2}) = \mu_1\left(\bigcup_{n \in \mathbb{N}} A_n \cap A_{x_2} \right) = \mu_1(A_{x_2}) = s_{1A}(x_2)$. The proof for s_{2A} is of course analogous.

8.3 The product measure

On the **product** $(X_1 \times X_2; \mathcal{A}_1 \otimes \mathcal{A}_2)$ of **two σ-finite measure spaces** $(X_i; \mathcal{A}_i; \mu_i)$ with $i \in \{1;2\}$ the expression $(\mu_1 \otimes \mu_2)(A) := \int \mu_1(A_{x_2})\, d\mu_2 = \int \mu_2(A_{x_1})\, d\mu_1$ for $A \in \mathcal{A}_1 \otimes \mathcal{A}_2$ defines a **σ-finite** measure **uniquely determined** by its **multiplicity** $(\mu_1 \otimes \mu_2)(A_1 \times A_2) = \mu_1(A_1) \cdot \mu_2(A_2)$ for every $A_1 \times A_2 \in \mathcal{A}_1 \times \mathcal{A}_2$.

Proof: On account of $\mu_1\left((A_1 \times A_2)_{x_2}\right) = \mu_1(A_1) \cdot \chi_{A_2}(x_2)$ and vice versa the two integrals coincide and the set function $\mu_1 \otimes \mu_2$ is **well defined** and obviously **uniquely determined** by its multiplicity on the family $\mathcal{A}_1 \times \mathcal{A}_2$ of all **cylinder sets**. Due to 8.2 both integrals are **well defined** on $\mathcal{A}_1 \otimes \mathcal{A}_2 = \sigma(\mathcal{A}_1 \times \mathcal{A}_2)$. The **first integral** is **σ-additiv**e on $\mathcal{A}_1 \otimes \mathcal{A}_2$ since for every sequence $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}_1 \times \mathcal{A}_2$ of pairwise disjoint measurable sets the **σ-additivity** of μ_1 and **monotone convergence** 5.13 applied to μ_2 yield $(\mu_1 \otimes \mu_2)\left(\mathring{\bigcup}_{n\in\mathbb{N}} A_n\right) = \int \mu_1\left(\left(\mathring{\bigcup}_{n\in\mathbb{N}} A_n\right)_{x_2}\right) d\mu_2 = \int \left(\sum_{n\in\mathbb{N}} \mu_1(A_n)_{x_2}\right) d\mu_2 = \sum_{n\in\mathbb{N}} \int \mu_1(A_{x_2})\, d\mu_2 = \sum_{n\in\mathbb{N}} (\mu_1 \otimes \mu_2)(A_n)$ in the case of the latter series converging to a finite limit. In the case of a diverging series $\sum_{n\in\mathbb{N}} \int \mu_1(A_{x_2})\, d\mu_2 = \infty$ there is an $N \in \mathbb{N}$ with $\int \left(\sum\limits_{n=0}^{N} \mu_1(A_n)_{x_2}\right) d\mu_2 = \sum\limits_{n=0}^{N} \int \mu_1(A_{x_2})\, d\mu_2 \geq C$ for every $C > 0$ and hence $(\mu_1 \otimes \mu_2)\left(\mathring{\bigcup}_{n\in\mathbb{N}} A_n\right) = \sum_{n\in\mathbb{N}} (\mu_1 \otimes \mu_2)(A_n) = \infty$. The same argument of course appplies to the **second integral** such that both are measures on $\mathcal{A}_1 \otimes \mathcal{A}_2$ coinciding on the **π-basis** $\mathcal{A}_1 \times \mathcal{A}_2$ and hence on all of $\mathcal{A}_1 \otimes \mathcal{A}_2$ due to 3.4. $\mu_1 \otimes \mu_2$ is **σ-finite** since for a cover $(A_{in})_{n\in\mathbb{N}} \subset \mathcal{A}_i$ of μ_i-sets A_{in} with $i \in \{1;2\}$ the sequence $(A_{1n} \times A_{2n})_{n\in\mathbb{N}} \subset \mathcal{A}_1 \otimes \mathcal{A}_2$ is a cover of $X_1 \times X_2$ from $\mu_1 \otimes \mu_2$-finite sets $A_{1n} \times A_{2n}$.

8.4 Cuts of null sets

Almost all cuts Z_{x_1} of a $\mu_1 \otimes \mu_2$**-null set** $Z \in \mathcal{A}_1 \otimes \mathcal{A}_2$ are μ_2**-null sets**: $(\mu_1 \otimes \mu_2)(Z) = 0 \Rightarrow \mu_2(Z_{x_1}) = 0$ for every $x_1 \in X_1 \setminus Z_1$ with $\mu_1(Z_1) = 0$ and analogously for Z_{x_2}.

Proof: By the **approximation property** 3.6 for every $\epsilon > 0$ there exists a sequence $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}_1 \times \mathcal{A}_2$ of **cylinder sets** with $Z \subset \bigcup_{n\in\mathbb{N}} A_n$ and $\sum_{n\in\mathbb{N}} (\mu_1 \otimes \mu_2)(A_n) \leq \frac{\epsilon}{n\cdot 2^n}$. Hence $Z_{x_1} \subset \bigcup_{n\in\mathbb{N}} A_{n,x_1}$ and for $T_n = \left\{x_1 \in X_1 : \sum_{n\in\mathbb{N}} \mu_2(A_{n,x_1}) \geq \frac{1}{n}\right\}$ we have $\frac{1}{n}\mu_1(T_n) \leq \int \sum_{n\in\mathbb{N}} \mu_2(A_{n,x_1})\, d\mu_1 \overset{5.12}{=} \sum_{n\in\mathbb{N}} \int \mu_2(A_{n,x_1})\, d\mu_1 \overset{8.3}{=} \sum_{n\in\mathbb{N}} (\mu_1 \otimes \mu_2)(A_n) \leq \frac{\epsilon}{n\cdot 2^n}$ whence $\mu_1(\bigcup_{n\in\mathbb{N}} T_n) \leq \sum_{n\in\mathbb{N}} \mu_1(T_n) \leq \epsilon$ and finally $\mu_1(\mu_2(Z_{x_1}) > 0) \leq \mu_1(\mu_2(\bigcup_{n\in\mathbb{N}} A_{n,x_1}) > 0) < \epsilon$ which proves the assertion for $Z_1 = \{x_1 \in X_1 : \mu_2(\bigcup_{n\in\mathbb{N}} A_{n,x_1}) > 0\}$.

8.5 Fubini's theorem

For two **σ-finite** measure spaces $(X_i; \mathcal{A}_i; \mu_i)$ with $i \in \{1;2\}$ every $\mathcal{A}_1 \otimes \mathcal{A}_2$**-measurable** function $f : X_1 \times X_2 \to Y$ into a **separable Banach space** $(Y; |\,|)$ is $\mu_1 \otimes \mu_2$- integrable iff either $f_{x_1} : X_2 \to Y$ with $f_{x_1}(x_2) = f(x_1; x_2)$ is μ_2 integrable for μ_1-a.e. $x_1 \in X_1$ and $\int \left(\int f_{x_1} d\mu_2\right) d\mu_1 < \infty$ or vice versa and in that case we have $\int f d(\mu_1 \otimes \mu_2) = \int \left(\int f_{x_1} d\mu_2\right) d\mu_1 = \int \left(\int f_{x_2} d\mu_1\right) d\mu_2$.

Proof:

Step I: The function $f_{x_1} : X_2 \to Y$ with $f_{x_1}(x_2) = f(x_1; x_2)$ is $\mathcal{A}_2$-measurable since due to 8.1 for every Borel measurable set $B \subset Y$ we have $f_{x_1}^{-1}[B] = \{(x_1; \xi_2) : f(x_1; \xi_2) \in B\} = \left(f^{-1}[B]\right)_{x_1} \in \mathcal{A}_2$. For **step functions** $\varphi = \sum\limits_{i=1}^{n} \alpha_i \chi_{F_{1,i} \times F_{2,i}} = \sum\limits_{i=1}^{n} \alpha_i \chi_{F_{1,i}} \cdot \chi_{F_{2,i}} \in \mathcal{S}(\mathcal{F}_1 \times \mathcal{F}_2; Y)$ with $\alpha_i \in Y$ and **w.l.o.g. pairwise disjoint cylinder sets** $F_{1,i} \times F_{2,i} \in \mathcal{F}_1 \times \mathcal{F}_2$ for the algebrae $\mathcal{F}_j$ of μ_j**-finite sets** such that $\mathcal{A}_j = \sigma(\mathcal{F}_j)$ with $j \in \{1;2\}$ the **step function** $\varphi_{x_1} = \sum\limits_{i=1}^{n} \alpha_i \chi_{F_{1,i}}(x_1) \cdot \chi_{F_{2,i}} \in \mathcal{S}(\mathcal{F}_2; Y)$ are obviously $\mathcal{A}_2$-measurable. On account of 8.3 the integration formula holds for these step functions since $\int \varphi d(\mu_1 \otimes \mu_2) = \sum\limits_{i=1}^{n} \alpha_i \cdot \mu_1(F_{1,i}) \cdot \mu_2(F_{2,i}) = \int \left(\int \varphi_{x_1} d\mu_2\right) d\mu_1$. Assuming $f \in L^1(X_1 \times X_2; Y)$

by 7.7 resp. 5.23 there is a sequence $(\varphi_n)_{n\in\mathbb{N}} \subset \mathcal{S}(\mathcal{F}_1 \times \mathcal{F}_2; Y)$ with $\lim\limits_{n\to\infty} \int |\varphi_n - f| \, d(\mu_1 \otimes \mu_2) = 0$ and in particular $\lim\limits_{n\to\infty} \int (\int \varphi_{n,x_1} d\mu_2)\, d\mu_1 = \lim\limits_{n\to\infty} \int \varphi_n d(\mu_1 \otimes \mu_2) = \int f d(\mu_1 \otimes \mu_2)$.

Step II: By 5.11 and w.l.o.g. transferring to a subsequence there is a $\mu_1 \otimes \mu_2$-null set $Z \in \mathcal{A}_1 \otimes \mathcal{A}_2$ with $\lim\limits_{n\to\infty} \varphi_n(x_1; x_2) = f(x_1; x_2)$ for every $(x_1; x_2) \in (X_1 \times X_2) \setminus Z$. Hence due to 8.4 we have $\lim\limits_{n\to\infty} \varphi_{n,x_1}(x_2) = f_{x_1}(x_2)$ for every $x_2 \in X_2 \setminus Z_{x_1}$ with $\mu_2(Z_{x_1}) = 0$ and $x_1 \in X_1 \setminus Z_1$ for a μ_1-null set Z_1. The sequence $(\Phi_n)_{n\in\mathbb{N}} \subset \mathcal{S}(X_1; \mathcal{S}(\mathcal{F}_2; Y))$ with $\Phi_n(x_1) = \varphi_{n,x_1}$ is $L^1(\mu_1)$-Cauchy since $\|\Phi_n - \Phi_m\|_1 = \int |\Phi_n - \Phi_m| \, d\mu_1 = \int (\int |\varphi_{n,x_1} - \varphi_{m,x_1}| \, d\mu_2)\, d\mu_1 = \int |\varphi_n - \varphi_m| \, d(\mu_1 \otimes \mu_2)$. By 5.11 and w.l.o.g. retreating to a subsequence there is a a $\Phi \in \overline{(\mathcal{S}(X_1; \mathcal{S}(\mathcal{F}_2; Y)); \|\|_1)} = L^1(X_1; \mathcal{S}(\mathcal{F}_2; Y))$ and a μ_1-null set W_1 such that $\lim\limits_{n\to\infty} \|\Phi_n(x_1) - \Phi(x_1)\|_1 = 0$ for every $x_1 \in X_1 \setminus (Z_1 \cup W_1)$. In particular $\|\varphi_{n,x_1}\|_1 = \|\Phi_n(x_1)\|_1 \leq \|\Phi_n(x_1) - \Phi(x_1)\|_1 + \|\Phi(x_1)\|_1 \leq 2\|\Phi(x_1)\|_1$ for n large enough whence $\Phi(x_1) \in L^1(X_2; Y)$ by 5.18. A third instance of 5.11 verifies that μ_2-a.e. and for $x_1 \in X_1 \setminus (Z_1 \cup W_1)$ we have $\Phi(x_1) = f_{x_1}$, i.e. $\lim\limits_{n\to\infty} \|\Phi_n(x_1) - \Phi(x_1)\|_1 = \lim\limits_{n\to\infty} \int |\varphi_{n,x_1} - f_{x_1}| \, d\mu_2 = 0$ and hence $\lim\limits_{n\to\infty} \int \varphi_{n,x_1} d\mu_2 = \lim\limits_{n\to\infty} \int f_{x_1} d\mu_2$.

Step III: Due to 4.9 the function $x_1 \mapsto \int f_{x_1} d\mu_2$ is $\mathcal{A}_1$-measurable. The step functions $(\Psi_n)_{n\in\mathbb{N}} \subset \mathcal{S}(X_1; Y)$ with $\Psi_n(x_1) = \int \varphi_{n,x_1} d\mu_2 = \sum\limits_{i=1}^{n} \alpha_i \chi_{F_{1,i}}(x_1) \cdot \mu_2(F_{2,i})$ are again $L^1(\mu_1)$-Cauchy since $\|\Psi_n - \Psi_m\|_1 = \int |\Psi_n - \Psi_m| \, d\mu_1 = \int (\int |\varphi_{n,x_1} - \varphi_{m,x_1}| \, d\mu_2)\, d\mu_1 = \int |\varphi_n - \varphi_m| \, d(\mu_1 \otimes \mu_2)$. Since by **step II** we have $\lim\limits_{n\to\infty} \Psi_n(x_1) = \lim\limits_{n\to\infty} \int \varphi_{n,x_1} d\mu_2 = \int f_{x_1} d\mu_2$ for every $x_1 \in X_1 \setminus (Z_1 \cup W_1)$ by 5.11 we conclude $\lim\limits_{n\to\infty} \int |\int \varphi_{n,x_1} d\mu_2 - \int f_{x_1} d\mu_2| \, d\mu_1 = \lim\limits_{n\to\infty} \|\Psi_n - \int f_{x_1} d\mu_2\|_1 = 0$. In particular by **step I** we have shown that $\int (\int f_{x_1} d\mu_2)\, d\mu_1 = \lim\limits_{n\to\infty} \int (\int \varphi_{n,x_1} d\mu_2)\, d\mu_1 = \lim\limits_{n\to\infty} \int \varphi_n d(\mu_1 \otimes \mu_2) = \int f d(\mu_1 \otimes \mu_2)$.

Step IV: By 5.16 we may assume $|f|_{x_1} \in L^1\left(\mu_2; \mathbb{R}_0^+\right)$ for every $x_1 \in X_1 \setminus V_1$ with $\mu_1(V_1) = 0$ resp. $\int \left(\int |f|_{x_1} d\mu_2\right) d\mu_1 < \infty$ and by steps I - III it suffices to show that $|f| \in L^1\left(\mu_1 \otimes \mu_2; \mathbb{R}_0^+\right)$. Since $|f| : X_1 \times X_2 \to \mathbb{R}_0^+$ is measurable 5.6 provides an w.l.o.g. increasing sequence $(\varphi_n)_{n\in\mathbb{N}} \subset \mathcal{S}(X_1 \times X_2; \mathbb{R})$ converging outside of a $\mu_1 \otimes \mu_2$-null set Z to f. As above resp. according to 8.4 we have $\lim\limits_{n\to\infty} \varphi_{n,x_1}(x_2) = |f|_{x_1}(x_2)$ for every $x_2 \in X_2 \setminus Z_{x_1}$ with $\mu_2(Z_{x_1}) = 0$ and $x_1 \in X_1 \setminus Z_1$ for a μ_1-null set Z_1. **Monotone convergence** 5.13 then yields $\lim\limits_{n\to\infty} \int \varphi_{n,x_1} d\mu_2 = \int |f|_{x_1} d\mu_2$ for every $x_1 \in X_1 \setminus (Z_1 \cup V_1)$. By definition 5.1 every step function $\varphi_n \in L^1(X_1 \times X_2; \mathbb{R})$ is integrable so that steps I - III yield $\int \varphi_n d(\mu_1 \otimes \mu_2) = \int (\int \varphi_{n,x_1} d\mu_2)\, d\mu_1$. Since the sequence $(\int \varphi_{n,x_1} d\mu_2)_{n\in\mathbb{N}}$ is increasing we may invoke **monotone convergence** a second time to obtain $\lim\limits_{n\to\infty} \int (\int \varphi_{n,x_1} d\mu_2)\, d\mu_1 = \int \left(\int |f|_{x_1} d\mu_2\right) d\mu_1$. A third instance of the **monotone convergence** theorem applied to $(\varphi_n)_{n\in\mathbb{N}} \subset L^1(X_1 \times X_2; \mathbb{R})$ delivers $\lim\limits_{n\to\infty} \int \varphi_n d(\mu_1 \otimes \mu_2) = \int |f| \, d(\mu_1 \otimes \mu_2)$ and hence the assertion.

8.6 Finite products of measure spaces

On the finite product $(\prod_{i\in J} X_i; \bigotimes_{i\in J} \mathcal{A}_i)$ of the σ-finite measure spaces $(X_i; \mathcal{A}_i; \mu_i)$ with a finite index set $J = \{1, ..., n\}$ the **product measure** $\bigotimes_{i\in J} \mu_i$ is **uniquely determined** by the **multiplicity** condition $\mu(\prod_{i\in J} A_i) = \prod_{i\in J} \mu_i(A_i)$ and is **constructed inductively** according to 8.3 by means of $\bigotimes_{1\leq j\leq i} \mu_j := \left(\bigotimes_{1\leq j<i} \mu_j\right) \otimes \mu_i$. The resulting product of measure spaces is denoted as $\bigotimes_{i\in J}(X_i; \mathcal{A}_i; \mu_i) := (\prod_{i\in J} X_i; \bigotimes_{i\in J} \mathcal{A}_i; \bigotimes_{i\in J} \mu_i)$. For a Borel measurable function $f : \prod_{i\in J} X_i \to Y$ with finite integrals $\int \left(... \left(\int f_{x_{j(2)}...x_{j(n)}} d\mu_{j(1)}\right) ...\right) d\mu_{j(k)}$ for every $1 \leq k \leq n$ and **some permutation** $j : J \to J$ we have $\int f d\mu = \int \left(... \left(\int f_{x_{j(2)}...x_{j(n)}} d\mu_{j(1)}\right) ...\right) d\mu_{j(n)}$ for **every permutation**. Hence the **convergence for one particular order of integration grants the integrability of all permutations**.

8.7 Completion of λ^n

The product $\lambda^n = \bigotimes_{1 \leq i \leq n} \lambda$ of the **complete** Lebesgue measures λ on the product $\mathcal{B}^n = \bigotimes_{1 \leq i \leq n} \mathcal{B}$ of the Lebesgue σ-algebrae $\mathcal{B}$ on $\mathbb{R}$ is **not complete** any more since for any λ-null set $A \in \mathcal{B}$ we have $\lambda^2 (A \times \mathbb{R}) = 0$ and for any non Lebesgue measurable $B \notin \mathcal{B}$ (cf. 3.11) evidently $A \times B \subset A \times \mathbb{R}$ holds but $A \times B \notin \mathcal{B}^2$. The completion of the product according to 3.9 will be included without change of notation in the extension obtained by means of the **Riesz representation theorem** 10.13 to the **Lebesgue measure** λ^n on the **Lebesgue σ-algebra** $\mathcal{B}^n$.

8.8 Translation invariance of λ^n

The **Lebesgue-Borel** measure λ^n on the **Borel** σ-algebra $\mathcal{B}^n$ on $\mathbb{R}^n$ is **uniquely determined** by its **translation invariance** on the **π-basis** of the **n-dimensional intervals** $\mathcal{I}^n$: For every translation $T_{\mathbf{c}} : \mathbb{R}^n \to \mathbb{R}^n$ with $T_{\mathbf{c}}(\mathbf{x}) = \mathbf{x} + \mathbf{c}$ for a $\mathbf{c} \in \mathbb{R}^{\mathbf{n}}$ and every interval $]\mathbf{a}; \mathbf{b}] := \prod\limits_{i=1}^{n} \,]a_i; b_i] \in \mathcal{I}^n$ with $a_i \leq b_i \in \mathbb{R}$ due to 4.3 and 8.3 we have $T_{\mathbf{c}} (\lambda^n) (]\mathbf{a}; \mathbf{b}]) = \lambda^n (T_{\mathbf{c}}^{-1} (]\mathbf{a}; \mathbf{b}])) = \lambda^n (]\mathbf{a} - \mathbf{c}; \mathbf{b} - \mathbf{c}]) = \lambda^n (]\mathbf{a}; \mathbf{b}]) = \prod\limits_{i=1}^{n} (b_i - a_i)$, i.e. the **$\sigma$-finite** measures $T_{\mathbf{c}} (\lambda^n)$ and λ^n coincide on the π-basis $\mathcal{I}^n$ and hence on $\sigma (\mathcal{I}^n) = \mathcal{B}^n$ due to 3.4.

8.9 The transformation formula

The image of the **Lebesgue-Borel measure** λ^n under a **homomorphism** $T \in GL(n; \mathbb{R})$ is $T \circ \lambda^n = \frac{\lambda^n}{|\det T|}$ such that $\lambda^n (T[A]) = |\det T| \cdot \lambda^n (A)$ for every Borel-measurable $A \in \mathcal{B}^n$.

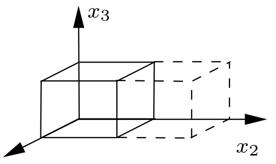
Proof: According to the **Gauss algorithm** every **automorphism** resp. every **invertible matrix** is the product of **elementary transformations** resp. **elementary matrices** of the two following types:

$$E_{kl} = \begin{array}{c} \begin{array}{ccccccc} 1 & \cdots & k & \cdots & l & \cdots & n \end{array} \\ \begin{pmatrix} 1 &&&&&& \\ & \ddots &&&&& \\ && 1 && 1 && \\ &&& \ddots &&& \\ &&&& 1 && \\ &&&&& \ddots & \\ &&&&&& 1 \end{pmatrix} \begin{array}{c} 1 \\ \vdots \\ k \\ \vdots \\ l \\ \vdots \\ n \end{array} \end{array}$$

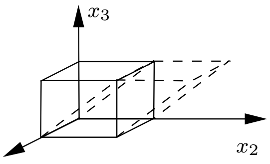
$$E_{k\alpha} = \begin{array}{c} \begin{array}{ccccc} 1 & \cdots & k & \cdots & n \end{array} \\ \begin{pmatrix} 1 &&&& \\ & \ddots &&& \\ && \alpha && \\ &&& \ddots & \\ &&&& 1 \end{pmatrix} \begin{array}{c} 1 \\ \vdots \\ k \\ \vdots \\ n \end{array} \end{array}$$

$$E_{22} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$E_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$



$E_{2;2}(H_3)$



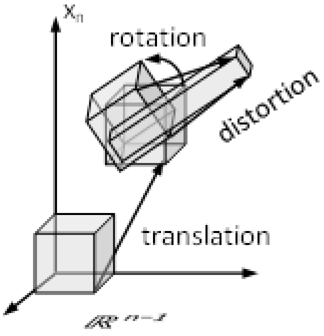
$E_{2;3}(H_3)$

Multiplication with E_{kl} results in an addition of the l-th row to the k-th row, i.e. a **shearing** so that the image of the **unit cube** $Q := [\mathbf{0}; \mathbf{1}[$ generated by the **basis vectors** $\mathbf{e}_1, ..., \mathbf{e}_n$ with the **measure** $\lambda^n(Q) = (1-0)^n = 1$ is $E_{kl}[Q] = \left\{\sum_{1 \le i \le n} x_i \mathbf{e}_i : 0 \le x_i \le 1; i \ne k \wedge x_l \le x_k < x_l + 1\right\}$. This **parallelepiped** can be split into two disjoint halves $L = \{\mathbf{x} \in E_{kl}[Q] : x_l \le x_k < 1\}$ and $R = \{\mathbf{x} \in E_{kl}[Q] : 1 \le x_k < x_l + 1\}$ such that $E_{kl}[Q] = L \mathring{\cup} R$ but also $Q = (R - \mathbf{e}_k) \mathring{\cup} L$ and due to the **translation invariance** of λ^n we obtain $\lambda^n(E_{kl}[Q]) = \lambda^n(K) + \lambda^n(L) = \lambda^n(K) + \lambda^n(L - \mathbf{e}_k) = \lambda^n(Q) = 1 \cdot \lambda^n(Q) = |\det E_{kl}| \cdot \lambda^n(Q)$.

Multiplication with $E_{k\alpha}$ results in a multiplication of the k th row with the factor $\alpha \in \mathbb{R}$ resulting in the **dilation** $E_{k\alpha}[Q] = \left\{\sum_{1 \le i \le n} x_i \mathbf{e}_i : 0 \le x_i < 1; i \ne k \wedge 0 \le x_k < \alpha\right\}$ with measure $\lambda^n(E_{k\alpha}(H)) = (1-0)^{n-1} \cdot (\alpha - 0) = \alpha = |\det E_{k\alpha}| \cdot \lambda^n(Q)$.

The assertion then follows from the **multiplicity of the determinant**: $|\det(A \cdot B)| = |\det(A)| \cdot |\det(B)|$**:**

8.10 Special cases of the transformation formula



1. A **dilation** along the axes by $T(e_i) = r_i \cdot e_i$ for $r_i \in \mathbb{R}$ and $0 \le i \le n$ results in $\lambda^n(T(A)) = \left|\prod_{i=1}^{n} r_i \cdot \lambda^n(A)\right| \cdot \lambda^n(A)$ and particularly a simple **scaling** of the set A by the **scaling factor** $r \in \mathbb{R}$ yields the volume $\lambda^n(rA) = |r^n| \cdot \lambda^n(A)$.
2. A **rotation** by an **orthogonal matrix** $T \in O(n; \mathbb{R})$ leaves the volume unacffected: $\lambda^n(T[A]) = |\det T| \cdot \lambda^n(A) = \lambda^n(A)$.
3. In the **three dimensions** of $\mathbb{R}^3$ the homomorphism T may be represented by a matrix with n linearly independent column vectors $\mathbf{x}_i = \sum_{k=1}^{n} x_{ki} \mathbf{e}_k \in \mathbb{R}^n$ for $1 \le i \le n$ generating a **parallelepiped** $T[Q] = \left\{\sum_{1 \le i \le n} t_i \mathbf{x}_i : 0 \le x_i < 1\right\}$ which is the image of the **unit cube** $Q = \left\{\sum_{1 \le k \le n} t_k \mathbf{e}_k : 0 \le t_k < 1\right\}$. Its volume is $\lambda^3(T[Q]) = |\det T| \cdot \lambda^3(Q) = \det\left((x_{ki})_{1 \le k, i \le n}\right) \cdot 1$.

5. With the **integral transformation formula** 5.8 we obtain the **linear form** of the **change-of-variables theorem** [9, th. 13.7] since $\int_{T[A]} f \cdot \frac{1}{|\det T|} d\lambda^n = \int_{T[A]} f d(T \circ \lambda^n) = \int_A (f \circ T) d\lambda^n$ implies $\int_{T[A]} f d\lambda^n = \int_A (f \circ T) \cdot |\det T| \cdot d\lambda^n$

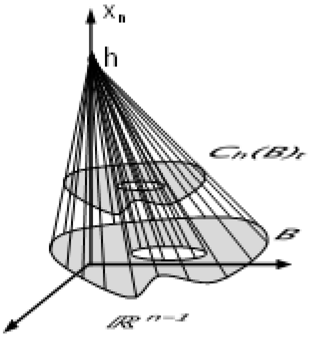
8.11 Cavalieri's principle

For a **compact** $K \subset \mathbb{R}^n$ and any **cut** $K_t = \{\boldsymbol{x} \in \mathbb{R}^{n-1} : (\boldsymbol{x}; t) \in K\}$ with $t \in \mathbb{R}$ we have $\lambda^n(K) = \int_{\mathbb{R}} \lambda^{n-1}(K_t) \, dt$.

Proof: Due to **Fubini's theorem** 8.5 we have $\lambda^n(K) = \int_{\mathbb{R}^n} \chi_K(\boldsymbol{x}) \, d\boldsymbol{x} = \int \left(\int_{\mathbb{R}^{n-1}} \chi_K(\boldsymbol{x}; t) \, d\boldsymbol{x}\right) dt = \int \left(\int_{\mathbb{R}^{n-1}} \chi_{K_t}(\boldsymbol{x}) \, d\boldsymbol{x}\right) dt = \int_{\mathbb{R}} \lambda^{n-1}(K_t) \, dt$.

8.12 The cone

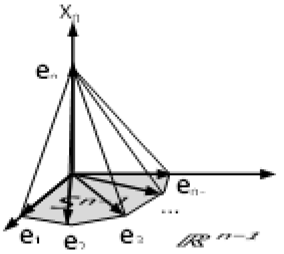
The cone $C_h(B) = \{((1-\lambda)\boldsymbol{\xi}, \lambda h) \in \mathbb{R}^n : \boldsymbol{\xi} \in B; 0 \le \lambda \le 1\}$ with **compact base** $B \subset \mathbb{R}^{n-1}$and **height** $h > 0$ has the **volume** $\lambda^n(C_h(B)) = \frac{h}{n} \cdot \lambda^{n-1}(B)$.



Proof: According to **Cavalieri's principle**8.11 and by the condition $\lambda h = t$ we obtain the **cuts** $C_h(B)_t = \begin{cases} \left(1-\frac{t}{h}\right)B & \text{for } 0 \le t \le h \\ 0 & \text{else} \end{cases}$ with $\lambda^{n-1}(C_h(B)_t) = \left(1-\frac{t}{h}\right)^{n-1} \cdot \lambda^{n-1}(B)$ due to 8.9 whence $\lambda^n(C_h(B)) = \int_{\mathbb{R}} \lambda^{n-1}(C_h(B)_t)\,dt = \lambda^{n-1}(B) \cdot \int_0^h = \left(1-\frac{t}{h}\right)^{n-1} dt = \frac{h}{n} \cdot \lambda^{n-1}(B)$.

8.13 The unit simplex

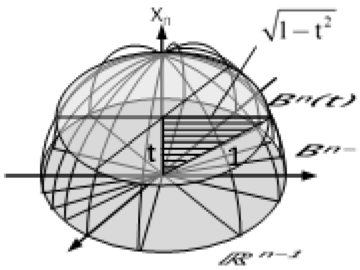
The unit simplex $S_1^n = \left\{\sum\limits_{i=1}^n \lambda_i \boldsymbol{e}_i : \sum\limits_{i=1}^n \lambda_i = 1\right\} \subset \mathbb{R}^n$ has the volume $\lambda^n(S_1^n) = \frac{1}{n!}$.



Proof: By **induction** over n we start with $\lambda^1(S_1^1) = \lambda^1([0;1]) = 1$ and proceed from $n-1$ to n by 8.12 with $\lambda^n(S_1^n) = \frac{1}{n} \cdot \lambda^{n-1}\left(S_1^{n-1}\right) = \frac{1}{n} \cdot \frac{1}{(n-1)!} = \frac{1}{n!}$.

8.14 The unit sphere

The unit sphere B_1^n has the volume $\tau_n = \lambda^n(B_1^n) = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)}$.



Proof: As above we procced by **induction** over n starting with $\lambda^1(B_1^1) = \lambda^1([-1;1]) = 2$ and proceed from $n-1$ to n by **Cavalieri's principle** 8.11 with $\lambda^n(B_1^n) = \int_{\mathbb{R}} \lambda^{n-1}\left(\left(B_1^{n-1}\right)_t\right) dt = \int_{\mathbb{R}} \lambda^{n-1}\left(\left(B^{n-1}_{\sqrt{1-t^2}}\right)\right) dt = \lambda^{n-1}\left(B_1^{n-1}\right) \cdot \int_{[-1;1]} (1-t^2)^{(n-1)/2} dt = \lambda^{n-1}\left(B_1^{n-1}\right) \cdot c_n$. By **substitution** and **integration by parts** we can simplify $c_n = \int_{-1}^1 (1-t^2)^{(n-1)/2} dt = 2\int_0^{\pi/2} \sin^n(\alpha) d\alpha = 2(n-1)\int_0^{\pi/2} \cos^2(\alpha) \cdot \sin^{(n-2)}(\alpha) d\alpha$. By expanding this expression to $(1-n)\int_0^{\pi/2} (\sin^2(\alpha) + \cos^2(\alpha)) \cdot \sin^{(n-2)}(\alpha) da + n\int_0^{\pi/2} \sin^n(\alpha) d\alpha = 0$ we can use **Pythagoras** to obtain

$$c_n = 2 \cdot \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2}(\alpha) d\alpha$$

$$= 2 \cdot \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \ldots \cdot \begin{cases} \frac{1}{2} \cdot \int_0^{\pi/2} 1 d\alpha = \frac{\pi}{4} & \text{for } n \text{ even} \\ \frac{2}{3} \cdot \int_0^{\pi/2} \sin(\alpha) d\alpha = \frac{2}{3} & \text{for } n \text{ odd} \end{cases}$$

$$= 2 \cdot \begin{cases} \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \ldots \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} & \text{for } n \text{ even} \\ \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \ldots \cdot \frac{4}{5} \cdot \frac{2}{3} & \text{for } n \text{ odd} \end{cases}$$

Hence we have $c_n \cdot c_{n-1} = \frac{2\pi}{n}$ so that with $\lambda^2(B_1^2) = \lambda^1(B_1^1) \cdot c_2 = \pi$ follows

$$\lambda^n(B_1^n) = \frac{2\pi}{n} \cdot \lambda^{n-2}\left(B_1^{n-2}\right) = \begin{cases} \frac{2\pi}{n} \cdot \frac{2\pi}{n-2} \cdot \ldots \cdot \frac{2\pi}{4} \cdot \pi = \frac{\pi}{n/2} \cdot \frac{\pi}{n/2-1} \cdot \ldots \cdot \frac{\pi}{2} \cdot \frac{\pi}{1} & \text{for } n \text{ even} \\ \frac{2\pi}{n} \cdot \frac{2\pi}{n-2} \cdot \ldots \cdot \frac{2\pi}{3} \cdot 2 = \frac{\pi}{n/2} \cdot \frac{\pi}{n/2-1} \cdot \ldots \cdot \frac{\pi}{3/2} \cdot \frac{\sqrt{\pi}}{1/2} \cdot \frac{1}{\sqrt{\pi}} & \text{for } n \text{ odd} \end{cases}$$

A comparison with the **Gamma function** (cf. [9, p. 2.1]) with the **functional equation** $\Gamma(x+1) = x \cdot \Gamma(x)$ for $0 < x < \infty$ and initial values $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \Rightarrow \Gamma\left(\frac{3}{2}\right) = \frac{1}{2} \cdot \sqrt{\pi} \Rightarrow \Gamma\left(\frac{5}{2}\right) = \frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi} \Rightarrow ...$ resp. $\Gamma(1) = 1 \Rightarrow \Gamma(2) = 1 \Rightarrow \Gamma(3) = 1 \cdot 2 \Rightarrow ...$ yields the desired formula.

8.15 Probability measures on function spaces

On the product $(X_I; \mathcal{A}_I)$ of **probability spaces** $(X_i; \mathcal{A}_i; \mu_i)_{i \in I}$ with **arbitrary index set** I exists a **probability measure** μ_I **uniquely determined** by its **multiplicity** $\mu_I|_{\mathcal{Z}_J} = \mu_J := \bigotimes_{\in J} \mu_i$ for all **finite** $J \subset I$, i.e. on **cylinder sets** $\pi_J^{-1}(A) \in \mathcal{Z}_J$ with $A \in \mathcal{A}_J = \bigotimes_{i \in J} \mathcal{A}_i = \pi_J(\mathcal{A}_I) = \pi_J(\mathcal{Z}_J)$ (cf. 7.4.3) it coincides with the corresponding **finite product measure** $\mu_J = \bigotimes_{i \in J} \mu_i$ on the **finite product-σ-algebrae** $\mathcal{A}_J$. The elements $x_I \in X_I$ with $x_I : I \to X_I$ are the **sample path**s or **realizations** of the **stochastic process** $(X_I; \mathcal{A}_I; \mu_I)$

Proof: The function $\mu_I : \mathcal{S}_I \to [0;1]$ given by $\mu_I\left(\pi_J^{-1}\left(\prod_{i \in J} A_i\right)\right) := \prod_{i \in J} \mu_i(A_i)$ for $A_j \in \mathcal{A}_j$ and **finite** $J \subset I$ is **well defined** and in particular independent of the representation of the **measurable rectangle** $S = \prod_{l \in L} S_l = \left(\pi_J^L\right)^{-1}\left(\prod_{j \in J} A_j\right) = \left(\pi_K^L\right)^{-1}\left(\prod_{k \in K} B_k\right) \in \mathcal{S}_L$ with $A_j \in \mathcal{A}_j$, $j \in J$ and $B_k \in \mathcal{A}_k$, $k \in K$ for **finite** $J, K, L \subset I$ with $J \cup K \subset L$. By the equality of the two representations we have $S_j = A_j = B_j$ for $j \in J \cap K$, $Z_j = A_j = X_j$ for $j \in J \setminus K$, $S_j = B_j = X_j$ for $j \in K \setminus J$ and finally $S_l = X_l$ for $l \in L \setminus (J \cup K)$. Hence the **multiplicity condition** with $\mu_i(X_i) = 1$ for all $i \in I$ yields $\mu_L(S) = \mu_J\left(\prod_{j \in J} A_j\right) = \prod_{j \in J \cap K} \mu_j(A_j) = \prod_{j \in J \cap K} \mu_j(B_j) = \mu_K\left(\prod_{k \in K} B_k\right)$. According to 8.6 for every **finite** $J \subset I$ there is a **uniquely determined product measure** $\mu_J = \bigotimes_{\in J} \mu_i$ on the **finite product-σ-algebra** $\mathcal{A}_J$ with $\mu_I\left(\prod_{i \in J} A_i\right) := \prod_{i \in J} \mu_i(A_i)$ for $A_j \in \mathcal{A}_j$. Hence the extension $\mu_I : \mathcal{Z}_I \to [0;1]$ given by $\mu_I(Z) := \mu_J(A_J)$ for $Z = \pi_J^{-1}(A_J)$ and $A_J \in \mathcal{A}_J$ with **finite** $J \subset I$ on the **algebra** $\mathcal{Z}_I$ is **well defined** and in particular independent of the representation of the **cylinder set** $Z = \pi_J^{-1}(A_J) = \pi_K^{-1}(B_K)$ with $A_J \in \mathcal{A}_J$ and $B_K \in \mathcal{A}_K$ for **finite** $J, K \subset I$. We now prove that μ_I is **$\emptyset$-continuous** on the algebra of cylinder sets.

To this end for a given **path** $x_J \in X_J$ and a given K-**cylinder set** $Z \in \mathcal{Z}_K$ with **finite** $J \subset K \subset I$ we examine the Z-**extensions** $Z^{x_J} = \left\{\xi_I \in X_I : \left(x_J; \pi_{K \setminus J}(\xi_I)\right) \in Z\right\} = \pi_{K \setminus J}^{-1}(A_{x_J}) \in \mathcal{Z}_K$ for $A = \pi_K(Z) \in \mathcal{A}_K = \mathcal{A}_J \otimes \mathcal{A}_{K \setminus J}$ and the **cuts** A_{x_J} of $A \in A_K$ being $\mathcal{A}_{K \setminus J}$-measurable due to 8.1. Hence he family Z^{x_J} consists of all **measurable extensions** $\xi_I \in X_I$ of the given path x_J with an **arbitrary course** during J (!) and **passing through** Z during $K \setminus J$. (cf. the set of all **paths** passing a given **tree** in [13, p. 15.5]). Owing to 8.3 we have $\mu_I(Z) = \mu_{I \setminus K}\left(\pi_{I \setminus K}(Z)\right) \cdot \mu_K(\pi_K(Z)) = 1 \cdot \mu_K(A) = \int \mu_{K \setminus J}(A_{x_J})\, d\mu_J = \int \mu_I(Z^{x_J})\, d\mu_J$.

Now let $(Z_n)_{n \geq 1} \subset \mathcal{Z}_I$ be a **decreasing** sequence of **cylinder sets** $Z_n = \pi_{J_n}^{-1}(A_n)$ with $A_n \in \mathcal{A}_{J_n}$ for finite $J_{n+1} \supset J_n$ and $Z_{n+1} \subset Z_n$ as well as $\mu_I(Z_n) \geq \alpha > 0$ for $n \geq 1$ such that $\inf_{n \geq 1} \mu_I(Z_n) \geq \alpha$. In order to show the **$\emptyset$-continuity** we have to prove that $\bigcap_{n \geq 1} Z_n \neq \emptyset$, i.e. we must find a path $x \in \bigcap_{n \geq 1} Z_n$. We start on the interval J_1 with a section x_{J_1} and proceed by **induction** to extend it to $\left(x_{J_1}; x_{J_2 \setminus J_1}; ...\right)$:

Due to 8.2 the mapping $x_{J_1} \mapsto \mu_I\left(Z_n^{x_{J_1}}\right) = \pi_{J_n \setminus J_1}^{-1}\left((A_n)_{x_{J_1}}\right)$ is measurable and hence the set $Q_n^{J_1} = \left(x_{J_1} \in X_{J_1} : \mu_I\left(Z_n^{x_{J_1}}\right) \geq \frac{\alpha}{2}\right) \in \mathcal{A}_{J_1}$ of all paths $x_{J_1} \in X_{J_1}$ which can be extended with a probability of at least $\frac{\alpha}{2}$ on Z_n is $\mathcal{A}_{J_1}$-measurable. According to the preceding paragraph we obtain the estimate $\alpha \leq \mu_I(Z_n) \leq \int_{Q_n^{J_1}} \mu_I\left(Z_n^{x_{J_1}}\right) d\mu_{J_1} + \int_{X_I \setminus Q_n^{J_1}} \mu_I\left(Z_n^{x_{J_1}}\right) d\mu_{J_1} \leq \mu_{J_1}\left(Q_n^{J_1}\right) + \frac{\alpha}{2}$ and hence $\mu_{J_1}\left(Q_n^{J_1}\right) \geq \frac{\alpha}{2}$ for all $n \geq 1$. Since μ_{J_1} is continuous from above and $Q_{n+1}^{J_1} \subset Q_n^{J_1}$ for all $n \geq 1$ there is an $x_{J_1} \in \bigcap_{n \geq 1} Q_n^{J_1} \neq \emptyset$, i.e. $\mu_I\left(Z_n^{x_{J_1}}\right) \geq \frac{\alpha}{2}$ for all $n \geq 1$.

We now extend the path x_{J_1}inductively with $Z_n^{x_{J_k}}$ taking the place of Z_n: Assuming there is an $x_{J_k} \in X_{J_k}$ with $\mu_I\left(Z_n^{x_{J_k}}\right) \geq \frac{\alpha}{2^k}$ for all $n \geq 1$ we have

$$Q_n^{J_{k+1}} = \left(x_{J_{k+1} \setminus J_k} \in X_{J_{k+1} \setminus J_k} : \mu_I\left(\left(Z_n^{x_{J_k}}\right)^{x_{J_{k+1} \setminus x_{J_k}}}\right) \geq \frac{\alpha}{2^{k+1}}\right) \in \mathcal{A}_{J_{k+1}}$$

$$\text{whence } \frac{\alpha}{2^k} \leq \mu_I\left(Z_n^{x_{J_k}}\right)$$
$$\leq \int_{Q_n^{J_{k+1}}} \mu_I\left(\left(Z_n^{x_{J_k}}\right)^{x_{J_{k+1}}\backslash x_{J_k}}\right) d\mu_{J_{k+1}} + \int_{X_I \backslash Q_n^{J_{k+1}}} \mu_I\left(\left(Z_n^{x_{J_k}}\right)^{x_{J_{k+1}}\backslash x_{J_k}}\right) d\mu_{J_{k+1}}$$
$$\leq \mu_{J_{k+1}}\left(Q_n^{J_{k+1}}\right) + \frac{\alpha}{2^{k+1}}$$

such that $\mu_{J_{k+1}}\left(Q_n^{J_{k+1}}\right) \geq \frac{\alpha}{2^{k+1}}$ for all $n \geq 1$. Consequently there must exist an extension $x_{J_{k+1}\backslash J_k} \in \bigcap_{n\geq 1} Q_n^{J_{k+1}} \neq \emptyset$, i.e. $\mu_I\left(\left(Z_n^{x_{J_k}}\right)^{x_{J_{k+1}}\backslash x_{J_k}}\right) \geq \frac{\alpha}{2^{k+1}}$ for all $n \geq 1$. If we add the new section to x_{J_k} we obtain $x_{J_{k+1}} := \left(x_{J_k}; x_{J_{k+1}\backslash J_k}\right) \in X_{J_{k+1}}$ with $Z_n^{x_{J_{k+1}}} = \left(Z_n^{x_{J_k}}\right)^{x_{J_{k+1}}\backslash x_{J_k}}$, particularly $\pi_{J_k}^{J_{k+1}}\left(x_{k+1}\right) = x_k$ and $\mu_I\left(Z_n^{x_{J_{k+1}}}\right) \geq \frac{\alpha}{2^{k+1}}$ for all $n \geq 1$. Thus we have found a path $x' = \left(x_{J_1}; x_{J_2\backslash J_1}; ...\right) \in \pi_{\bigcup_{n\geq 1} J_n}\left(\bigcap_{n\geq 1} Z_n\right) \subset X_{\bigcup_{n\geq 1} J_n}$ and by an arbitrary extension on the remaining time $I \setminus \bigcup_{n\geq 1} J_n$ we get the desired $x \in \bigcap_{n\geq 1} Z_n \neq \emptyset$ with $\pi_{\bigcup_{n\geq 1} J_n}(x) = x'$.

Hence μ_I is **$\emptyset$-continuous** and since due to 8.6 it is **finitely additive** as well as **bounded** according to 2.2.4 its **σ-additivity** follows. Due to the **extension theorem** 3.5 the **pre-measure** μ_I on the **algebra** $\mathcal{Z}_I$ of the **cylinder sets** can be extended in a unique way to a **measure** μ_I on the **σ-algebra** $\sigma\left(Z_I\right) = A_I$. This completes the proof.

9 Measures with densities

9.1 Complex measure and total variation

A **complex measure** is a **complex** and **σ-additive** set function $\mu : \mathcal{A} \to \mathbb{C}$ on a measurable space $(X; \mathcal{A})$. Contrary to the **positive measure** $\mu : \mathcal{A} :\to [0; \infty]$ defined in 3.1 the complex measure is **finite**. According to the **theorem of Lévy und Steinitz** ([10, th. 8.18]) the **σ-additivity** $\mu\left(\mathring{\bigcup}_{n\in\mathbb{N}} A_n\right) = \sum_{n\in\mathbb{N}} \mu\left(A_n\right) < \infty$ resp. the **interchangeability of the union** imply the **absolute convergence** of the series.

So its **total variation** $|\mu| : \mathcal{A} :\to \mathbb{R}$ with $|\mu|(A) := \sup\left\{\sum_{n\in\mathbb{N}} \left|\mu\left(A_n\right)\right| : \left(A_n\right)_{n\in\mathbb{N}} \subset \mathcal{A} : \mathring{\bigcup}_{n\in\mathbb{N}} A_n = A\right\}$ is well defined as well as **σ-additive**: On the one hand for every $A_m \in \mathcal{A}$ and $\epsilon > 0$ there is a partition $\left(A_{mn}\right)_{n\in\mathbb{N}} \subset \mathcal{A}$ with $|\mu|\left(A_m\right) - \epsilon \cdot 2^{-m-1} < \sum_{n\in\mathbb{N}} \left|\mu\left(A_{mn}\right)\right| \leq |\mu|\left(A_m\right)$ such that $\sum_{m\in\mathbb{N}} |\mu|\left(A_m\right) - \epsilon < \sum_{m,n\in\mathbb{N}} \left|\mu\left(A_{mn}\right)\right| \leq \sum_{m\in\mathbb{N}} |\mu|\left(A_m\right)$ and hence $\sum_{m\in\mathbb{N}} |\mu|\left(A_m\right) \leq |\mu|\left(\mathring{\bigcup}_{m\in\mathbb{N}} A_m\right)$. On the other hand for **every** partition $\left(B_n\right)_{n\in\mathbb{N}} \subset \mathcal{A}$ with $\mathring{\bigcup}_{n\in\mathbb{N}} B_n = \mathring{\bigcup}_{m\in\mathbb{N}} A_m$ the intersections $\left(B_n \cap A_m\right)_{n\in\mathbb{N}}$ partition A_m while the intersections $\left(B_n \cap A_m\right)_{m\in\mathbb{N}}$ partition B_n such that due to the σ-additivity of μ holds $\sum_{n\in\mathbb{N}} \left|\mu\left(B_n\right)\right| \leq \sum_{m,n\in\mathbb{N}} \left|\mu\left(A_m \cap B_n\right)\right| \leq \sum_{m\in\mathbb{N}} |\mu|\left(A_m\right)$. This estimate extends to the suprema such that $|\mu|\left(\mathring{\bigcup}_{m\in\mathbb{N}} A_m\right) \leq \sum_{m\in\mathbb{N}} |\mu|\left(A_m\right)$. Hence $|\mu|$ is a **measure**.

9.2 The minimal range of a set of complex numbers

For any n complex $z_1, ..., z_n$ there is a subset $S \subset \{1; ...; n\}$ with $\left|\sum_{k\in S} z_k\right| \geq \frac{1}{\pi} \sum\limits_{i=1}^{n} \left|z_i\right|$.

Proof: For $z_i = \left|z_i\right| \cdot e^{i\alpha_i}$ and $-\pi \leq \vartheta \leq \pi$ let $S(\vartheta) := \left\{1 \leq k \leq n : \cos\left(\alpha_k - \vartheta\right) > 0\right\}$. Then for every such ϑ we have $\left|\sum_{k\in S} z_k\right| = \left|\sum_{k\in S} e^{-i\vartheta} \cdot z_k\right| \geq \operatorname{Re}\left(\sum_{k\in S} e^{-i\vartheta} \cdot z_k\right) = \sum_{k\in S} \left|z_k\right| \cdot \cos\left(\alpha_k - \vartheta\right) \geq \sum\limits_{i=1}^{n} \left|z_i\right| \cdot \cos^{+}\left(\alpha_k - \vartheta\right)$ and the maximal value of the sum on the right hand side attained for say $\vartheta = \vartheta_0$ is not less than the average $\frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\sum\limits_{i=1}^{n} \left|z_i\right| \cdot \cos^{+}\left(\alpha_k - \vartheta\right)\right) d\vartheta = \frac{1}{\pi} \sum\limits_{i=1}^{n} \left|z_i\right|$ which proves the lemma for $S := S\left(\vartheta_0\right)$.

9.3 The total variation of complex measures

The total variation $|\mu|$ of a complex measure μ is **finite**.

Proof: Assuming $|\mu|(X) = \infty$ there must be a partition $(A_i)_{i\in\mathbb{N}} \subset \mathcal{A}$ of X and an $n \in \mathbb{N}$ with $\frac{1}{\pi}\sum\limits_{i=1}^{n} |\mu(A_i)| > |\mu(X)| + 1$. Due to 9.2 there is a subset $S \subset \{1;...;n\}$ such that for $B_1 := \bigcup_{k\in S} A_i$ on the one hand $|\mu(B_1)| = |\sum_{k\in S} \mu(A_i)| > |\mu(X)| + 1 \geq 1$ and on the other hand $|\mu(X \setminus B_1)| = |\mu(X) - \mu(B_1)| \geq |\mu(B_1)| - |\mu(X)| \geq 1$. According to the hypothesis we have either $|\mu|(B_1) = \infty$ or $|\mu|(X \setminus B_1) = \infty$ and assuming this being the case for $X \setminus B_1$ we can repeat the argument from above to split off a subset $B_2 \subset X \setminus B_1$ with $|\mu|(X \setminus (B_1 \cup B_2)) = \infty$ and $|\mu(B_2)| \geq 1$. Hence by induction we obtain a sequence $(B_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ of paiwise disjoint sets B_n with $|\mu(B_n)| \geq 1 \forall n \in \mathbb{N}$ and consequently $\left|\mu\left(\mathring{\bigcup}_{n\in\mathbb{N}} B_n\right)\right| = |\sum_{n\in\mathbb{N}} \mu(B_n)| = \infty$ contrary tor the finite character of μ according to definition 9.1.

9.4 The Banach space of complex measures

The set $\mathcal{M}(\mathcal{A}, \mathbb{C})$ of complex measures on a measurable space $(X; \mathcal{A})$ with the operations $(\lambda + \mu)(A) := \lambda(A) + \mu(A)$ resp. $(c \cdot \lambda)(A) := c \cdot \lambda(A)$ for $A \in \mathcal{A}$, $c \in \mathbb{C}$, $\lambda, \mu \in \mathcal{M}$ and the **norm** $\|\mu\| := |\mu|(X)$ is a **Banach space**.

Proof: The vector space axioms are clearly satisfied. The **positive definiteness** $\|\mu\| = 0 \Rightarrow \mu = 0$ follows from the **monotonicity** $A \subset B \Rightarrow |\mu|(A) \leq |\mu|(B)$ of the total variation. With regard to the **completeness** for every Cauchy sequence $(\mu_n)_{n\in\mathbb{N}} \subset \mathcal{M}(\mathcal{A}, \mathbb{C})$ and every measurable set $A \in \mathcal{A}$ we have $|\mu_n(A) - \mu_m(A)| = |(\mu_n - \mu_m)(A)| \leq |\mu_n - \mu_m|(A) \leq |\mu_n - \mu_m|(X) = \|\mu_n - \mu_m\|$ such that the corresponding Cauchy sequence $(\mu_n(A))_{n\in\mathbb{N}} \subset \mathbb{C}$ converges to a complex number $\mu(A)$ hence defining a complex set function $\mu : \mathcal{A} \to \mathbb{C}$. For a sequence of disjoint measurable sets $(A_k)_{k\in\mathbb{N}} \subset \mathcal{A}$ and every $k \in \mathbb{N}$ there is an $n_k \in \mathbb{N}$ with $|\mu_n(A_k) - \mu(A_k)| \leq \epsilon 2^{-k}$ for every $n \geq n_k$ such that for every $N \geq \max\{n_k : k \leq m\}$ and $\sum\limits_{k=0}^{m} \mu_N(A_k) = \mu_N\left(\bigcup\limits_{k=0}^{m} A_k\right)$ we have $\left|\sum\limits_{k=0}^{m} \mu(A_k) - \mu\left(\bigcup\limits_{k=0}^{m} A_k\right)\right| = \left|\sum\limits_{k=0}^{m} \mu(A_k) - \sum\limits_{k=0}^{m} \mu_N(A_k) + \mu_N\left(\bigcup\limits_{k=0}^{m} A_k\right) - \mu\left(\bigcup\limits_{k=0}^{m} A_k\right)\right| \leq \epsilon 2^{-m+1} + \left|\mu_N\left(\bigcup\limits_{k=0}^{m} A_k\right) - \mu\left(\bigcup\limits_{k=0}^{m} A_k\right)\right| \leq \epsilon 2^{-m+2}$ for a suitably large N. Since ϵ and m are arbitrary we have shown the σ-additivity $\sum\limits_{k=0}^{\infty} \mu(A_k) = \mu\left(\bigcup\limits_{k=0}^{\infty} A_k\right)$, i.e. $\mu \in \mathcal{M}$. Assuming there is an $\epsilon > 0$ with $\|\mu - \mu_n\| = \sup\left\{\sum_{k\in\mathbb{N}} |(\mu - \mu_n)(A_k)| : (A_k)_{k\in\mathbb{N}} \subset \mathcal{A} : \mathring{\bigcup}_{k\in\mathbb{N}} A_k = X\right\} \leq \epsilon$ for every $n \in \mathbb{N}$ we find an $B_n = \bigcup\limits_{k=0}^{K_n} A_k \in \mathcal{A}$ with $|(\mu - \mu_n)(B_n)| \geq \frac{\epsilon}{2}$ whence $|(\mu - \mu_n)(B)| \geq \frac{\epsilon}{2}$ for $B = \bigcup_{n\in\mathbb{N}} B_n$ and every $n \in \mathbb{N}$ contrary to $(\mu_n(B))$ converging to $\mu(B)$. Hence $\lim\limits_{n\to\infty} \|\mu - \mu_n\| = 0$

9.5 Continuous and singular measures

A complex or positive measure μ is **λ-absolutely continuous** with respect to the **positive** measure λ on the same measurable space $(X, \mathcal{A})$ with the notation $\mu \lessdot \lambda$ iff $\lambda(A) = 0 \Rightarrow \mu(A) = 0 \forall A \in \mathcal{A}$. The measure μ is **concentrated** on the set $A \in \mathcal{A}$ iff $\lambda(B) = \mu(B \cap A) \forall B \in \mathcal{A}$ resp. $\mu(B) = 0 \Leftrightarrow A \cap B = \emptyset$. The measures μ and λ are **mutually singular** with the notation $\mu \perp \lambda$ iff μ and λ are concentrated on two disjoint sets. These relations have the following properties:

1. If μ is concentrated on A the so is $|\mu|$ since for every partition $(E_m)_{m\in\mathbb{N}}$ of the set $E \in \mathcal{A}$ with $E \cap A = \emptyset$ we have $\mu(E_m) = 0 \, \forall m \in \mathbb{N}$.
2. $\mu \perp \lambda \Rightarrow |\mu| \perp |\lambda|$ due to 1.
3. $\mu \lessdot \lambda \Rightarrow |\mu| \lessdot \lambda$ since from $\lambda(A) = 0$ for every partition $(A_m)_{m\in\mathbb{N}}$ of A follows $\mu(A_m) = \lambda(A_m) = 0 \, \forall m \in \mathbb{N}$.

4. $\mu \perp \lambda \wedge \mu \lessdot \lambda \Rightarrow \mu = 0$ is obvious.
5. $\mu_1 \perp \lambda \wedge \mu_2 \perp \lambda \Rightarrow \mu_1 + \mu_2 \perp \lambda$ since if μ_1, μ_2 and λ are concentrated on A_1, A_2 resp. B with $A_1 \cap B = A_2 \cap B = \emptyset$ the measure $\mu_1 + \mu_2$ is concentrated on $A_1 \cup A_2$ with $(A_1 \cup A_2) \cap B = \emptyset$.
6. $\mu_1 \lessdot \lambda \wedge \mu_2 \lessdot \lambda \Rightarrow \mu_1 + \mu_2 \lessdot \lambda$ is obvious.
7. $\mu_1 \perp \lambda \wedge \mu_2 \lessdot \lambda \Rightarrow \mu_1 \perp \mu_2$ since if μ_1 is concentrated on A we have $\mu_1 (A) \neq 0$ and hence $\mu_2 (A) = \lambda (A) = 0$, i.e. μ_2 is concentrated on $X \setminus A$.

9.6 ϵ-δ-definition of absolute contiuity

A **complex** measure μ is **absolutely continuous** with respect to the **positive** measure λ iff for every $\epsilon > 0$ exists a $\delta > 0$ such that for every $A \in \mathcal{A}$ holds: $\lambda (A) < \delta \Rightarrow |\mu| (A) < \epsilon$.

Proof:

$\Rightarrow$: Assuming an $\epsilon > 0$ and a sequence $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ with $\lambda (A_n) < 2^{-n}$ but $|\mu (A_n)| \geq \epsilon$ then $(B_m)_{m\in\mathbb{N}} \subset \mathcal{A}$ with $B_m = \bigcup_{n\geq m} A_n$ is a decreasing sequence of measurable sets with $\lambda (B_m) < 2^{-m+1}$ and $\lambda \left(\bigcap_{m\in\mathbb{N}} B_m\right) = 0$ on account of the **continuity from above** 2.2.3. But the measure $|\mu|$ is also continuous from above such that $|\mu| \left(\bigcap_{m\in\mathbb{N}} B_m\right) = \lim_{m\to\infty} |\mu| (B_m) \geq \inf_{m\in\mathbb{N}} |\mu| (A_m) \geq \epsilon$ contrary to the hypothesis $|\mu| \lessdot \lambda$ resp. 9.5.3.

$\Leftarrow$: $\lambda (A) = 0 \Rightarrow |\mu| (A) < \epsilon \forall \epsilon > 0 \Rightarrow |\mu (A)| \leq |\mu| (A) = 0$.

9.7 The Jordan decomposition of signed measures

The real and complex parts of complex measures are **finite** and are called **signed measures** to distinguish them from the **positive measures**. The **Jordan decomposition** $\mu = \mu^+ - \mu^-$ resp. $|\mu| = \mu^+ + \mu^-$ of a signed measure μ splits it into its **positive and negative variations** $\mu^+ = \frac{1}{2} (|\mu| + \mu)$ resp. $\mu^- = \frac{1}{2} (|\mu| - \mu)$ both being **finite** and **positive**. On account of the **σ-additivity** the **total variation** of a **positive signed measure** coincides with the measure itself: $|\mu^+| = \mu^+$ bzw. $|\mu^-| = \mu^-$.

9.8 The theorem of Lebesgue Radon-Nikodym

For a **positive, σ-finite measure** $\lambda : \mathcal{A} \to [0;\infty]$ and a **complex measure** $\mu : \mathcal{A} \to \mathbb{C}$ on a common measurable space $(X;\mathcal{A})$ exist:

1. a uniquely determined **Lebesgue decomposition** of $\mu = \mu_a + \mu_s$ with respect to λ into two **complex** measures μ_a and μ_s such that $\mu_a \lessdot \lambda$ and $\mu_s \perp \lambda$.
2. a uniquely determined **Radon-Nikodym density** or **derivative** $\frac{d\mu_a}{d\lambda} \in L^1 (\lambda)$ with $\mu_a (A) = \int_A \frac{d\mu_a}{d\lambda} \, d\lambda$ for every $A \in \mathcal{A}$.

Proof: The **Lebesgue decomposition** is uniquely determined since for every other decomposition μ'_a and μ'_s we have $\mu'_a - \mu_a \overset{9.4.6}{\lessdot} \sum \lambda$ bzw. $\mu_s - \mu'_s \overset{9.4.5}{\perp} \lambda$ and hence $\mu'_a - \mu_a \overset{9.4.4}{=} \mu_s - \mu'_s = 0$. The uniqueness of the **Radon-Nikodym density** follows from 5.6.3 resp. 9.6.

We start the **construction of the decomposition** with $w = \sum_{n\in\mathbb{N}} \frac{\chi_{A_n}}{2^{n+1}\cdot(1+\lambda(A_n))} : X \to]0;1[$ for a countable cover $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ of X with $\lambda (A_n) < \infty \forall n \in \mathbb{N}$ such that the measure ν with $\nu (A) := \int_A w \, d\lambda$ is **finite** and due to $w > 0$ possesses the same **null sets** as λ. Then $\varphi = |\mu| + \nu$ is again a **positive** and **finite** measure with $\int f \, d\varphi = \int f \, d|\mu| + \int f w \, d\lambda$ for every **step function** f and due to 5.4 for **positive measurable** f. Applying 9.5.1, the **Schwarz inequality** 6.4.1 and the finite character of φ for every $f \in L^2 (\varphi)$ we obtain $|\int f \, d|\mu|| \leq \int |f| \, d\mu \leq \int |f| \, d\varphi \leq \left(\int |f|^2 \, d\varphi\right)^{\frac{1}{2}} \cdot (\varphi (X))^{\frac{1}{2}} < \infty$. In particular for every null sequence $(f_n) \subset L^2 (\varphi)$ with $(\|f_n\|_2)_n \to 0$ we have $(|\int f_n \, d|\mu||)_n \to 0$, i.e. the **linear functional** $I_\mu : L^2(\varphi) \to [0;\infty[$ with $I_\mu f = \int f\, d|\mu|$ is **continuous at the origin**. According to [13, p. 20.11] it is also **bounded** resp. **uniformly continuous** and hence a member of the **dual space** $\left(L^2(\varphi)\right)^*$. Due to [7, p 308 Th 12.5] I_μ possesses a φ-a.e. uniquely determined representant $g \in L^2(\varphi)$ with respect to the **inner product** $\int f\, d|\mu| = I_\mu f = \langle f, g\rangle = \int fg\, d\varphi$ resp. $\int (1-g) f\, d|\mu| = \int fgw\, d\lambda$ for every positive measurable f. We keep this result in mind as equation (X). Choosing $f = \chi_A$ for every $A \in \mathcal{A}$ with $\varphi(A) > 0$ we obtain $0 \le \int_A g\, d\varphi = |\mu|(A) \le \varphi(A)$ and hence φ-a.e. $0 \le g \le 1$. The **Lebesgue decomposition** of the **total variation** $|\mu| = \mu_a + \mu_s$ can now be given by $\mu_a = |\mu|_{\{g<1\}}$ and $\mu_s = |\mu|_{\{g=1\}}$: Substituting $f = \chi_{\{g=1\}}$ in equation (X) yields $0 = \int_{\{g=1\}} w\, d\lambda$ such that on account of $w(x) > 0$ follows $\lambda(\{g = 1\}) = 0$ and hence $\mu_s \perp \lambda$. The **Radon-Nikodym density** is $\frac{d\mu_a}{d\lambda} = w \sum\limits_{n=1}^{\infty} g^n$ such that $\frac{d\mu_a}{d\lambda}(x) = \frac{w(x)\cdot g(x)}{1-g(x)}$ in the case of $g(x) < 1$ and $\frac{d\mu_a}{d\lambda}(x) = \infty$ else: Substituting $f = \chi_A \cdot \sum\limits_{n=0}^{m} g^n$ in equation (X) we obtain $\int_A \left(1 - g^{m+1}\right) d|\mu| = \int_A w \cdot \sum\limits_{n=1}^{m+1} g^n\, d\lambda$ and taking recourse to **monotone convergence** 5.13 for $m \to \infty$ leads to $\mu_a(A) = \int_A \frac{d\mu_a}{d\lambda}\, d\lambda$ which also yields $\mu_a \lessdot \lambda$. The boundedness of $|\mu|$ transfers to μ_a such that $\frac{d\mu_a}{d\lambda} \in L^1(\lambda)$. The Lebesgue decomposition for the **complex** measure $\mu = \mathrm{Re}\mu + i\mathrm{Im}\mu = (\mathrm{Re}\mu)^+ - (\mathrm{Re}\mu)^- + i\left((\mathrm{Im}\mu)^+ - (\mathrm{Im}\mu)^-\right)$ is accomplished by applying the above construction four times to the positive resp. negative variation of the rea resp. imaginary part of μ.

9.9 Polar representation of complex measures

For every **complex** measure μ exists a measurable complex function $\frac{d\mu}{d|\mu|} : X \to \mathbb{C}$ with $\left|\frac{d\mu}{d|\mu|}\right| = 1$ and $d\mu = \frac{d\mu}{d|\mu|}\, d|\mu|$.

Proof: According to the **Lebesgue-Radon-Nikodym theorem**9.8 and on account of $\mu \lessdot |\mu|$ there is a $\frac{d\mu}{d|\mu|} \in L^1$ with $d\mu = \frac{d\mu}{d|\mu|}\, d|\mu|$ which only has to be adapted to the absolute value $\left|\frac{d\mu}{d|\mu|}\right| = 1$: For a partition $(A_n)_{n\in\mathbb{N}}$ of the set $A = \left\{\left|\frac{d\mu}{d|\mu|}\right| < r\right\}$ holds $|\mu|(A) \le \sum_{n\in\mathbb{N}} |\mu(A_n)| = \sum_{n\in\mathbb{N}} \left|\int_{A_n} \frac{d\mu}{d|\mu|}\, d|\mu|\right| \le \sum_{n\in\mathbb{N}} r \cdot |\mu|(A_n) = r \cdot |\mu|(A)$, i.e. for $r < 1$ we have $|\mu|(A) = 0$ resp. μ-a.e. $\left|\frac{d\mu}{d|\mu|}\right| \ge 1$. On the other hand for every $A \in \mathcal{A}$ with $|\mu|(A) > 0$ holds $\left|\frac{1}{|\mu|(A)} \int_A \frac{d\mu}{d|\mu|}\, d|\mu|\right| = \frac{|\mu(A)|}{|\mu|(A)} \le 1$ so that we can apply the **mean value theorem** 5.20 with $S = \overline{B}_1(0)$ to obtain μ-a.e. $\left|\frac{d\mu}{d|\mu|}\right| \le 1$. Hence the assertion holds μ-a.e. and by redefining $\frac{d\mu}{d|\mu|} := 1$ on the μ-null set $\left\{\frac{d\mu}{d|\mu|} \ne 1\right\}$ we obtain the desired absolute value for every $x \in X$.

9.10 The density of the total variation

For a **positive** measure λ and $h \in L^1(\lambda)$ with $d\mu = \frac{d\mu}{d\lambda}\, d\lambda$ we have $d|\mu| = \left|\frac{d\mu}{d\lambda}\right|\, d\lambda$.

Proof: Owing to 9.9 there is a $\frac{d\mu}{d|\mu|}$ with $\left|\frac{d\mu}{d|\mu|}\right| = 1$ so that $d\mu = \frac{d\mu}{d|\mu|}\, d|\mu|$ and hence $\frac{d\mu}{d|\mu|}\, d|\mu| = \frac{d\mu}{d\lambda}\, d\lambda$ resp. $d\,|\mu| = \overline{\frac{d\mu}{d|\mu|}} \frac{d\mu}{d\lambda}\, d\lambda$. From $|\mu| \ge 0$ and $\lambda \ge 0$ follows λ-a.e. $\overline{\frac{d\mu}{d|\mu|}} \frac{d\mu}{d\lambda} \ge 0$ and hence $\overline{\frac{d\mu}{d|\mu|}} \frac{d\mu}{d\lambda} = \left|\frac{d\mu}{d\lambda}\right|$.

9.11 Decomposition of complex measures

Every **complex** measure μ can be decomposed into four **positive** and **finite** measures according to $\mu = \mathrm{Re}\mu^+ - \mathrm{Re}\mu^- + i\left(\mathrm{Im}\mu^+ - \mathrm{Im}\mu^-\right)$.

Proof: Owing to 9.9 and the additivity of the integral for every measurable A we have $\mu(A) = \int \chi_A (\mathrm{Re}h)^+\, d|\mu| - \int \chi_A (\mathrm{Re}h)^-\, d|\mu| + i\left(\int \chi_A (\mathrm{Im}h)^+\, d|\mu| - \int \chi_A (\mathrm{Im}h)^-\, d|\mu|\right)$. Each of the four summands is a positive and finite measure with the σ-additivity resulting from the **monotone convergence** 5.13 in the form of $\mu\left(\mathring{\bigcup}_{n\in\mathbb{N}} A_n\right) = \int \left(\sum_{n\in\mathbb{N}} \chi_A\right) g d|\mu| = \sum_{n\in\mathbb{N}} \int \chi_A g d|\mu| = \sum_{n\in\mathbb{N}} \mu(A_n)$ for every **positive** and **real measurable** g.

9.12 The Hahn decomposition for signed measures

The **Jordan decomposition** of a **signed** measure $\mu = \mu^+ - \mu^-$ extends to the measure space $(X;\mathcal{A};\mu)$: There is a **Hahn decomposition** of X into two disjoint subsets $M^+ \cup M^- = X$ with $M^+ \cap M^- = \emptyset$ and $\mu^+ (A) = \mu (A \cap M^+)$ resp. $\mu^- (A) = \mu (A \cap M^-)$ for every $A \in \mathcal{A}$.

Proof: Due to 9.10 there is a measurable $\frac{d\mu}{d|\mu|} : X \to \{-1;1\}$ with $d\mu = \frac{d\mu}{d|\mu|}\, d|\mu|$ such that $M^+ := \left\{\frac{d\mu}{d|\mu|} = 1\right\}$ and $M^- := \left\{\frac{d\mu}{d|\mu|} = -1\right\}$ are measurable . On account of $\frac{1}{2}\left(1 + \frac{d\mu}{d|\mu|}\right) = \chi_{M^+}$ follows $\mu^+ (A) = \frac{1}{2}\left(|\mu| (A) + \mu (A)\right) = \int_A \frac{1}{2}\left(1 + \frac{d\mu}{d|\mu|}\right) d|\mu| = \mu (A \cap M^+)$ resp. $\mu^- (A) = \mu (A \cap M^-)$.

9.13 The dual space of $L^p (\lambda)$

For every **σ-finite** and **positive** measure λ and $1 < p < \infty$ the **bounded linear functional** $M : L^p (\lambda) \to \mathbb{C}$ can be expressed **uniquely** as an **integral** $Mf = \int f \frac{d\mu}{d\lambda}\, d\lambda$ for $f \in L^p (\lambda)$ with the **Radon-Nikodym density** of the measure μ defined by $\mu (A) = M\chi_A$ with respect to λ. Furthermore we have $\frac{d\mu}{d\lambda} \in L^q (\lambda)$ for $\frac{1}{p} + \frac{1}{q} = 1$ and the **norm** $\|M\|^* = \sup\left\{\left|M\left(\frac{f}{\|f\|_p}\right)\right| : f \in L^p (\lambda)\right\}$ of the linear functional satisfies $\|M\|^* = \left\|\frac{d\mu}{d\lambda}\right\|_q$, i.e. the dual space $(L^p (\lambda))^*$ is **isometric** and hence **isomorphic** to $L^q (\lambda)$.

Proof: The λ-a.e. **uniqueness** of the representant $\frac{d\mu}{d\lambda} = g$ follows from the comparison of two possible candidates g and g' with $f_1 = \chi_{\{g<g'\}}$ resp. $f_2 = \chi_{\{g>g'\}}$ by means of $\int f_1 g'\, d\lambda = \int f_1 g\, d\lambda$ and $\int f_2 g'\, d\lambda = \int f_2 g\, d\lambda$ from 5.6.3.

Before we can use 9.8 we have to show that μ is a complex measure and absolutely continuous with respect to λ. Since we need the **continuity from above** 2.2.3 in this **first part** of the proof we have to restrict our reasoning to the case $\lambda (X) < \infty$. In a **second part** we will adapt the case $\lambda (X) = \infty$ to the first part making use of the σ-finiteness of λ:

For a sequence $(A_k)_{k\in\mathbb{N}} \subset \mathcal{A}$ of pairwise disjoint measurable sets with $B_n = \mathring{\bigcup}_{0\le k\le n} A_k$ and $B = \mathring{\bigcup}_{k\in\mathbb{N}} A_k$ the **continuity from above** 2.2.3 of the measure λ yields $\lim\limits_{n\to\infty} \|\chi_B - \chi_{B_n}\|_p = \lim\limits_{n\to\infty} \left\|\chi_{B\setminus B_n}\right\|_p$ $= \lim\limits_{n\to\infty} (\lambda (B \setminus B_n))^{\frac{1}{p}} = 0$ whence from the **continuity of the functional** M follows $\lim\limits_{n\to\infty} \mu (B_n) = \mu (B)$. Hence μ is **σ-additive** and thus a **complex measure**. For a λ-null set E we have $\|\chi_E\|_p = 0$ and since $M0 = 0$ the continuity of M implies $\mu (E) = 0$, i.e. $\mu \lessdot \lambda$. Hence 9.8 provides $\frac{d\mu}{d\lambda} \in L^1 (\lambda)$ with $M\chi_A = \int \chi_A \frac{d\mu}{d\lambda}\, d\lambda$ for all $A \in \mathcal{A}$. The linearity of M guarantees $M\varphi = \int \varphi \frac{d\mu}{d\lambda}\, d\lambda$ for **step functions** $\varphi \in \mathcal{S} (X;\mathbb{C})$. According to 6.11 the **step functions** $\mathcal{S} (X;\mathbb{C})$ are dense in $L^p (\lambda)$ for every $1 \le p \le \infty$ and $\lambda (X) < \infty$. For now we apply only the case $p = \infty$, i.e. we extend the proposition to $f \in L^\infty (\lambda)$: On the left hand side a λ-a.e. bounded $f \in L^\infty (\lambda)$ is a limit of a uniformly convergent sequence $(\varphi_n)_{n\in\mathbb{N}} \subset \mathcal{S}(X)$ converging also in the p-th mean on account of $\|f\|_p \le \|f\|_\infty \cdot (\lambda (X))^{\frac{1}{p}}$ whence follows the convergence of $(M\varphi_n)_{n\in\mathbb{N}}$. On the right hand side the uniform convergence directly entails the convergence of the integral on $L^\infty (\lambda)$ due to $\left|\int f \frac{d\mu}{d\lambda}\, d\lambda\right| \le \|f\|_\infty \cdot \|g\|_1$. In order to extend the validity of the proposition to $f \in L^p (\lambda)$ we show that $g := \frac{d\mu}{d\lambda} \in L^q (\lambda)$: Let $E_n = \{|g| \ge n\}$ for $n \in \mathbb{N}$ and $f = \frac{|g|^q}{g} \cdot \chi_{E_n} \in L^\infty (\lambda)$ for $n \in \mathbb{N}$ such that $|f|^p \cdot \chi_E = |g|^{(q-1)p} \cdot \chi_E = |g|^q \cdot \chi_E = fg$. Hence we have $\int_{E_n} |g|^q\, d\lambda = \int fg\, d\lambda = \Lambda(f) \le \|\Lambda\|^* \cdot \|f\|_p = \|\Lambda\|^* \cdot \left(\int_{E_n} |g|^q\, d\lambda\right)^{\frac{1}{p}} \Leftrightarrow \left(\int_{E_n} |g|^q\, d\lambda\right)^{1-\frac{1}{p}} \le \|\Lambda\|^* \Leftrightarrow \int_{E_n} |g|^q\, d\lambda \le \|\Lambda\|^{*q}$ such that with **monotone convergence** 5.13 we obtain $\|g\|_q \le \|\Lambda\|^* < \infty$ and in particular $g = \frac{d\mu}{d\lambda} \in L^q (\lambda)$. The **Hölder inequality** 6.4.1 combined with $\left\|\frac{d\mu}{d\lambda}\right\|_q < \infty$ asserts the continuity of the mapping $f \mapsto \int f \frac{d\mu}{d\lambda}\, d\lambda$ on $L^p (\lambda)$ and since it coincides on the dense subset $\mathcal{E} (X) \subset L^p (\lambda)$ with the continuous mapping M the assertion follows for $\lambda(X) < \infty$. Another look at **Hölder** yields $\|M\|^* \le \left\|\frac{d\mu}{d\lambda}\right\|_q$ and hence the second assertion $\|M\|^* = \left\|\frac{d\mu}{d\lambda}\right\|_q$.

In the case of $\lambda(X) = \infty$ as in the proof of 9.8 we define $w = \sum_{n\in\mathbb{N}} \frac{\chi_{A_n}}{2^n\cdot(1+\lambda(A_n))} : X \to]0;1[$ for a countable cover $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ of X with $\lambda(A_n) < \infty\, \forall n \in \mathbb{N}$ such that the measure ν with $\nu(A) := \int_A w\, d\lambda$ is **finite** and on account of $w > 0$ has the same null sets as λ. Then the bijection $\omega_p : L^p(\lambda) \to L^p(\nu)$ with $\omega_p(f) = w^{-\frac{1}{p}} \cdot f$ is a **linear isometry** and $M \circ \omega_p^{-1} : L^p(\nu) \to \mathbb{C}$ is a bounded linear functional with $\left\| M \circ \omega_p^{-1} \right\|^* = \sup\left\{ \left| M\left(\frac{w^{\frac{1}{p}}\cdot\omega_p(f)}{\left(\int |\omega(f)|_p^p \cdot w d\lambda\right)^{\frac{1}{p}}} \right) \right| : \omega_p(f) \in L^p(\nu) \right\} = \sup\left\{ \left| M\left(\frac{f}{\left(\int |f|^p d\lambda\right)^{\frac{1}{p}}} \right) \right| : f \in L^p(\lambda) \right\} = \|M\|^*$. According to the **first part** of the proof there is an $\omega_q\left(\frac{d\mu}{d\lambda}\right) \in L^q(\nu)$ with $\left(M \circ \omega_p^{-1}\right)(\omega_p(f)) = \int \omega_p(f) \cdot \omega_q\left(\frac{d\mu}{d\lambda}\right) w d\lambda$ for all $\omega_p(f) \in L^p(\nu)$ resp. $Mf = \int fg\, d\lambda$ for all $f \in L^p(\lambda)$.

9.14 The case $p = q = 2$

The special case of the **Hilbert space** with $p = q = 2$ is the central argument in the proof of the **Lebesgue-Radon-Nikodym theorem** 9.8 where [7, p 308 Th 12.5] is used to find a uniquely determined representant $g \in L^2(\varphi)$ with $Mf = \langle f, g\rangle = \int fg\, d\varphi$ for the bounded functional $M \in \left(L^2(\varphi)\right)^*$ with $Mf = \int f\, d|\lambda|$. Alas the **isometry** of the two spaces is **not** an issue in this proof.

9.15 Scheffé's theorem

For **bounded positive measures** μ_n resp. μ on a common measurable space $(X; \mathcal{A})$ with $\mu_n(X) = \mu(X) < \infty$ for $n \geq 1$ and **λ-a.e. converging densities** $\lim\limits_{n\to\infty} \frac{d\mu_n}{d\lambda} = \frac{d\mu}{d\lambda}$ we have $\lim\limits_{n\to\infty} |\mu(A) - \mu_n(A)| = 0$ for every measurable $A \in \mathcal{A}$.

Proof: By the hypothesis we have $\lim\limits_{n\to\infty} \int g_n d\mu = 0$ for $g_n = \frac{d\mu}{d\lambda} - \frac{d\mu_n}{d\lambda}$. Furthermore we have the integrable **majorant** $\frac{d\mu}{d\lambda} \geq g_n^+ \geq 0$ for the positive part whence $\lim\limits_{n\to\infty} \int g_n^+ d\mu = 0$ by the **dominated convergence theorem** 5.15 and consequently $\lim\limits_{n\to\infty} \int g_n^- d\mu = \lim\limits_{n\to\infty} \int \left(g_n - g_n^+\right) d\mu = 0$. Hence $\lim\limits_{n\to\infty} |\mu(A) - \mu_n(A)| = \lim\limits_{n\to\infty} \int |g_n|\, d\mu = \lim\limits_{n\to\infty} \left(\int g_n^+ d\mu - \int g_n^- d\mu\right) = \lim\limits_{n\to\infty} \int g_n^+ d\mu - \lim\limits_{n\to\infty} \int g_n^- d\mu = 0$.

10 Measures on locally compact spaces

In this section X will always be a **locally compact space** furnished with the **Borel** σ-algebra $\mathcal{B}(X) = \sigma(\mathcal{O})$ induced by its topology $\mathcal{O}$.

10.1 Linear functionals on locally compact spaces

1. The **dual space** $(\mathcal{C}_c(X, \mathbb{C}))^*$ of the **complex linear functionals** $\Lambda : \mathcal{C}_c(X, \mathbb{C}) \to \mathbb{C}$ on the **Banach space** $\mathcal{C}_c(X, \mathbb{C})$ of **complex continuous functions** $f : X \to \mathbb{C}$ **with compact support** under the **supremum norm** $\|\|$ is furnished with the **dual norm** $\|\,\|^*$ defined by $\|\Lambda\|^* = \sup\left\{ \left| \Lambda\left(\frac{f}{\|f\|_\infty} \right) \right| : f \in \mathcal{C}_c(X, \mathbb{C}) \right\} = \sup\{|\Lambda f| : f \in \mathcal{C}_c(X, [0;1])\}$ according to 9.13 and considering $f \in \mathcal{C}_c(X, \mathbb{C}) \Rightarrow \frac{|f|}{\|f\|_\infty} \in \mathcal{C}_c(X, [0;1])$. According to [10, th. 1.10] every complex functional Λ is **bounded** and in particular **uniformly continuous** with regard to this norm whence due to [10, th. 1.10] the vector space $(\mathcal{C}_c(X, \mathbb{C}))^*$ is a **Banach space.**
2. The $\mathbb{C}$-linearity of a complex functional Λ implies $\Lambda(\text{Re}f + i\text{Im}f) = \Lambda\text{Re}f + i\Lambda\text{Im}f = \text{Re}\Lambda\text{Re}f - \text{Im}\Lambda\text{Im}f + i\text{Re}\Lambda\text{Im}f + i\text{Im}\Lambda\text{Re}f$ such that it suffices to examine **complex linear functionals** $\Lambda : \mathcal{C}_c(X, \mathbb{R}) \to \mathbb{C}$ with **real valued** arguments as e.g. in the case of $\Lambda f = \int fd(\text{Re}\mu) + i\int fd(\text{Im}\mu) = \int fd\mu$ with a **complex measure** $\mu = \text{Re}\mu + i\text{Im}\mu$ according to 9.11. A **complex linear functional** $\Lambda : \mathcal{C}_c(X, \mathbb{C}) \to \mathbb{C}$ is **positive** resp. $\Lambda \in (\mathcal{C}_c(X, \mathbb{C}))_+^*$ iff for **positive** f

the value Λf also is **positive**, i.e. **positive real part** $\text{Re}\Lambda \in (\mathcal{C}_c(X,\mathbb{R}))^*_+$ and **vanishing imaginary part** $\text{Im}\Lambda = 0$, the directly available example being the integral $\Lambda f = \int f d\lambda$ with a **positive** measure $\lambda \in \mathcal{M}(\mathcal{B}(X);\mathbb{R}^+)$.

3. Due to their **positive character** the two classes $(\mathcal{C}_c(X,\mathbb{C}))^*_+$ and $\mathcal{M}(\mathcal{B}(X);\mathbb{R}^+)$ are **not vector spaces** any more but since we still have have $\alpha\Gamma + \beta\Lambda \in (\mathcal{C}_c(X,\mathbb{C}))^*$ for every $\Gamma;\Lambda \in (\mathcal{C}_c(X,\mathbb{C}))^*$ and $\alpha;\beta \geq 0$ they are **convex cones**. The **Riesz representation theorem** 10.13 resp. 10.14 states that in fact every **positive** resp. **complex** functional can be represented as an integral with regard to a measure with corresponding properties:

10.2 Measures on locally compact spaces

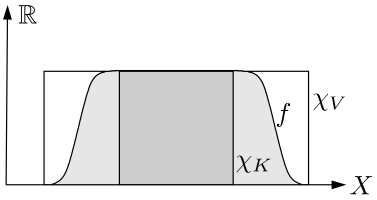
A **positive Borel measure** μ is **outer regular** iff $\mu(A) = \inf\{\mu(O) : A \subset O$ **open**$\}$ and **inner regular** iff $\mu(A) = \sup\{\mu(K) :$ **compact** $K \subset A\}$ respectively for every **measurable** $A \in \mathcal{B}(X)$. It is **regular** iff both conditions hold for every **measurable** set A and **σ-regular** if the latter condition holds for measurable sets which are either **open** or **σ-finite**. A set is **σ-finite** iff it is a countable union of sets with finite measure. Hence every **inner regular** measure is **σ-regular** and on a **σ-finite** space X every **σ-regular** measure is already **regular**. A **complex Borel measure** is regular iff its **variation** $|\mu|$ is regular.

Examples:

1. On a **Hausdorff** space X the **Dirac measure** $\epsilon_x(A) = \chi_A(x)$ for any point $x \in X$ and a **Borel** set $A \in \mathcal{B}(X)$ is **regular**.
2. The measure $\mu(A) := \begin{cases} 0 & \text{for } A \text{ countable} \\ \infty & \text{else} \end{cases}$ defined in 2.3.2 on the σ-algebra $\mathcal{B}(X) = \sigma(\mathcal{O}) = \mathcal{O} = \mathcal{P}(X)$ of a **discrete** space X is a **locally finite** and **outer regular Borel** measure. It is **inner regular** iff X is **countable**.
3. The **Lebesgue measure** $\lambda^n := \bigotimes_{1\leq i\leq n} \lambda$ on the Borel σ-algebra $\mathcal{B}^n$ of $\mathbb{R}^n$ is a **σ-finite Borel measure** owing to 7.7 resp. the **Heine-Borel theorem** [13, p. 9.10]. Its **regularity** is a consequence of the **locally compact** character of $\mathbb{R}^n$ and follows from the **Riesz representation theorem** 10.13 applied to the positive functional Λ with $\Lambda f = \int f d\lambda^n$ for $f \in C_c(\mathbb{R}^n,\mathbb{R})$.

10.3 Separation properties on locally compact spaces

For **real** $f \in C_c(X,\mathbb{R})$, **open** $V \subset X$ and **compact** $K \subset X$ we write $K \prec f$ iff $\chi_K \leq f \leq 1$ and $f \prec V$ iff $0 \leq f \leq \chi_V$. In these terms the **separation property** [13, th. 10.5] for **locally compact** spaces states that for every **compact** K and **open** $V \supset K$ there is an $f \in C_c(X,\mathbb{R})$ with $K \prec f \prec V$ resp. $\mu(K) \leq \int f d\mu \leq \mu(V)$. Since in a locally compact space the compact neighbourhoods form a **neighbourhood basis** we can strengthen this proposition to $\chi_V = \sup\{f \in C_c(X,\mathbb{R}) : f \prec V\}$.



10.4 The $\| \|_p$-closure of $\mathcal{C}_c(X,\mathbb{C})$

For every **positive σ-regular Borel** measure λ and $1 \leq p < \infty$ we have $\overline{\mathcal{C}_c(X,\mathbb{C})} = L^p(\lambda)$ with regard to $\| \|_p$.

Proof: According to 6.11.1 it suffices to find for every $A \in \mathcal{B}(X)$ with $\lambda(A) < \infty$ a function $g \in \mathcal{C}_c(X,\mathbb{R})$ such that $\|\chi_A - g\|_p = \|i\chi_A - ig\|_p < \epsilon$. Since λ is σ-regular and $\lambda(A) < \infty$ there is a compact K and an open V with $K \subset A \subset V$ and $\lambda(K) < \lambda(V) + \epsilon$ as well as a $g \in C_c(X,\mathbb{R})$

with $K \prec g \prec V$ such that $\lambda(K) \leq \int g d\lambda \leq \lambda(V)$ whence $\|\chi_A - g\|_p \leq \|\chi_A - \chi_K\|_p + \|\chi_K - g\|_p < \epsilon^{1/p} + \epsilon^{1/p}$.

10.5 The $\| \|_\infty$ closure of $\mathcal{C}_c(X, \mathbb{C})$

The closure $\mathcal{C}_0(X, \mathbb{C}) = \overline{\mathcal{C}_c(X, \mathbb{C})} \subset \mathcal{C}_0(X, \mathbb{C})$ with regard to the **supremum norm** $\| \|_\infty$ is the vector space of the **continuous functions vanishing at infinity**. These functions can be characterized by the following three equivalent conditions for every **bounded continuous** $f \in \mathcal{C}_0(X, \mathbb{C})$:

1. $f \in \mathcal{C}_0(X, \mathbb{C})$.
2. $f \in \mathcal{C}(X, \mathbb{C})$ and the sets $\{|f| \geq \epsilon\}$ are **compact** for every $\epsilon > 0$.
3. The extension $\overline{f} : \overline{X} \to \mathbb{C}$ on the **Alexandrov-compactification** $\overline{X} = X \cup \{\infty\}$ defined by $\overline{f}|_X = f$ and $\overline{f}(\infty) = 0$ is **uniformly continuous.**

Proof:

1. $\Rightarrow$ 2.: For the given $\epsilon > 0$ exists a $g \in \mathcal{C}_c(X, \mathbb{C})$ with $\|f - g\| < \frac{\epsilon}{2}$ whence the **closed** set $\{|f| \geq \epsilon\} \subset \left\{|g| \geq \frac{\epsilon}{2}\right\} \subset \text{supp} g$ is **compact** owing to [13, th. 9.4].

2. $\Rightarrow$ 3.: $\overline{f}$ is **continuous** in $x = \infty$ since according to [13, th. 10.2] the **open** sets $\left\{\left|\overline{f}\right| < \epsilon\right\} = \overline{X} \setminus \{|f| \geq \epsilon\}$ are contained in the neighbouhood basis of ∞.

3. $\Rightarrow$ 1.: For every $\epsilon > 0$ exists a **compact** $K \subset X$ with $|f(x)| = \left|\overline{f}(x) - \overline{f}(\infty)\right| \leq \epsilon$ for every $x \in X \setminus K$ and due to 10.3 there is a $g \in \mathcal{C}_c(X, \mathbb{C})$ with $K \prec g \prec X$. Then we have $f \cdot g \in \mathcal{C}_c(X, \mathbb{C})$ with $|f(x) \cdot g(x) - f(x)| = |f(x)| \cdot (1 - g(x)) \leq \epsilon$ for all $x \in X$, i.e. $\|f \cdot g - f\| \leq \epsilon$ which proves the assertion.

10.6 Lusin's Theorem

For every **positive** σ**-regular Borel** measure λ and $f : X \to \mathbb{C}$ with $\lambda(f \neq 0) < \infty$ for every $\epsilon > 0$ exists a $g \in \mathcal{C}_c(X, \mathbb{C})$ such that $\lambda(f \neq g) < \epsilon$ and $\|g\| \leq \|f\|$.

Proof: Due to $\bigcap_{n \geq 1} \{|f| \geq n\} = \emptyset$ and the **continuity** of λ **from above** there is an $n_\epsilon \in \mathbb{N}$ with $\lambda(A_1) < \frac{\epsilon}{4}$ for $A_1 = \{|f| \geq n_\epsilon\}$ and hence $f \in L^1(\lambda')$ with $\lambda' = \lambda|_{X \setminus A_1}$. According to 10.4 there is a sequence $(f_n)_{n \in \mathbb{N}} \subset \mathcal{C}_c(X \setminus A_1; \mathbb{C})$ converging **in mean** to f and according to 5.11 we have a subsequence **uniformly** converging on $X \setminus (A_1 \cup A_2)$ with $\lambda(A_2) < \frac{\epsilon}{4}$ to f and consequently $f \in \mathcal{C}(X \setminus (A_1 \cup A_2); \mathbb{C})$. By the σ-regularity we find a **compact** $K \subset \{f \neq 0\} \setminus (A_1 \cup A_2)$ with $\lambda(A_3) < \frac{\epsilon}{4}$ for $A_3 = \{f \neq 0\} \setminus (K \cup A_1 \cup A_2)$ and $f \in \mathcal{C}(K, \mathbb{C})$. Since in a locally compact space the compact neighbourhoods form a **neighbourhood basis** we find an **open** set $V \subset K$ with **compact closure** $\overline{V}$ which due to the **outer regularity** of λ we can choose such that w.l.o.g. $\lambda(A_4) < \frac{\epsilon}{4}$ for $A_4 = V \setminus K$. The compact set $\overline{V}$ is also **normal** such that we can apply **Tietze's extension theorem** [13, p. 8.5] to find $\text{Re} g^*$ resp. $\text{Im} g^* \in \mathcal{C}\left(\overline{V}, \mathbb{R}\right)$ coinciding with $\text{Re} f$ resp. $\text{Im} f$ on K and vanishing on the closed boundary $\overline{V} \setminus V$. Extending $g^* = \text{Re} g^* + i \text{Im} g^*$ to X by assigning the value 0 outside $\overline{V}$ we obtain a $g \in \mathcal{C}_c(X, \mathbb{C})$ coinciding with f on $X \setminus A_\epsilon \subset K \cup X \setminus (A_1 \cup A_2 \cup \{f \neq 0\} \cup V)$ with $A_\epsilon = A_1 \cup A_2 \cup A_3 \cup A_4$ and $\lambda(A_\epsilon) < \epsilon$. In order to **scale** g according to $\|g\| \leq \|f\|$ we define a continuous $h : \mathbb{C} \to \mathbb{C}$ by $h(z) = z$ if $|z| \leq \|f\|$ and $h(z) = \|f\| \cdot \frac{z}{|z|}$ otherwise such that $\|h \circ g\| \leq \|f\|$.

10.7 The Vitali-Carathéodory theorem

For every **Lebesgue integrable** $f \in L^1(X; \mathbb{R})$with **positive** σ**-regular Borel** measure λ and $\epsilon > 0$ there are **bounded** and **upper** resp. **lower semicontinuous** (cf. [13, p. 3.3]) functions $u; v : X \to \mathbb{R}$ such that $u \leq f \leq v$ and $\int (v - u) d\lambda < \epsilon$.

Proof: We start with $f \geq 0$ which due to 5.4 has an **approximating sequence** $(\varphi_n)_{n\in\mathbb{N}} \subset \mathcal{S}\left(X;\mathbb{R}_0^+\right)$ of step functions such that for every $k \geq 1$ there is an $n_k \geq 1$ with $|\varphi_{n_k}(x) - f(x)| < \frac{1}{k}$ for every $x \in Z_k \subset X$ with $\mu(Z_k) < \frac{1}{k}$ such that the sequence $\sup\limits_{k\leq m}\left(\varphi_{n_k} \cdot \chi_{Z_k} - \frac{2}{k}\right)_{m\in\mathbb{N}} \subset \mathcal{S}\left(X;\mathbb{R}_0^+\right)$ is **increasing** and μ-a.e. converges to f. Hence we have measurable sets $(A_i)_{i\in\mathbb{N}} \subset \mathcal{B}(X)$ and positive $c_i > 0$ such that $f = \sum_{i\geq 1} c_i\chi_{A_i}$and $\sum_{i\geq 1} c_i \cdot \lambda(A_i) = \int f d\lambda < \infty$. Due to the regularity of λ there are **compact** K_i and **open** V_i with $K_i \subset A_i \subset V_i$ and $\lambda(V_i \setminus K_i) < \frac{\epsilon}{c_i \cdot 2^{i+1}}$. Then $u =\sum\limits_{i=1}^{N} c_i\chi_{K_i}$ with $\sum_{i\geq N} c_i{\cdot}\lambda(A_i) < \frac{\epsilon}{2}$ is **upper semicontinuous**, $v = \sum_{i\geq 1} c_i\chi_{V_i}$ is **lower semicontinuous** , $u \leq f \leq v$ and $\int (v-u)\,d\lambda = \sum\limits_{i=1}^{N} c_i \cdot \lambda(V_i \setminus K_i) + \sum_{i\geq N} c_i\lambda(V_i) \leq \sum_{i\geq 1} c_i \cdot \lambda(V_i \setminus K_i) + \sum_{i\geq N} c_i\lambda(A_i) < \frac{\epsilon}{2} + \frac{\epsilon}{2}$. In the general case we apply the first step to the positive and negative parts of $f = f^+ - f^-$ to find corresponding $u^+ \leq f^+ \leq v^+$ resp. $u^- \leq f^- \leq v^-$ such that $u = u^+ - v^- \leq f$ is upper semicontinuous, $v = v^+ - u^- \geq f$ is lower semicontinuous and $\int (v-u)\,d\lambda = \int (v^+ - u^+)\,d\lambda - \int (v^- - u^-)\,d\lambda < \epsilon$.

10.8 Positive functionals are bounded on compact sets

Every **positive** $\Lambda : \mathcal{C}_c(X,\mathbb{C}) \to \mathbb{C}$ is **bounded** on $\mathcal{C}_K(X,\mathbb{C})$ for every **compact** K.

Proof: Due to the **separation property** [13, p. 10.5] of locally compact spaces already cited in 10.1 there is a continuous $g : X \to [0;1]$ with $g^{-1}(\{1\}) = K$ and compact support. Then for $f \in \mathcal{C}_K(X,\mathbb{C})$ we have $\|\mathrm{Re}f\| \cdot g \pm \mathrm{Re}f \geq 0$ whence $\Lambda(\|\mathrm{Re}f\| \cdot g) \pm \Lambda(\mathrm{Re}f) = \|\mathrm{Re}f\| \cdot \Lambda g \pm \Lambda(\mathrm{Re}f) \geq 0$, i.e. $\Lambda\left(\frac{\mathrm{Re}f}{\|f\|}\right) \leq \Lambda\left(\frac{\mathrm{Re}f}{\|\mathrm{Re}f\|}\right) \leq \Lambda g$ and since the same is true for $\mathrm{Im}f$ we obtain $\left|\Lambda\left(\frac{f}{\|f\|}\right)\right| = \left|\Lambda\left(\frac{\mathrm{Re}f}{\|f\|}\right) + i\Lambda\left(\frac{\mathrm{Im}f}{\|f\|}\right)\right| \leq \sqrt{2} \cdot \Lambda g < \infty$.

10.9 Decomposition of complex and bounded real functionals

1. Every **bounded real** functional $\Lambda \in (\mathcal{C}_c(X,\mathbb{R}))^*$ has a **decomposition** $\Lambda = \Lambda^+ - \Lambda^-$ with **positive real and bounded** $\Lambda^+;\Lambda^- \in (\mathcal{C}_c(X,\mathbb{R}))^*$.
2. Every **complex** functional $\Lambda \in (\mathcal{C}_c(X,\mathbb{R}))^*$ allows the decomposition into four **positive real and bounded** functionals $\mathrm{Re}\Lambda^+;\mathrm{Re}\Lambda^-;\mathrm{Im}\Lambda^+;\mathrm{Im}\Lambda^- \in (\mathcal{C}_c(X,\mathbb{R}))^*$ such that $\Lambda f = \mathrm{Re}\Lambda^+ f - \mathrm{Re}\Lambda^- f + i\left(\mathrm{Im}\Lambda^+ f + \mathrm{Im}\Lambda^- f\right)$.

Note: Recall that according to the definition in 10.1 a **positive complex linear functional** $\Lambda : C_c(X,\mathbb{C}) \to \mathbb{C}$ has a **positive real part** $\mathrm{Re}\Lambda \in (C_c(X,\mathbb{R}))_+^*$ and a **vanishing imaginary part** $\mathrm{Im}\Lambda = 0$ whence the decompostion from 2. extends to **positive** functionals $\Lambda : \mathcal{C}_c(X,\mathbb{C}) \to \mathbb{C}$ as in the following theorem:

Proof:

1. For **positive** $f \in \mathcal{C}_c(X,\mathbb{R})$ define $\Lambda^+ f := \sup\{\Lambda g : g \in \mathcal{C}_c(X,\mathbb{R});0 \leq g \leq f\}$ such that $0 \leq \Lambda^+ f \leq \|\Lambda\|^* \|f\|$, i.e. $\Lambda^+$is **positive and bounded**. For **positive** $c \in \mathbb{R}$ we have $g \leq cf \Leftrightarrow g = cg' : g' \leq f$ for any positive $g;g' \in \mathcal{C}_c(X,\mathbb{R})$ such that $\Lambda^+(cf) = c\Lambda^+ f$ thus establishing conformity with **scalar muplitplication**. With regard to additivity we take any positive $f_1;f_2;g_1;g_2;g \in \mathcal{C}_c(X,\mathbb{R})$ with $g_1 \leq f_1$, $g_2 \leq f_2$ resp. $g \leq f_1 + f_2$ in order to note that $\Lambda^+ f_1 + \Lambda^+ f_2 = \sup\Lambda^+ g_1 + \sup\Lambda^+ g_2 = \sup\left(\Lambda^+ g_1 + \Lambda^+ g_2\right) = \sup\Lambda^+(g_1 + g_2) \leq \sup\Lambda^+ g = \Lambda^+(f_1 + f_2)$ and conversely $\inf(g;f_1) \leq f_1$ resp. $g - \inf(g;f_1) \leq f_2$ hence $\Lambda^+ g \leq \Lambda^+ f_1 + \Lambda^+ f_2$,i.e. $\Lambda^+(f_1 + f_2) = \sup\Lambda^+ g \leq \Lambda^+ f_1 + \Lambda^+ f_2$ thus demonstrating **additivity**. We extend Λ^+ to **real** $f \in \mathcal{C}_c(X,\mathbb{R})$ with decomposition $f = f^+ - f^-$with **positive** $f^+;f^- \in \mathcal{C}_c(X,\mathbb{R})$ by means of $\Lambda^+ f := \Lambda^+ f^+ - \Lambda^+ f^-$ being independent of the choice of the decomposition and hence **well defined** as well as **linear** on account of the linearity of the components. The same is true for $\Lambda^- := \Lambda - \Lambda^+$which completes the proof.
2. directly follows from 1.

10.10 The outer measure of a positive functional

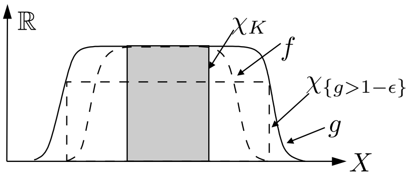
For every **positive** functional $\Lambda : \mathcal{C}_c(X, \mathbb{C}) \to \mathbb{C}$ the set function $\mu : \mathcal{P}(X) \to [0;\infty]$ defined by $\mu(V) = \sup\{\Lambda g : g \prec V\}$ for **open** $V \subset X$ and $\mu(A) = \inf\{\mu(V) : A \subset V \,\textbf{open}\}$ for arbitrary $A \subset X$ is an **outer measure** according to 10.1 with the **additional regularity property** $\mu(K) \le \Lambda g \le \mu(V)$ for any **compact** K, **open** V and $g \in C_c(X, \mathbb{R})$ with $K \prec g \prec V$.

Proof: Obviously we have $\mu(\emptyset) = 0$ and $\mu(A) \le \mu(B)$ if $A \subset B$. The **subadditivity** requires more attention. We start with $\mu(U \cup V) \le \mu(U) + \mu(V)$ for **open** U and V: Let $f \prec U \cup V$ and $\Phi = \{\sup(g;h) : g \prec U; h \prec V\}$ and $\Phi_f = \left\{\inf\left(f;\overline{f}\right) : \overline{f} \in \Phi\right\}$. Then $f = \sup \Phi_f \le \sup \Phi = \chi_{U \cup V}$ such that on account of **Dini's theorem** [13, p. 9.12] and the **continuity** of Λ we have

$$\begin{aligned} \Lambda f &= \Lambda \sup \Phi_f \\ &= \sup \Lambda \Phi_f \\ &= \sup\{\Lambda(\inf(f; \sup(g;h))) : g \prec U; h \prec V\} \\ &\le \sup\{\Lambda(\inf(f;g) + \inf(f;h)) : g \prec U; h \prec V\} \\ &\le \sup\{\Lambda(g+h) : g \prec U; h \prec V\} \le \mu(U) + \mu(V). \end{aligned}$$

Since this estimate holds for every $f \prec U \cup V$ we obtain the subadditivity for open sets. In order to show the σ-subadditivity 3.2.3 we take a sequence $(A_n)_{n\in\mathbb{N}}$ of arbitrary subsets with $A = \bigcup_{n\in\mathbb{N}} A_n$, open sets V_n with $A_n \subset V_n$ and $\mu(A_n) \le \mu(V_n) + \epsilon 2^{-n}$ such that $A \subset V = \bigcup_{n\in\mathbb{N}} V_n$. Since any $g \prec V$ has a compact support there is an $n \in \mathbb{N}$ with $g \prec \bigcup_{k\le n} V$ and hence $\Lambda g \le \mu\left(\bigcup_{k\le n} V_n\right) \le \sum_{k\le n} \mu(V_n)$ due to the subadditivity inductively extended to finite unions. Again we use the validity of this estimate for every $g \prec V$ to infer $\mu(A) \le \mu(V) \le \sum_{n\in\mathbb{N}} \mu(A_n) + \epsilon$ thus proving the main assertion.

Concerning the **additional regularity property** we only have to show the left inequality: For any $\epsilon > 0$ we have $K \subset \{g > 1 - \epsilon\}$ and hence a $f \in C_c(X)$ with on the one hand $K \prec f \prec \{g > 1-\epsilon\}$ such that $(1-\epsilon) f \le g$, i.e. $(1-\epsilon)\Lambda f \le \Lambda g$ and on the other hand $\Lambda f \ge \mu(\{g > 1-\epsilon\}) - \epsilon \ge \mu(K) - \epsilon$ whence $(\mu(K) - \epsilon)(1-\epsilon) \le \Lambda g$ which proves the assertion.



10.11 σ-Additivity of the outer measure on sets of finite measure

The **outer measure** μ determined by Λ according to the preceding lemma 10.10 is **σ-additive** and hence a **pre-measure** on the **algebra** $\mathcal{A}(X)$ of all sets $A \subset X$ with $\mu(A) = \sup\{\mu(K) : A \supset K \text{ compact}\} < \infty$. Furthermore $\mathcal{A}(X)$ contains all **open** sets.

Proof: For brevity in this proof we omit the argument and write $\mathcal{A}$ for $\mathcal{A}(X)$.

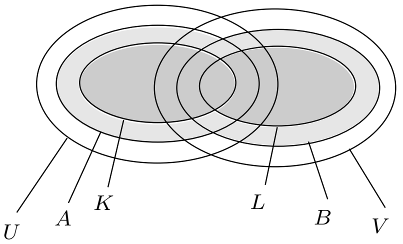
Step I. Every **compact** set K has a **finite measure** and hence belongs to $\mathcal{A}$: There is an open $V \subset K$ with compact closure $\overline{V}$ such that the separation property of locally compact spaces ensures the existence of $f, g \in C_c(X)$ with $K \prec f \prec V$ resp. $\overline{V} \prec g \prec X$ hence $g - f \ge 0 \Rightarrow \Lambda(g-f) \ge 0 \Rightarrow \Lambda f \le \Lambda g < \infty$ due to the positiv and linear character of Λ. Furthermore we can choose f such that $\mu(V) \le \Lambda f + \epsilon$ whence $\mu(K) \le \mu(V) \le \Lambda f + \epsilon \le \Lambda g + \epsilon < \infty$.

Step II. $\mathcal{A}$ contains every **open** set V: In the case of $\mu(V) = 0$ the definition of μ immediately yields $\mu(K) = \inf\{\mu(V) : K \subset V\text{open}\} = 0$ for every compact $K \subset V$. Hence we can assume $\mu(V) > 0$ and for every $\epsilon > 0$ the existence of an $f \prec V$ with $\mu(V) - \epsilon < \Lambda f < \mu(V)$ and compact support $K = \overline{\{f > 0\}}$. For every open $W \supset K$ we have $f \prec W$ and hence $\Lambda f \le \mu(K)$ and consequently $\mu(V) - \epsilon < \Lambda f \le \mu(K) < \mu(V) < \infty$ on account of $K \subset V$ and 10.10.

Step III. μ is **finitely additive** for **compact** sets: For **disjoint** and **compact** sets K, L and $\epsilon > 0$ according to the **separation property** [13, p. 10.5] of locally compact spaces choose **disjoint** and **open** $U \supset K$, $V \supset L$ and an open $W \supset K \cup L$ with $\mu(W) < \mu(K \cup L) + \epsilon$ as well as $f \prec U \cap W$ resp. $g \prec V \cap W$ with $\Lambda f > \mu(U \cap W) - \epsilon$ resp. $\Lambda g > \mu(V \cap W) - \epsilon$. We then have $\mu(K) + \mu(L) \le \mu(W \cap U) + \mu(W \cap V) \le \Lambda f + \Lambda g + 2\epsilon = \Lambda(f+g) + 2\epsilon \le \mu(W) + 2\epsilon \le \mu(K \cup L) + 3\epsilon$. Since the reverse inequality follows from the montonicity of μ we have proved the asssertion.

Step IV. μ is **σ-additive** on $\mathcal{A}$: For a sequence $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ with $A = \bigcup_{n\in\mathbb{N}} A_n$ there are compact $K_n \subset A_n$ with $\mu(A_n) \leq \mu(K_n)+\epsilon 2^{-n}$ whence $\sum\limits_{k=1}^{n} \mu(A_k) \leq \sum\limits_{k=1}^{n} \mu(K_k)+\epsilon = \mu\left(\bigcup\limits_{k=1}^{n} K_k\right)+\epsilon \leq \mu(A)+\epsilon$. Since this estimate remains valid for $n \to \infty$ and $\epsilon \to 0$ we obtain $\sum_{n\in\mathbb{N}} \mu(A_n) \leq \mu(A)$ and with the reverse inequality following from property 3.2.3 of the outer measure we have proved the assertion. Furthermore we note that for every sequence $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ the union $A = \bigcup_{n\in\mathbb{N}} A_n$ also belongs to $\mathcal{A}$ if it has finite measure, i.e. for **finite** μ the algebra $\mathcal{A}$ is a **σ-algebra**. This will be used of in the subsequent lemma to construct the actual σ-algebra $\mathcal{M}$ carrying the measure μ determined by Λ.

Step V. $\mathcal{A}$ is an **algebra**: Clearly $\emptyset \in \mathcal{A}$. For $A, B \in \mathcal{A}$ we can find compact K, L and open U, V such that $K \subset A \subset V$ resp. $L \subset B \subset V$ and $\mu(K) \leq \mu(A) \leq \mu(U) < \mu(K) + \epsilon$ resp. $\mu(L) \leq \mu(B) \leq \mu(V) < \mu(L) + \epsilon$. By the finite additivity of μ follows $\mu(U \setminus K), \mu(V \setminus L) < \epsilon$ and with $(U \cup V) \setminus (K \cup L) \subset (U \setminus K) \cup (V \setminus L)$ we get $\mu(A \cup B) < \mu(K \cup L) + 2\epsilon$ and hence $A \cup B \in \mathcal{A}$. Regarding the intersection we note that $K \setminus V \subset A \setminus B \subset V \setminus L$ and the two outer sets are **open** with $(V \setminus L) \setminus (K \setminus V) \subset (U \setminus K) \cup (V \setminus L)$ so that $A \setminus B \in \mathcal{A}$ and finally $A \cap B = B \setminus (B \setminus A) \in \mathcal{A}$.



10.12 σ-Additivity of the outer measure on $\mathcal{L}(X)$

The outer measure μ determined by Λ according to lemma 10.10 is **σ-additive** and hence a **measure** on the **Lebesgue σ-algebra** $\mathcal{L}(X) = \bigcap\limits_{K \text{ compact}} \mathcal{L}_K(X)$ with $\mathcal{L}_K(X) = \{A \subset X : A \cap K \in \mathcal{A}(X)\}$ including the **Borel σ-algebra** $\mathcal{B}(X)$ as well as the **algebra** $\mathcal{A}(X)$ of sets of finite measure introduced in the preceding lemma 10.11. $\mathcal{A}(X)$ consists precisely of all sets of **finite measure** in $\mathcal{L}(X)$. In particular μ is **complete, outer regular and σ-regular** on $\mathcal{L}(X)$.

Proof: Again we abbreviate $\mathcal{A} = \mathcal{A}(X)$ etc. Obviously we have $\mathcal{A} \subset \mathcal{L}$. According to the **step IV** of the proof of the preceding lemma the families $\mathcal{L}_K$ are σ-algebrae and so is $\mathcal{L}$. Every $\mathcal{L}_K$ contains all **closed** sets (cf. [13, p. 9.4] and hence $\mathcal{B}(X) \subset \mathcal{L}$. Every **$\mu$-null set** $A \subset X$ with $\mu(A) = 0$ is either empty or contains a point $x \in A \subset X$ and hence a compact set $\{x\} \subset A$ which must have the measure $\mu(\{x\}) = 0$ due to the **monotonicity** of μ. Hence $A \in \mathcal{L}$ and in particular μ is **complete.**

For $A \in \mathcal{L}$ with $\mu(A) < \infty$ there is an **open** $V \supset A$ with $\mu(V) < \infty$. Furthermore according to **step II** in the proof of 10.11 we can find a **compact** $K \subset V$ such that $\mu(V) < \mu(K) + \epsilon$. Since $A \cap K \in \mathcal{A}$ there is a **compact** $K_A \subset A \cap K$ such that $\mu(A \cap K) < \mu(K_A) + \epsilon$. With $A \subset (A \cap K) \cup V \setminus K$ we obtain $\mu(A) \leq \mu(A \cap K) + \mu(V \setminus K) \leq \mu(K_A) + 2\epsilon$ and since ϵ was arbitrary we have $\mu(A) = \sup\{\mu(K) : A \supset K \text{ compact}\}$ whence follows $A \in \mathcal{A}$. Finally the **σ-additivity** of μ extends from $\mathcal{A}$ to $\mathcal{L}$ since for a disjoint sequence $(A_n)_{n\in\mathbb{N}} \subset \mathcal{L}$ with $\mu(A_n) < \infty$ for all $n \in \mathbb{N}$ we have $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ such that the preceding lemma applies. In the case of $\mu(A_n) = \infty$ for an $n \in \mathbb{N}$ the σ-additivity follows from the **monotonicity** of μ. Due to its definition in 10.10 μ is **outer regular** on $\mathcal{L}$. According to 10.11 it is **inner regular** for all sets **open** or with **finite measure**. For $\epsilon > 0$ and a **σ-finite** set $A = \bigcup_{n\in\mathbb{N}} A_n$ with $\mu(A_n) < \infty$ and w.l.o.g $A_n \subset A_{n+1}$ for $n \in \mathbb{N}$ we find compact $K_n \subset A_n$ with $\mu(K_n) \geq \mu(A_n) - \frac{\epsilon}{2}$ for $n \in \mathbb{N}$. In the case of $\mu(A) < \infty$ there is an $m \in \mathbb{N}$ with $\mu(A_m) \geq \mu(A) - \frac{\epsilon}{2}$ and hence $\mu(K_m) \geq \mu(A) - \epsilon$. In the case of $\mu(A) < \infty$ for every $N \in \mathbb{N}$ there is an $m \in \mathbb{N}$ with $\mu(A_m) \geq N + \frac{\epsilon}{2}$ and hence $\mu(K_m) \geq N$. Hence we have shown that $\mu(A) = \sup\{\mu(K) : K \text{ compact with } K \subset A\}$.

10.13 The Riesz representation theorem for positive functionals

The **normed, complete** and **closed convex cone** $(\mathcal{C}_c(X,\mathbb{C}))^*_+$ of the **positive** functionals on $\mathcal{C}_c(X,\mathbb{C})$ with the **norm** $\|\|^*$defined by $\|\Lambda\|^* = \sup\left\{\left|\Lambda\left(\frac{f}{\|f\|_\infty}\right)\right| : f \in C_c(X,\mathbb{R}^+)\right\}$ is **positively isometric** and **isomorphic** to the **normed, complete** and **convex cone** $\mathcal{M}_{\sigma 0}(\mathcal{L}(X);\mathbb{R}^+)$ of the **complete, outer** and **σ-regular positive Borel** measures on a σ-algebra $\mathcal{L}(X)$ including the **Borel σ-algebra** $\mathcal{B}(X) \subset \mathcal{L}(X)$ under the **norm** $\|\|$ with $\|\mu\| = \mu(X)$ by $\mu \simeq \Lambda$ iff $\Lambda f = \int f d\mu$ for every $f \in \mathcal{C}_c(X,\mathbb{C})$.

Notes:

1. Corresponding to the restriction of the **algebraic closure** to positive scalars in a **convex cone** as given in 10.1 we define a **positive vector isomorphism** between convex cones $\mathcal{C}^*$ and $\mathcal{M}$ as a bijection $\mu : \mathcal{C}^* \to \mathcal{M}$ with $\mu_{\alpha\Lambda+\beta\Gamma} = \alpha\mu_\Lambda + \beta\mu_\Gamma$ for every $\Gamma;\Lambda \in \mathcal{C}^*$ and $\alpha;\beta \geq 0$.
2. The norm $\|\|$ on $\mathcal{M}_{\sigma 0}(\mathcal{L}(X);\mathbb{R}^+)$ induces a subclass of the **weak topology** which will be examined in 11.8.
3. Apart from above used **outer** or **principal measure** defined in 10.10 by $\mu(V) = \sup\{\Lambda g : g \prec V\}$ for **open** $V \subset X$ and $\mu(A) = \inf\{\mu(V) : A \subset V\,\textbf{open}\}$ for arbitrary $A \subset X$ there are other representation measures, among them the **inner** or **essential measure** defined in 11.1 by $\mathring{\mu}(K)$ $= \inf\{\Lambda g : K \prec g\}$ for **compact** $K \subset X$ resp. $\mathring{\mu}(A) = \sup\{\mathring{\mu}(K) : \textbf{compact}\, K \subset A\}$ for arbitrary $A \subset X$. The inner measure is **not necessarily complete** but obviously **finite on compact sets** and **inner regular**, i.e. a **Radon measure** according to the definition in 10.1. Furthermore it is **uniquely defined** by these properties whence we have a **second variation of the Riesz representation theorem**: The **convex cone** $(\mathcal{C}_c(X,\mathbb{C}))^*_+$ of the **positive** functionals on $\mathcal{C}_c(X,\mathbb{C})$ is **positively isometric** and **isomorphic** to the **convex cone** $\mathcal{M}^c_{i0}(\mathcal{B}(X);\mathbb{R}^+)$ of the **Radon** measures on $\mathcal{B}(X)$ with $\mu \simeq \Lambda$ iff $\Lambda f = \int f d\mu$ for every $f \in \mathcal{C}_c(X,\mathbb{C})$. According to 11.1 the **outer and the inner measure coincide on σ-compact spaces**.

Proof:

Step I. Uniqueness: Assuming that there are two σ-regular and complete positive Borel measures μ_1 and μ_2 such that $\int f d\mu_1 = \int f d\mu_2$ for all $f \in \mathcal{C}_c(X,\mathbb{C})$ for every $\epsilon> 0$, **compact** $K \subset X$ **open** $V \supset K$ with $\mu_2(V) < \mu_2(K) + \epsilon$ and $f \in C_c(X,\mathbb{R})$ with $K \prec f \prec V$, i.e. $\chi_K \leq f \leq \chi_V$ follows $\mu_1(K) \leq \int f d\mu_1 = \int f d\mu_2 \leq \mu_2(V) \leq \mu_2(K) + \epsilon$. and vice versa. Hence the two measures coincide on the **compact** sets and due to their **σ-regularity** this identity extends first to the **open** sets and by the **uniqueness theorem** 3.4 to all **measurable** sets.

Step II. Existence: Since the $f \in \mathcal{C}_c(X,\mathbb{C})$ are **continuous** and in particular **Borel measurable** we can restrict the measure μ determined by Λ according to lemma 10.10 on the σ-algebra $\mathcal{L}(X)$ from 10.12 to the **Borel σ-algebra** $\mathcal{B}(X) \subset \mathcal{L}(X)$. On account of $\mathrm{Re}\Lambda f = \Lambda\mathrm{Re}f$ resp. $\mathrm{Im}\Lambda f = \Lambda\mathrm{Im}f$ for positive functionals it suffices to show the equation for real f. Since $f \in \mathcal{C}_c(X) \Leftrightarrow -f \in \mathcal{C}_c(X)$ we only have to show $\Lambda f \leq \int f d\mu$ for **every** $f \in \mathcal{C}_c(X,\mathbb{R})$. Since the **step functions** defining the integral are **not continuous** we have to take recourse to a corresponding **partition of unity** consisting of **continuous** functions of compact support being amenable to Λ and providing a result which can be compared to the integral. Furthermore the general case only provides for **pointwise convergence** so that we need the **compactness** of the support $K = \overline{\{f \neq 0\}}$ in order to find elementary functions **uniformly converging** to f: For $\epsilon > 0$ let $A_k = \{k\epsilon \leq f < (k+1)\epsilon\}$ with $-n \leq k \leq n = \left[\frac{\|f\|}{\epsilon}\right]$ such that $(A_k)_{|k|\leq n}$ is a partition of the compact support $K = \bigcup\limits_{k=-n}^{n} A_k$ and $e = \sum\limits_{k=-n}^{n} k\epsilon\chi_{A_k} \in \mathcal{S}(X)$ according to 5.2 and 5.4 such that $e \leq f \leq e + \epsilon$ whence $\int e d\mu \leq \int f d\mu \leq \int e d\mu + \epsilon \cdot \mu(K)$. Due to 10.10 for every $|k| \leq n$ there is an open V_k with $A_k \subset V_k \subset \{f < e + \epsilon\}$ and $\mu(V_k) \leq \mu(A_k) + \frac{\epsilon}{n\|f\|}$. On account of [13, 8.9, 9.5 and 10.5] we can find a partition of unity $(h_k)_{|k|\leq n} \subset \mathcal{C}_c(X,\mathbb{R})$ subordinate to $(V_k)_{|k|\leq n}$ with $fh_k \prec V_k$ and $fh_k \leq (k+1)\epsilon h_k$ as well as $K \prec \sum\limits_{k=-n}^{n} h_k$ such that $\mu(K) \leq \sum\limits_{k=-n}^{n} \Lambda h_k$.

Thus we have

$$
\begin{aligned}
\Lambda f &= \sum_{k=-n}^{n} \Lambda f h_k \\
&\leq \sum_{k=-n}^{n} (k+1)\,\epsilon \Lambda h_k \\
&= \sum_{k=-n}^{n} (k\epsilon + \epsilon + \|f\|)\,\Lambda h_k - \|f\| \sum_{k=-n}^{n} \Lambda h_k \\
&\leq \sum_{k=-n}^{n} (k\epsilon + \epsilon + \|f\|)\,\mu(V_k) - \|f\|\,\mu(K) \\
&\leq \sum_{k=-n}^{n} (k\epsilon + \epsilon + \|f\|)\left(\mu(A_k) + \frac{\epsilon}{n\,\|f\|}\right) - \|f\|\,\mu(K) \\
&\leq \int e d\mu + \epsilon\mu(K) + \|f\|\,\mu(K) + 2\,(n\,(n+1)\,\epsilon + 2n\,\|f\|)\,\frac{\epsilon}{n\,\|f\|} - \|f\|\,\mu(K) \\
&= \int e d\mu + \epsilon\mu(K) + \frac{2\,(n+1)\,\epsilon^2}{\|f\|} + 2\epsilon \\
&\leq \int f d\mu + \epsilon\mu(K) + 6\epsilon.
\end{aligned}
$$

Step III. The map $\mu \mapsto \Lambda$ is **isometric**: In the case of $\mu(X) < \infty$ on the one hand we have $\|\Lambda\|^* = \sup\left\{\left|\frac{\int f d\mu}{\sup|f|}\right| : f \in \mathcal{C}_c(X, \mathbb{R})\right\} = \sup\left\{\left|\int f d\mu\right| : f \in \mathcal{C}_c(X, \mathbb{R}^+), \sup f = 1\right\} \leq \mu(X) = \|\mu\|$. On the other hand according to **Lusin's theorem** 10.6 for every $\epsilon > 0$ there exists a $g \in \mathcal{C}_c(X, \mathbb{C})$ such that $\mu(g \neq 1) < \epsilon$ and $\|g\| \leq 1$. This implies $\|\Lambda\|^* \geq \left|\int_X g d\mu\right| \geq |\mu(X \setminus \{g \neq 1\}) - \mu(g \neq 1)| \geq \mu(X) - 2\epsilon$ whence $\|\Lambda\|^* \geq \|\mu\|$. Hence in the finite case we conclude that $\|\Lambda\|^* = \|\mu\|$ and the estimates also show that $\|\Lambda\|^* < \infty$ iff $\mu(X) < \infty$.

Step IV. The **convex cone** $(\mathcal{C}_c(X, \mathbb{C}))^*_+$ is **normed, complete** and **closed**: For every $\Lambda \in (\mathcal{C}_c(X, \mathbb{C}))^* \setminus (\mathcal{C}_c(X, \mathbb{C}))^*_+$ there is an $f \in \mathcal{C}_c(X, [0; 1])$ and an $\epsilon > 0$ such that $\Lambda f < -\epsilon$. Due to [10, th. 1.10] the **bounded** functional Λ is **uniformly continuous** such that there is a $\delta > 0$ with $\Lambda[B_\delta(f)] \subset B_{\epsilon/2}(\Lambda f) \subset \mathbb{R}$. Then for every $\Gamma \in B_{\epsilon \cdot \|f\|}(\Lambda) = \left\{\Gamma \in (\mathcal{C}_c(X, \mathbb{C}))^* : \|\Gamma\|^* = \sup_{\|g\| \leq 1} \Gamma g < \epsilon \cdot \|f\|\right\}$ we have $\Gamma\left(\frac{f}{\|f\|}\right) < \epsilon$, i.e. $B_{\epsilon \cdot \|f\|}(\Lambda) \subset (\mathcal{C}_c(X, \mathbb{C}))^* \setminus (\mathcal{C}_c(X, \mathbb{C}))^*_+$. Hence $(\mathcal{C}_c(X, \mathbb{C}))^*_+$ is a closed subset of the vector space $(C_c(X, \mathbb{C}))^*$ and since according to [10, th. 7.1] the set $(\mathcal{C}_c(X, \mathbb{C}))^*_+$ is a **Banach space** the completeness follows from [13, th. 14.2.2]. The corresponding properties of $\mathcal{M}_{\sigma 0}(\mathcal{L}(X); \mathbb{R}^+)$ are a consequence of the **isometry** between the two spaces.

10.14 The Riesz representation theorem for complex functionals

The **Banach space** $(C_c(X, \mathbb{C}))^*$ with the **norm** $\|\|^*$defined by $\|\Lambda\|^* = \sup\left\{\left|\Lambda\left(\frac{f}{\|f\|_\infty}\right)\right| : f \in C_c(X, \mathbb{R}^+)\right\}$ is **isometric** and **isomorphic** to the **Banach space** $\mathcal{M}_0(\mathcal{B}(X); \mathbb{C})$ of **complex regular Borel measures** on $\mathcal{B}(X)$ under the **norm** $\|\|$ with $\|\mu\| = |\mu|(X)$ defined in 9.1 with $\mu \simeq \Lambda$ iff $\Lambda f = \int f d\mu = \int f \frac{d\mu}{d|\mu|} d|\mu|$ for every $f \in \mathcal{C}_c(X, \mathbb{C})$ (cf. 9.8).

Note: According to [10, th. 7.1] the **completeness** of the **dual space** $(C_c(X, \mathbb{C}))^*$ follows from the completeness of $\mathbb{C}$ while the corresponding property of the space $\mathcal{M}_0(\mathcal{B}(X); \mathbb{C})$ is a consequence of the **isometry** with $(C_c(X, \mathbb{C}))^*$. The **topology** on $\mathcal{M}_{\sigma 0}(\mathcal{L}(X); \mathbb{R}^+)$ induced a norm $\|\|$ will be examined in 11.8.

Proof: According to [13, th. 20.6.6] and 9.4 the **closed** vector subspace $\mathcal{M}_0(\mathcal{L}(X); \mathbb{C})$ of the **Banach** space $\mathcal{M}(\mathcal{L}(X); \mathbb{C})$ is again a **Banach** space.

The map $\mu \mapsto \Lambda$ is **well defined** and **$\mathbb{C}$-linear**: The **complete** and **regular complex measure**

$\mu = \mathrm{Re}\mu^+ - \mathrm{Re}\mu^- + i\left(\mathrm{Im}\mu^+ - \mathrm{Im}\mu^-\right)$ with

$$\mu(A) = \int \chi_A \mathrm{Re}h^+ d|\mu| - \int \chi_A \mathrm{Re}h^- d|\mu| + i\left(\int \chi_A \mathrm{Im}h^+ d|\mu| - \int \chi_A \mathrm{Im}h^- d|\mu|\right)$$

represented by four **complete** and **regular positive measures** according to 9.11 is mapped to the **complex** functional Λ with

$$\Lambda f = \int f d\mu = \int f\mathrm{Re}h^+ d|\mu| - \int f\mathrm{Re}h^- d|\mu| + i\left(\int f\mathrm{Im}h^+ d|\mu| - \int f\mathrm{Im}h^- d|\mu|\right)$$

constructed of four **positive bounded functionals** matching the four summands in the decomposition of Λ in 10.9.2. Since the range of μ resp. Λ has been extended to $\mathbb{C}$ the map is now completely $\mathbb{C}$-linear.

The map $\mu \mapsto \Lambda$ is **surjective**: For every complex functional $\Lambda = \mathrm{Re}\Lambda^+ - \mathrm{Re}\Lambda^- + i\left(\mathrm{Im}\Lambda^+ + \mathrm{Im}\Lambda^-\right)$ each **positive bounded functional** of the decomposition according to 10.9.2 is represented by an integral, e.g. $\mathrm{Re}\Lambda^+ f = \int f d\left(\mathrm{Re}\mu^+\right)$ for every $f \in \mathcal{C}_c(X, \mathbb{C})$ resp. a **complete** and **σ-regular positive Borel** measure $\mathrm{Re}\mu^+$ etc. due to the preceding version 10.13 of the **Riesz representation** theorem such that $\mu = \mathrm{Re}\mu^+ - \mathrm{Re}\mu^- + i\left(\mathrm{Im}\mu^+ - \mathrm{Im}\mu^-\right)$ is the uniquely determined **complete** and **σ-regular complex Borel** measure with $\Lambda f = \int f d\mu$ for every $f \in \mathcal{C}_c(X, \mathbb{C})$. For any **complete** and **σ-regular positive Borel** measure λ determined by a **positive bounded** functional Γ, every compact K and $f \in \mathcal{C}_c(X, [0;1])$ with $K \prec f$ according to 10.13 we have $\lambda(K) \leq \int f d\lambda \overset{11.9}{=} \Gamma f \overset{11.1}{\leq} \|\Gamma\|^* \cdot \|f\| = \|\Gamma\|^*$ and on account of the **regularity condition** follows $\|\lambda\| = \mu(X) = \sup\{\lambda(K) : K \text{ compact}\} \leq \|\Gamma\|^*$. Hence every component of μ is **finite** and since this condition transfers to μ itself it is also **regular**.

The map $\mu \mapsto \Lambda$ is **injective**: Assuming $\Lambda = 0$, i.e. $\Lambda f = \int f h d|\mu| = 0$ for every $f \in \mathcal{C}_c(X, \mathbb{C})$. Since according to 10.4 the space $\mathcal{C}_c(X, \mathbb{C})$ is **dense** in $L^1(|\mu|)$ this implies $\int \chi_A h d|\mu| = \int_A h d|\mu| = 0$ for every measurable A and hence $|\mu|$-a.e. $h = 0$. But on the other hand we have $|h| = 1$ which only leaves $|\mu|(X) = 0$, i.e. $\mu = 0$. Thus the kernel of the isomorphism $\mu \mapsto \Lambda$ contains only the trivial element 0 which implies the assertion.

The map $\mu \mapsto \Lambda$ is **isometric**: On the one hand we have $\|\Lambda\|^* = \sup\left\{\left|\frac{\int f h d|\mu|}{\sup|f|}\right| : f \in \mathcal{C}_c(X, \mathbb{R})\right\} = \sup\left\{\left|\int f h d|\mu|\right| : f \in \mathcal{C}_c\left(X, \mathbb{R}^+\right), \sup f = 1\right\} \leq |\mu|(X) = \|\mu\|$. On the other hand according to **Lusin's theorem** 10.6 for every $\epsilon > 0$ there exists a $g \in \mathcal{C}_c(X, \mathbb{C})$ such that $|\mu|\left(\overline{h} \neq g\right) < \epsilon$ and $\|g\| \leq 1$. This implies $\|\Lambda\|^* \geq \int_X g h d|\mu| \geq |\mu|\left(X \setminus \left\{\overline{h} \neq g\right\}\right) - |\mu|\left(\overline{h} \neq g\right) \geq |\mu|(X) - 2\epsilon$ whence $\|\Lambda\|^* \geq \|\mu\|$.

11 Vague convergence on locally compact spaces

11.1 The inner measure of a positive functional

On a **locally compact** space X for every **positive** functional $\Lambda : \mathcal{C}_c(X, \mathbb{C}) \to \mathbb{C}$ the **inner measure** $\mathring{\mu} : \mathcal{P}(X) \to [0;\infty]$ defined by

1. $\mathring{\mu}(K) = \inf\{\Lambda g : K \prec g\}$ for **compact** $K \subset X$ resp. $\mathring{\mu}(A) = \sup\{\mathring{\mu}(K) : \textbf{compact}\, K \subset A\}$ for arbitrary $A \subset X$

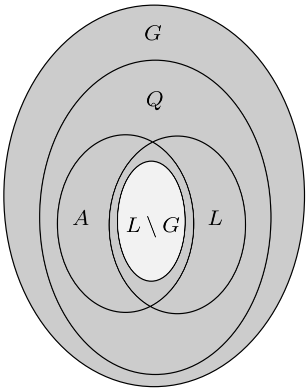
coincides with the **outer measure** μ defined in 10.10 by

2. $\mu(O) = \sup\{\Lambda g : g \prec O\}$ for **open** $O \subset X$ resp. $\mu(A) = \inf\{\mu(O) : A \subset O\, \textbf{open}\}$ for arbitrary $A \subset X$

on **Borel sets of finite outer measure** : $\mathring{\mu}(A) = \mu(A)\ \forall A \in \mathcal{B}(X)$ with $\mu(A) < \infty$.

In particular the **inner and the outer measure coincide** on the Borel σ-algebra $\mathcal{B}(X)$ of every **locally** and **σ-compact** space X.

Proof: Due to 10.3 for every **compact** K and **open** O with $K \subset O$ there is $g \in C_c(X, \mathbb{R})$ with $K \prec g \prec O$.



Step I: $\mathring{\mu}(O) = \mu(O)$ for every **open** $O \subset X$ since obviously $\mathring{\mu}(O)$ $= \sup\{\inf\{\Lambda g : K \prec g\} : \textbf{compact}\, K \subset O\} \leq \sup\{\Lambda g : g \prec O\} = \mu(O)$. Conversely the assumption $\mathring{\mu}(O) < \mu(O)$ implies the existence of a $g \prec O$ such that for every **compact** $K \subset O$ there is an $K \prec f$ with $\Lambda f < \Lambda g$. Since one of these f must coincide with g we have a contradiction whence follows the equality.

Step II: $\mathring{\mu}(K) = \mu(K)$ for every **compact** $K \subset X$ since obviously $\mathring{\mu}(K)$ $= \inf\{\Lambda g : K \prec g\} \leq \inf\{\sup\{\Lambda g : g \prec O\} : K \subset O\, \textbf{open}\} = \mu(K)$. Conversely the assumption $\mathring{\mu}(K) < \mu(K)$ implies the existence of a $K \prec g$ such that for every **open** $O \supset K$ there is an $f \prec O$ with $\Lambda g < \Lambda f$. Since one of these f must coincide with g we have a contradiction whence follows the equality.

Step III: $\mathring{\mu}(A) = \mu(A)$ for **arbitrary** $A \in \mathcal{B}(X)$ since the assumption $\mathring{\mu}(A) > \mu(A)$ implied the existence of **compact** K and **open** O with $K \subset A \subset O$ and $\mathring{\mu}(K) > \mu(O)$ whence from **step II** followed $\mu(K) = \mathring{\mu}(K) > \mu(O)$ in contradiction to the **monotonicity** of μ. Conversely for every $\epsilon > 0$ there is an **open** $U \supset A$ with $\mu(U \setminus A) = \mu(U) - \mu(A) < \frac{\epsilon}{2}$ and due to **step I** resp. **step II** there is a **compact** $L \subset U$ with $\mu(U \setminus L) = \mu(U) - \mu(L) = \mathring{\mu}(U) - \mathring{\mu}(L) < \frac{\epsilon}{2}$. Hence we have $\mu(Q) < \epsilon$ for $Q = (U \setminus A) \cup (U \setminus L) = U \setminus (A \cap L)$ and according to the definition of the outer measure an **open** $G \supset Q$ with $\mu(G) < \epsilon$. Then $K = L \setminus G \subset A$ is **compact** with $A \setminus K \subset G$ such that $\mu(A) - \mu(K) = \mu(A \setminus K) \leq \mu(G) < \epsilon$ whence $\mu(A) < \mu(K) + \epsilon = \mathring{\mu}(K) + \epsilon \leq \mathring{\mu}(A) + \epsilon$.

11.2 Radon measures

On a **locally** and **σ-compact** space X every **Radon measure** $\mu \in \mathcal{M}^c_{i0}(\mathcal{B}(X); \mathbb{R}^+)$ defined as an **inner regular positive Borel measure** μ on $\mathcal{B}(X)$ with $\mu(K) < \infty$ for every **compact** $K \subset X$ is **σ-finite** and **regular**.

Note: According to 10.2 on a **locally** and **σ-compact** space X the Radon measures $\mathcal{M}^c_{i0}(\mathcal{B}(X); \mathbb{R}^+)$ coincide with the **normed, complete** and **convex cone** of the **positive regular** measures $\mathcal{M}_0(\mathcal{B}(X); \mathbb{R}^+)$ under the **norm** $\|\|$ with $\|\mu\| = \mu(X)$. The corresponding norm topology will be examined in 11.8.

Proof: Since X is **σ-compact** and $\mu(K) < \infty$ for every **compact** $K \subset X$ the measure μ is **σ-finite**. The **regularity** is a direct consequence of the preceding theorem 11.1 and the **Riesz representation theorem for positive functionals** 10.13.

11.3 The Lebesgue measure

Since $\mathbb{R}^n$ is **σ-compact** we can apply the preceding theorem to the Lebesgue-Borel measure λ^n and obtain its **σ-finite**, **regular** and **complete** extension, the **Lebesgue measure** λ^n on the extended σ-algebra $\mathcal{L}(\mathbb{R}^n)$ of the **Lebesgue measurable** sets. A set A is **Lebesgue measurable** iff there are an F_σ-set F and a G_δ-set G such that $F \subset A \subset G$ and $\lambda^n(G \setminus F) = 0$. This follows from 10.11 resp. 10.12 and the σ-compactness of $\mathbb{R}^n$ together with the observation that for **any** σ-compact set A with $A = \bigcup_{n \in \mathbb{N}} K_n$ for a sequence of compact K_n and any other given compact K the intersection $A \cap K \in \mathcal{A}(X)$ since $\lambda^n(A \cap K) = \sup\{\lambda^n(K_n \cap K)\} < \infty$. Consequently **every Lebesgue set is the union of a Borel measurable G_δ-set and a λ^n-null set**. Thus every **Lebesgue measurable function** f coincides λ^n-a.e. with a Borel measurable function f_0 and identical integral $\int_A f d = \int_A f_0 d$ for every Lebesgue measurable A. The **translation invariance** 8.8 as well as the **transformation formula** 8.9 extend from $\mathcal{B}(X)$ to $\mathcal{L}(X)$due to the **regularity** of λ^n.

11.4 The vague topology

For a **topological space** X the **vague topology** is the **initial topology** on the family of **positive Borel measures** $\mathcal{M}\left(\mathcal{B}\left(X\right);\mathbb{R}^{+}\right)$ with regard to the maps $\{\mu \mapsto \int f d\mu : f \in \mathcal{C}_c\left(X;\mathbb{R}\right)\}$, i.e. it is the **weakest** or **smallest** topology on $\mathcal{M}\left(\mathcal{B}\left(X\right);\mathbb{R}^{+}\right)$ such that these maps are **continuous**.

In the case of a **locally compact** space X according to the **Riesz representation theorem for positive functionals** 10.13 these maps are **isometric isomorphisms** on $\mathcal{M}_{\sigma 0}\left(\mathcal{L}\left(X\right);\mathbb{R}^{+}\right)$ with regard to the much **stronger** and **larger** topology of **uniform convergence** induced by the **supremum norm**.

The vague topology on the **convex cone** of the **Radon measures** $\mathcal{M}_{i0}^{c}\left(\mathcal{B}\left(X\right);\mathbb{R}^{+}\right)$ on a **locally compact** space X is generated by the **subbasis**

$$\mathcal{S} = \left\{V_{f;\epsilon}\left(\mu\right) : \nu \in \mathcal{M}_{i0}^{c}\left(\mathcal{B}\left(X\right);\mathbb{R}^{+}\right); f \in \mathcal{C}_c\left(X;\mathbb{R}\right); \epsilon > 0\right\}$$

formed by the **neighbourhoods**

$$V_{f;\epsilon} = \left\{\mu;\nu \in \mathcal{M}_{i0}^{c}\left(\mathcal{B}\left(X\right);\mathbb{R}^{+}\right) : \left|\int f d\mu - \int f d\nu\right| < \epsilon\right\}.$$

On a **locally compact** space X a sequence $\left(\mu_n\right)_{n\geq 1} \subset \mathcal{M}_{i0}^{c}\left(\mathcal{B}\left(X\right);\mathbb{R}^{+}\right) \subset \mathcal{M}_{\sigma 0}\left(\mathcal{L}\left(X\right);\mathbb{R}^{+}\right)$ of **Radon measures vaguely converges** to a $\mu \in \mathcal{M}_{i0}^{c}\left(\mathcal{B}\left(X\right);\mathbb{R}^{+}\right)$ iff one of the following two equivalent conditions holds:

1. $\lim\limits_{n\to\infty} \int f d\mu_n = \int f d\mu$ for every $f \in \mathcal{C}_c\left(X;\mathbb{R}\right)$.
2. $\limsup\limits_{n\to\infty}\mu_n\left(K\right) \leq \mu\left(K\right)$ for every **compact** $K \subset X$ and $\liminf\limits_{n\to\infty}\mu_n\left(G\right) \geq \mu\left(G\right)$ for every **open** $G \subset X$.

Examples:

3. Since X is a **Hausdorff** space and [13, th. 9.4] for every **convergent** sequence $\left(x_n\right)_{n\geq 1} \subset X$ with $\lim\limits_{n\to\infty} x_n = x \in X$ the sequence $\left(\delta_{x_n}\right)_{n\geq 1}$ **vaguely** converges to δ_x. Obviously we have $\lim\limits_{n\to\infty} \delta_{x_n}\left(A\right) = 1 \neq 0 = \delta_x\left(A\right)$ for every open $A \subset X$ with $x \in \delta A$.
4. For every sequence $\left(x_n\right)_{n\geq 1} \subset X$ without **accumulation points** (cf. [13, th. 2.7]) in X every **compact** K contains only **finitely many** x_j (cf. [13, th. 9.2.3]) such that the sequence $\left(\delta_{x_n}\right)_{n\geq 1}$ **vaguely** converges to the **null measure** 0.

Proof:

1. $\Rightarrow$ 2. : According to 10.1 for every **compact** K there is an $f \in \mathcal{C}_c\left(X,\mathbb{R}\right)$ with $K \prec f$ whence follows $\limsup\limits_{n\to\infty}\mu_n\left(K\right) \leq \lim\limits_{n\to\infty} \int f d\mu_n = \int f d\mu$. Due to 11.1.1 and the **Riesz representation theorem** 10.13 we have $\mu\left(K\right) = \inf\left\{\int f d\mu : K \prec f\right\}$ and consequently $\limsup\limits_{n\to\infty}\mu_n\left(K\right) \leq \mu\left(K\right)$. According to 10.1 for every **open** $V \supset K$ there is an $f \in C_c\left(X,\mathbb{R}\right)$ with $f \prec V$ whence follows $\int f d\mu = \lim\limits_{n\to\infty} \int f d\mu_n \leq \liminf\limits_{n\to\infty}\mu_n\left(V\right)$. Due to 11.1.2 and 10.13 we have $\mu\left(V\right) = \sup\left\{\int f d\mu : f \prec V \,\text{open}\right\}$ and consequently $\mu\left(V\right) \leq \liminf\limits_{n\to\infty}\mu_n\left(V\right)$.

2. $\Rightarrow$ 1. :By the decomposition $f = f^{+} - f^{-}$ with $f^{+}; f^{-} \in \mathcal{C}_c\left(X;\mathbb{R}^{+}\right)$ w.l.o.g. we assume $f \in \mathcal{C}_c\left(X;\mathbb{R}^{+}\right)$. For every $m \geq 1$ we consider the **compact** sets $K_0 = \left\{f \geq 0\right\} = \mathrm{supp} f$, $K_i = \left\{f \geq \frac{i}{m}\left\|f\right\|\right\}$ and $A_i = K_{i-1} \setminus K_i$ resp. the **open** sets $G_0 = \left\{f > 0\right\} = \mathring{K}_0$, $G_i = \left\{f > \frac{i}{m}\left\|f\right\|\right\}$ and $B_i = G_{i-1} \setminus G_i$ for $1 \leq i \leq m+1$. Note that $K_{m+1} = G_m = G_{m+1} = \emptyset$. On account of $\sum\limits_{i=1}^{m+1} \frac{i-1}{m}\left\|f\right\|\chi_{A_i} \leq f < \sum\limits_{i=1}^{m+1} \frac{i}{m}\left\|f\right\|\chi_{A_i}$ resp. $\sum\limits_{i=1}^{m} \frac{i-1}{m}\left\|f\right\|\chi_{B_i} < f \leq \sum\limits_{i=1}^{m} \frac{i}{m}\left\|f\right\|\chi_{B_i}$ for every **Radon measure** $\nu \in \mathcal{M}_0^{c}\left(\mathcal{B}\left(X\right);\mathbb{R}^{+}\right)$ follows $\sum\limits_{i=1}^{m+1} \frac{i-1}{m}\left\|f\right\|\nu\left(A_i\right) \leq \int f d\nu < \sum\limits_{i=1}^{m+1} \frac{i}{m}\left\|f\right\|\nu\left(A_i\right)$ resp.

$\sum\limits_{i=1}^{m} \frac{i-1}{m} \|f\| \nu(B_i) < \int f d\nu \leq \sum\limits_{i=1}^{m} \frac{i}{m} \|f\| \nu(B_i)$. For $1 \leq i \leq m+1$ we have $\nu(A_i) = \nu(K_{i-1}) - \nu(K_i)$ resp. $\nu(B_i) = \nu(G_{i-1}) - \nu(G_i)$ whence $\frac{1}{m} \sum\limits_{i=0}^{m+1} \nu(K_i) - \frac{1}{m}\nu(K_0) = \frac{1}{m} \sum\limits_{i=1}^{m+1} \nu(K_i) \leq \int f d\nu < \frac{1}{m} \sum\limits_{i=0}^{m+1} \nu(K_i)$ resp. $\frac{1}{m} \sum\limits_{i=0}^{m} \nu(G_i) - \frac{1}{m}\nu(G_0) = \frac{1}{m} \sum\limits_{i=1}^{m} \nu(G_i) < \int f d\nu \leq \frac{1}{m} \sum\limits_{i=0}^{m} \nu(G_i)$. For $\nu = \mu_n$ the **right** resp. **left** hand sides yield $\int f d\mu_n < \frac{1}{m} \sum\limits_{i=0}^{m+1} \nu(K_i)$ resp. $\frac{1}{m} \sum\limits_{i=1}^{m} \nu(G_i) < \int f d\mu_n$ for every $n \geq 1$ whence $\frac{1}{m} \sum\limits_{i=1}^{m} \nu(G_i) \leq \liminf\limits_{n\to\infty} \int f d\mu_n \leq \limsup\limits_{n\to\infty} \int f d\mu_n < \frac{1}{m} \sum\limits_{i=0}^{m+1} \nu(K_i)$. For $\nu = \mu$ the **left** resp. **right** hand sides yield $\frac{1}{m} \sum\limits_{i=0}^{m+1} \nu(K_i) \leq \int f d\mu + \frac{1}{m}\nu(K_0)$ resp. $\int f d\mu - \frac{1}{m}\nu(G_0) \leq \frac{1}{m} \sum\limits_{i=1}^{m} \nu(G_i)$. By combining these four estimates we obtain $\int f d\mu - \frac{1}{m}\nu(G_0) \leq \liminf\limits_{n\to\infty} \int f d\mu_n \leq \limsup\limits_{n\to\infty} \int f d\mu_n \leq \int f d\mu + \frac{1}{m}\nu(K_0)$ for every $m \geq 1$ whence follows the assertion.

11.5 Vague convergence on continuous functions vanishing at infinity

On a **locally compact** space X for every sequence $(\mu_n)_{n\geq 1} \subset \mathcal{M}_{i0}^{c}(\mathcal{B}(X); \mathbb{R}^+)$ of **Radon measures vaguely converg**ing to a $\mu \in \mathcal{M}_{i0}^{c}(\mathcal{B}(X); \mathbb{R}^+)$ with $\sup\limits_{n\geq 1} \|\mu_n\| < \infty$ we have $\lim\limits_{n\to\infty} \int f d\mu_n = \int f d\mu$ for every **continuous** $f \in C_0(X; \mathbb{R})$ **vanishing at infinity**.

Proof: Due to 11.4.2 we have $\|\mu\| \leq \alpha = \sup\limits_{n\geq 1} \|\mu_n\| < \infty$ which implies $\mu \in \mathcal{M}_{i0}^{b}(\mathcal{B}(X); \mathbb{R}^+)$. According to 10.5 every $f \in C_0(X; \mathbb{R})$ and $\epsilon > 0$ exists a $g \in C_c(X; \mathbb{R})$ with $\|f - g\| \leq \epsilon$ which implies $|\int f d\mu_n - \int g d\mu_n| \leq \alpha \cdot \epsilon$ and also $|\int f d\mu - \int g d\mu| \leq \alpha \cdot \epsilon$. Owing to 11.4.1 exists an $N \in \mathbb{N}$with $|\int g d\mu - \int g d\mu_n| \leq \alpha \cdot \epsilon$ for all $n \geq N$.By the **triangle equation** we obtain $|\int f d\mu_n - \int f d\mu| \leq |\int f d\mu_n - \int g d\mu_n| + |\int g d\mu_n - \int g d\mu| + |\int g d\mu - \int f d\mu| \leq 3\alpha\epsilon$ whence follows the assertion.

11.6 Vague approximation of the Dirac measure

For every $\psi_n \in L^1$ with $\int \psi_n(\mathbf{x}) d\mathbf{x} = 1$ like e.g. the **characteristic function** of the **unit cube** $\psi_n = \chi_{[\mathbf{0};\mathbf{1}]}$ and $\psi_{n;k}(\mathbf{x}) = k^n(k\mathbf{x})$ we have $\lim\limits_{k\to\infty} \int f \cdot \psi_{n;k} d\lambda = f(\mathbf{0})$ for every $f \in C_c(\mathbb{R}^n; \mathbb{R})$, i.e. the sequence $(\psi_{n;k} \circ \lambda)_{k\geq 1}$ **vaguely converges** to the **Dirac measure** $\delta_{\mathbf{0}}$.

Note: Similar to the **approximate identities** $\psi_n \in L^1$ with $\int \psi_n(\mathbf{x}) d\mathbf{x} = 1$ and $\psi_{n;k}(\mathbf{x}) = k^n \psi_n(k\mathbf{x})$ such that $\lim\limits_{k\to\infty} \|f - f * \psi_{n;k}\|_1 = 0$ for every $f \in L^1$ defined in [9, th. 7.13] used in the **Fourier inversion formula** [9, th. 7.14].

Proof: For every $f \in \mathcal{C}_c(\mathbb{R}^n; \mathbb{R})$ and $k \geq 1$ holds $\left|f\left(\frac{1}{k}\mathbf{x}\right) \cdot \psi(\mathbf{x})\right| \leq \|f\|_\infty \cdot K(\mathbf{x})$ such that by a **change of variable** [9, th. 3.7] $\mathbf{y} = k\mathbf{x}$ with $\left|\det\left(\frac{d\mathbf{y}}{d\mathbf{x}}\right)\right| = k^n$ resp. **dominated convergence** 5.15 yields $\lim\limits_{k\to\infty} \int f \cdot \psi_{n;k} d\lambda = \lim\limits_{k\to\infty} \int f(\mathbf{x}) \cdot k^n \psi_n(k\mathbf{x}) d\mathbf{x} = \lim\limits_{k\to\infty} \int f\left(\frac{1}{k}k\mathbf{x}\right) \cdot \psi_n(k\mathbf{x}) d(k\mathbf{x}) = \lim\limits_{k\to\infty} \int f\left(\frac{1}{k}\mathbf{y}\right) \cdot \psi_n(\mathbf{y}) d\mathbf{y} = \int \lim\limits_{k\to\infty} f\left(\frac{1}{k}\mathbf{y}\right) \cdot \psi_n(\mathbf{y}) d\mathbf{y} = \int f(\mathbf{0}) \cdot \psi_n(\mathbf{y}) d\mathbf{y} = f(\mathbf{0})$.

11.7 Vague limits of discrete Radon measures

On a **locally compact** space X the linear combinations $\delta = \sum\limits_{i=1}^{k} \alpha_i \delta_{\mathbf{x}_i}$ of **Dirac measures** $\delta_{\mathbf{x}_i}$ on $\mathbf{x}_i \in X$ with $\alpha_i \geq 0$ for $1 \leq i \leq k$ and some $k \geq 1$ are called **discrete Radon measures**. Then with regard to **vague convergence**

1. the family of all **discrete Radon measures** is **dense** in $\mathcal{M}_{i0}^{c}(\mathcal{B}(X); \mathbb{R}^+)$
2. the family of all **discrete probability measures** is **dense** in $\mathcal{M}_{i0}^{c}(\mathcal{B}(X); [0; 1])$

Proof:

1. According to [13, th. 10.3 and 13.5] the locally compact space X is **uniformizable** whence due to to **Heine's theorem** [13, th. 12.9] every $f \in \mathcal{C}_c(X;\mathbb{R})$ is **uniformly continuous** on its **compact support** $K = \operatorname{supp} f$. Thus for every $\epsilon > 0$ exists a **finite cover** of **relatively compact** neighbourhoods $B(x_i)$ with $x_i \in K$ for $1 \leq i \leq k$ and $K \subset \bigcup_{1 \leq i \leq k} B(x_i)$. Hence the **disjoint** sets $A_j = K \cap B(x_j) \setminus \bigcup_{1 \leq i < j} B(x_j)$ still cover K and are **relatively compact**. Then for every $\mu \in \mathcal{M}_0^c(\mathcal{B}(X);\mathbb{R}^+)$ and any collection $(y_j)_{1 \leq j \leq k} \subset K$ with $y_j \in A_j$ for $1 \leq j \leq k$ the discrete Radon measure $\delta = \sum_{1 \leq j \leq k} \mu(A_j) \cdot \delta_{y_j}$ satisfies $\left|\int f d\mu - \int f d\delta\right| = \left|\sum_{1 \leq j \leq k} \left(\int_{A_j} f d\mu - \mu(A_j) \cdot f(y_j)\right)\right| = \left|\sum_{1 \leq j \leq k} \int_{A_j} (f - f(y_j))\, d\mu\right| \leq \sum_{1 \leq j \leq k} \int_{A_j} |f - f(y_j)|\, d\mu$ $\leq \epsilon \sum_{1 \leq j \leq k} \mu(A_j) = \epsilon \cdot \mu(K) \leq \epsilon$ which proves the assertion.
2. Follows from 1. with $\delta_0 = \mu(X \setminus K) \cdot \delta_{y_0} + \sum_{1 \leq j \leq k} \mu(A_j) \cdot \delta_{y_j}$ for some $y_0 \in X \setminus K$ since $\sum_{1 \leq j \leq k} \mu(A_j) = \mu(K) \leq 1$.

11.8 The weak topology

On a topological space X the **weak topology** is defined as the **initial topology** on the family of **positive Borel measures** $\mathcal{M}(\mathcal{B}(X);\mathbb{R}^+)$ with regard to the maps $\{\mu \mapsto \int f d\mu : f \in \mathcal{C}_b(X;\mathbb{R})\}$.

On a **locally compact** space X the **weak topology** on the **convex cone** of **bounded Radon measures** $\mathcal{M}_{i0}^b(\mathcal{B}(X);\mathbb{R}^+) \subset \mathcal{M}_{\sigma 0}(\mathcal{L}(X);\mathbb{R}^+)$ is generated by the **subbasis**

$$\mathcal{S} = \left\{W_{f;\epsilon}(\mu) : \nu \in \mathcal{M}_{i0}^c\left(\mathcal{B}(X);\mathbb{R}^+\right); f \in \mathcal{C}_b(X;\mathbb{R}); \epsilon > 0\right\}$$

of the **neighbourhoods**

$$W_{f;\epsilon} = \left\{\mu; \nu \in \mathcal{M}_{i0}^c\left(\mathcal{B}(X);\mathbb{R}^+\right) : \left|\int f d\mu - \int f d\nu\right| < \epsilon\right\}$$

Due to $C_c(X;\mathbb{R}) \subset C_b(X;\mathbb{R})$ resp. $1 \in C_b(X;\mathbb{R})$ the weak topology is both **stronger** than the **vague topology** defined in 11.4 and the topology of the **norm** $\| \, \|$ given by $\|\mu\| = \mu(X) = \int 1 d\mu$ in the **Riesz representation theorem** 10.13.2 on the subset of the **bounded Radon measures** $\mathcal{M}_{i0}^b(\mathcal{B}(X);\mathbb{R}^+) \subset \mathcal{M}_{i0}(\mathcal{B}(X);\mathbb{R}^+)$. The following theorem in essential states that on the **convex cone of the bounded Radon measures** the **union of these two topologies generates the weak topology**.

On a **locally compact** space X a sequence $(\mu_n)_{n \geq 1} \subset \mathcal{M}_{i0}^b(\mathcal{B}(X);\mathbb{R}^+)$ of **bounded Radon measures weakly converges** to a $\mu \in \mathcal{M}_{i0}^b(\mathcal{B}(X);\mathbb{R}^+)$ iff one of the following equivalent conditions is satisfied:

1. $\lim\limits_{n \to \infty} \int f d\mu_n = \int f d\mu$ for every $f \in C_b(X;\mathbb{R})$
2. $\lim\limits_{n \to \infty} \int f d\mu_n = \int f d\mu$ for every $f \in C_c(X;\mathbb{R})$ and $\lim\limits_{n \to \infty} \|\mu_n\| = \|\mu\|$.

Proof:

1. $\Rightarrow$ 2.: Due to $C_c(X;\mathbb{R}) \subset C_b(X;\mathbb{R})$ the first part is obvious and the second part follows from 11.1.2 since $\|\mu\| = \mu(X) = \sup\{\int f d\mu : f \prec X\} = \sup\{\int f d\mu : f \in C_c(X;\mathbb{R})\}$.

2. $\Rightarrow$ 1.: According to 11.1.2 for every $\epsilon > 0$ there is an $g \in C_c(X;[0;1])$ with $\mu(X) - \int g d\mu = \int (1 - g)\, d\mu < \epsilon$. Hence for every $f \in C_b(X;\mathbb{R})$ holds $\left|\int f \cdot (1 - g)\, d\mu\right| \leq \|f\| \cdot \int (1 - g)\, d\mu \leq \|f\| \cdot \epsilon$. The hypothesis implies $\lim\limits_{n \to \infty} \int g d\mu_n = \int g d\mu$ and $\lim\limits_{n \to \infty} \int 1 d\mu_n = \int 1 d\mu$ such that there isn an $m \geq 1$ with $\int (1 - g)\, d\mu < \epsilon$ for every $n \geq m$. For these n and every $f \in C_b(X;\mathbb{R})$ follows $\left|\int f \cdot (1 - g)\, d\mu_n\right|$ $\leq \|f\| \cdot \int (1 - g)\, d\mu_n \leq \|f\| \cdot \epsilon$ such that by the **triangle equation** we obtain $\left|\int f d\mu_n - \int f d\mu\right| \leq 2\|f\| \cdot \epsilon + \left|\int g d\mu_n - \int g d\mu\right|$ whence follows the assertion.

11.9 The Portmanteau theorem for locally compact spaces

On a **locally compact** space X a sequence $(P_n)_{n\geq 1} \subset \mathcal{M}_{i0}(X;[0;1])$ of **inner regular probability measures weakly converges** to an inner regular probability measure $P \in \mathcal{M}_{i0}(X;[0;1])$ iff one of the following equivalent conditions is satisfied:

1. $\lim\limits_{n\to\infty} \int f dP_n = \int f dP$ for every $f \in C_b(X;\mathbb{R})$.
2. $\lim\limits_{n\to\infty} \int f dP_n = \int f dP$ for every $f \in C_c(X;\mathbb{R})$.
3. $\limsup\limits_{n\to\infty} P_n(K) \leq P(K)$ for every **closed** $K \subset X$.
4. $\liminf\limits_{n\to\infty} P_n(O) \geq P(O)$ for every **open** $O \subset X$.
5. $\lim\limits_{n\to\infty} \int f dP_n = \int f dP$ for every **Borel measurable**, **bounded** and μ-a.e. **continuous** $f : X \to \mathbb{R}$.

Note: Condition 5. implies the condition $\lim\limits_{n\to\infty} P_n(A) = P(A)$ for every P**-continuous** $A \subset X$ with $P(\delta A) = 0$ corresponding to 12.6.5 in the **Portmanteau theorem for metric spaces**.

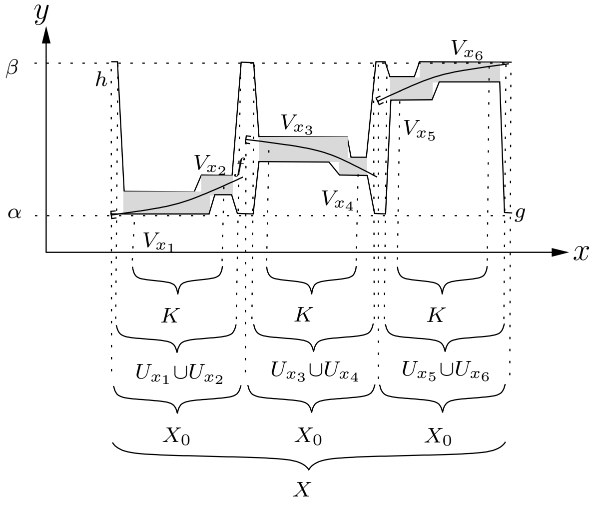
Proof:

1. $\Rightarrow$ 2.: obvious since $C_c(X;\mathbb{R}) \subset C_b(X;\mathbb{R})$

2. $\Rightarrow$ 3.: follows from 11.4.2

3. $\Rightarrow$ 4.: obvious since $\mu(O) = \mu(X) - \mu(X \setminus O) = 1 - \mu(X \setminus O)$

4. $\Rightarrow$ 5.: Due to the hypothesis there is a set $X_0 \subset X$ with $f \in C_b(X_0;\mathbb{R})$ and $\mu(X \setminus X_0) = 0$. Since μ is inner regular for every $\epsilon > 0$ exists a compact $K \subset X_0$ with $\mu(X_0 \setminus K) < \epsilon$. Then for every $x \in K$ there is an **open** neighbourhood U_x with $|f(y_1) - f(y_2)| < \epsilon$ for all $y_1; y_2 \in U_x$ and a **compact** $x \in V_x \subset U_x$. According to 10.3 for the finite cover $(V_{x_i})_{1\leq i\leq n}$ of $K \subset \bigcup_{1\leq i\leq n} V_{x_i}$ exist $\mathring{f}_i \in \mathcal{C}_b(X_0;\mathbb{R})$ with $V_{x_i} \prec \mathring{f}_i \prec X \setminus \bigcup_{1\leq i\leq n} U_{x_i}$. By $g_i(x) = \alpha_i \cdot \mathring{f}_i + \alpha$ resp. $h_i(x) = \beta_i \cdot \mathring{f}_i + \beta$ with $\alpha = \inf f[X]$; $b = \sup f[X]$; $\alpha_i = \inf f[U_{x_i}]$; $\beta_i = \sup f[U_{x_i}]$ we obtain $g_i; h_i \in \mathcal{C}_b(X_0;\mathbb{R})$ with $\alpha \leq g_i \leq \alpha_i \leq f \leq \beta_i \leq h_i \leq \beta$ and finally $g = \min\limits_{1\leq i\leq n} g_i$ resp. $h = \max\limits_{1\leq i\leq n} h_i$ with $\alpha \leq g_i(y) \leq g(y) \leq f(y) \leq h_i(y) \leq h(y) \leq \beta$ for every $y \in V_{x_i}$. In particular we have $h - g \leq \epsilon$ such that $\int (h-g)\, d\mu = \int_K (h-g)\, d\mu + \int_{X\setminus K} (h-g)\, d\mu$ $\leq \epsilon \cdot \mu(K) + (\beta - \alpha)\, \mu(X \setminus K) \leq \epsilon \cdot (\mu(X) + \beta - \alpha)$.



Also on the one hand we have $\int g d\mu = \lim\limits_{n\to\infty} \int g d\mu_n \leq \liminf\limits_{n\to\infty} \int f d\mu_n \leq \limsup\limits_{n\to\infty} \int f d\mu_n \leq \lim\limits_{n\to\infty} \int h d\mu_n =$ $\int h d\mu$ while on the other hand holds $\int g d\mu \leq \int g d\mu \leq \int h d\mu$ such that $\left\{\liminf\limits_{n\to\infty} \int f d\mu_n; \int g d\mu; \limsup\limits_{n\to\infty} \int f d\mu_n\right\} \subset$ $[\int g d\mu; \int g d\mu + \epsilon \cdot (\mu(X) + \beta - \alpha)]$ whence follows the assertion.

5. $\Rightarrow$ 1.: obvious.

11.10 Vaguely compact sets

On a **locally compact** space X every family $H \subset \mathcal{M}^c_{i0}(X;\mathbb{R}^+)$ of **Radon measures** is **relatively compact** with regard to the **vague topology** iff it is **vaguely bounded** with $\sup\limits_{\mu\in H} |\int f d\mu| < \infty$ for every $f \in \mathcal{C}_c(X;\mathbb{R})$.

Note: According to [13, def. 9.1] a set A is **relatively compact** iff its **closure** $\overline{A}$ is **compact**.

Proof:

$\Rightarrow$: Due to the **Heine-Borel theorem** [13, th. 9.10] for every $f \in \mathcal{C}_c(X;\mathbb{R})$ the image $\{\int f d\mu : \mu \in H\} \subset \mathbb{R}$ of a **relatively compact** $H \subset \mathcal{M}_{i0}^c(X;\mathbb{R}^+)$ under the **continuous** map $\mu \mapsto \int f d\mu$ is again **relatively compact** and in particular **bounded** which implies the assertion.

$\Leftarrow$:

Step I. The set $\mathcal{M}_{i0}^c(X;\mathbb{R}^+)$ is **homeomorphic** to $\Phi\left[\mathcal{M}_{i0}^c(X;\mathbb{R}^+)\right] \subset \mathcal{J}$: According to **Tychonov's theorem** [13, th. 9.9] the product $\mathcal{J} = \prod_{f\in\mathcal{C}_c(X;\mathbb{R})} J_f \subset \mathbb{R}^{\mathcal{C}_c(X;\mathbb{R})}$ of the **compact** intervals $J_f = [-\alpha_f;\alpha_f]$ for $\alpha_f = \sup\limits_{\mu\in H} |\int f d\mu|$ is again **compact**. By **Riesz' representation theorem for positive functionals** 10.13 the map $\Phi : H \to \mathcal{J} = \prod_{f\in\mathcal{C}_c(X;\mathbb{R})} J_f \subset \mathbb{R}^{\mathcal{C}_c(X;\mathbb{R})}$ defined by $\Phi(\mu) = (\int f d\mu)_{f\in\mathcal{C}_c(X;\mathbb{R})}$ is **injective** and due to the continuity of ist components $\Phi_f : \mu \mapsto \int f d\mu$ and [13, th. 4.2] it is also **continuous**. Its is also **open** since for every $\mu \in \mathcal{M}_{i0}^c(X;\mathbb{R}^+)$; $f \in \mathcal{C}_c(X;\mathbb{R})$; $\delta > 0$ and $0 < \eta < \delta$ there is a $\nu_\eta = \frac{\int f d\mu + \eta}{\int f d\mu}\mu \in \mathcal{M}_{i0}^c(X;\mathbb{R}^+)$ with $\int f d\nu_\eta = \int f d\mu + \eta$ such that every **neighbourhood** $B_\delta(\mu) = \{\nu \in \mathcal{M}_{i0}^c(X;\mathbb{R}^+) : |\int f d\mu - \int f d\nu| < \delta\}$ has an **open** image $\Phi[B_\delta(\mu)] = \pi_f^{-1}[B_\delta(\int f d\mu)] \subset \mathcal{J}$ with the component $B_\delta(\int f d\mu) \subset J_f = \pi_f[\mathcal{P}_c]$ in the **product topology** of $\mathcal{J} \subset \mathbb{R}^{\mathcal{C}_c(X;\mathbb{R})}$.

Step II. $\Phi\left[\overline{H}\right] \subset \mathcal{J}$: For every $\mu \in \overline{H}$ with regard to the vague topology holds $|\int f d\mu| \leq \alpha_f$ since for every $f \in \mathcal{C}_c(X;\mathbb{R})$ and $\epsilon > 0$ there is a $\nu \in \mathcal{M}_{i0}^c(X;\mathbb{R}^+)$ with $|\int f d\mu - \int f d\nu| < \epsilon$ whence follows $|\int f d\mu| \leq |\int \int f d\nu| + |\int f d\mu - \int f d\nu| < \alpha_f + \epsilon$ such that $|\int f d\mu| \leq \alpha_f$. Hence for every $f \in \mathcal{C}_c(X;\mathbb{R})$ we have $(\pi_f \circ \Phi)\left[\overline{H}\right] \subset J_f$ whence follows the proposition.

Step III. $\Phi\left[\mathcal{M}_{i0}^c(X;\mathbb{R}^+)\right]$ is **closed** in $\mathbb{R}^{\mathcal{C}_c(X;\mathbb{R})}$: For every element $I \in \overline{\Phi\left[\mathcal{M}_{i0}^c(X;\mathbb{R}^+)\right]} \subset \mathbb{R}^{\mathcal{C}_c(X;\mathbb{R})}$ being regarded as a map $I : \mathcal{C}_c(X;\mathbb{R}) \to \mathbb{R}$ defined by $If = \pi_f(I)$, every $f;g \in \mathcal{C}_c(X;\mathbb{R})$ and every $\epsilon > 0$ the set $\pi_f^{-1}[B_\epsilon(I)] \cap \pi_g^{-1}[B_\epsilon(I)] \cap \pi_{f+g}^{-1}[B_\epsilon(I)]$ is a **neighbourhood** of $I \in \mathbb{R}^{\mathcal{C}_c(X;\mathbb{R})}$. It therefore contains an $I' \subset \Phi\left[\mathcal{M}_{i0}^c(X;\mathbb{R}^+)\right]$ whence $|I(f+g) - If - Ig| \leq |I(f+g) - I'(f+g)| + |I'f - If| + |I'g - Ig| < 3\epsilon$ and consequently $I(f+g) = If + Ig$. Similarly for $\alpha \in \mathbb{R}$ there is an $I' \in \pi_f^{-1}[B_\epsilon(I)] \cap \pi_{\alpha f}^{-1}[B_\epsilon(I)] \cap \Phi\left[\mathcal{M}_{i0}^c(X;\mathbb{R}^+)\right]$ whence $|I(\alpha f) - \alpha If| \leq |I(\alpha f) - I'(\alpha f)| + |I'(\alpha f) - \alpha I'f| + |\alpha I'f - \alpha If| < \epsilon + 2\cdot|\alpha|\cdot\epsilon$. Hence we have proved that $I \in (\mathcal{C}_c(X;\mathbb{R}))^*$ is a **linear functional**. A third application of this argument delivers $If \geq 0$ for every $f \geq 0$ whence follows $I \subset \Phi\left[\mathcal{M}_{i0}^c(X;\mathbb{R}^+)\right]$.

Step IV. Due to **steps I** and **II** the homeomorphic image $\Phi\left[\overline{H}\right]$ is **closed** in $\Phi\left[\mathcal{M}_{i0}^c(X;\mathbb{R}^+)\right]$. By **step III** $\Phi\left[\overline{H}\right]$ is a **closed** subset of the **compact** set $\mathcal{J} \subset \mathbb{R}^{\mathcal{C}_c(X;\mathbb{R})}$ and hence **compact**.

11.11 Vague compactness of open balls

For every $\epsilon > 0$ the **open ball** $\overline{B_\epsilon(0)} = \{\mu \in \mathcal{M}_{i0}(X;\mathbb{R}^+) : |\mu(X)| \leq \epsilon\}$ of **bounded Radon measures** is **vaguely compact**.

Proof: Owing to $|\int f d\mu| \leq \int |f| d\mu \leq \epsilon \cdot \|f\|$ for every $f \in \mathcal{C}_c(X;\mathbb{R})$ and $\mu \in B_\epsilon(0)$ the set $B_\epsilon(0)$ is **vaguely bounded** whence by the preceding theorem 11.10 follows its **vague relative compactness**. According to the **Riesz representation theorem for positive functionals** 10.13 we have $B_\epsilon(0) = \{\mu \in \mathcal{M}_{i0}(X;\mathbb{R}^+) : \int f d\mu \leq \epsilon \forall f \in \mathcal{C}_c(X;[0;1])\} = \bigcap_{f\in\mathcal{C}_c(X;[0;1])} M_{f;\epsilon}$ with **vaguely closed** $M_{f;\epsilon} = \{\mu \in \mathcal{M}_{i0}(X;\mathbb{R}^+) : \int f d\mu \leq \epsilon\}$ whence $B_\epsilon(0)$ is **vaguely closed** and hence **compact**.

11.12 Separability of $\mathcal{C}_c(X;\mathbb{R})$

A **locally compact** space X is **second countable** iff $\mathcal{C}_c(X;\mathbb{R})$ is **separable** with regard to **uniform convergence**.

Proof:

$\Rightarrow$: For the **countable basis** $\mathcal{G}$ of the topology on X and every $n \geq 1$ the products $U_1 \times ... \times U_n \subset X^n$ with $U_i \in \mathcal{G}$ and $I_1 \times ... \times I_n \subset \mathbb{R}^n$ with $I_i \in \mathcal{R} = \{]a;b[\subset \mathbb{R} : a < b \in \mathbb{Q}\}$ for $1 \leq i \leq n$ are **compatible** iff there is at least one **compatibility function** $f \in \mathcal{C}_c(X;\mathbb{R})$ with $f[U_i] \subset I_i$ and $\mathrm{supp} f \subset \bigcup_{1 \leq i \leq n} U_j$. For every compatible product $U_1 \times ... \times U_n \times I_1 \times ... \times I_n$ we choose **one** possible compatible function such that the resulting set $\mathcal{F}$ of these functions is **countable**. For every $g \in \mathcal{C}_c(X;\mathbb{R})$ with **compact** $K = \mathrm{supp} g$; $x \in K$ and $\epsilon > 0$ exists an **open neighbourhood** $x \in U_x \in \mathcal{G}$ with $g[U_x] \subset B_\epsilon(g(x))$ and a **finite subcover** $(U_{x_i})_{1 \leq i \leq n}$ with $K \subset \bigcup_{1 \leq i \leq n} U_{x_i}$. Also there are $I_i =]a_i; b_i[\in \mathcal{R}$ with **length** $b_i - a_i < 3\epsilon$ with $g(U_{x_i}) \subset I_i$ for $1 \leq i \leq n$. Hence g is a compatible function and there exists an $f \in \mathcal{F}$ for the same compatible product $U_{x_1} \times ... \times U_{x_n} \times I_1 \times ... \times I_n$ with $|f(x) - g(x)| \leq \lambda(I_i) < 3\epsilon$ for every $x \in \bigcup_{1 \leq i \leq n} U_j$ and $f(x) = g(x) = 0$ for $x \in X \setminus \bigcup_{1 \leq i \leq n} U_j$. Hence $\mathcal{F}$ is **dense** in $\mathcal{C}_c(X;\mathbb{R})$.

$\Leftarrow$: For a **countable dense** subset $\mathcal{F} \subset \mathcal{C}_c(X;\mathbb{R})$ the **countable** family $\mathcal{G} = \left\{\left\{f > \frac{1}{2}\right\} : f \in \mathcal{F}\right\}$ is a **basis** for the topology on X since according to 10.3 for every open U and every $x \in U$ exists a $g \in \mathcal{C}_c(X;\mathbb{R})$ with $\{x\} \prec g \prec U$ and due to the hypothesis an $f \in \mathcal{F}$ with $\|f - g\| < \frac{1}{2}$ such that $x \in \left\{f > \frac{1}{2}\right\} \subset \{g > 0\} \subset \mathrm{supp} g \subset U$.

11.13 Embedding of X into $\mathcal{M}_{i0}^c(X;\mathbb{R}^+)$

Every **locally compact** space X by $\varphi : X \to \mathcal{M}_{i0}^c(X;\mathbb{R}^+)$ with $\varphi(x) = \delta_x$ is **homeomorphic** to the family of all **Dirac measures** $\varphi[X] = \{\delta_x : x \in X\} \subset \mathcal{M}_{i0}^c(X;\mathbb{R}^+)$.

Proof: φ obviously is **injective** and also **continuous** since due to $\int f d\delta_x = f(x)$ for every **subbasis** set $V_{f;\epsilon}(\delta_x) = \{\delta_y \in \varphi[X] : |f(x) - f(y)| < \epsilon\} \in \mathcal{S} \cap \varphi[X]$ with $x \in X$; $f \in C_c(X;\mathbb{R})$ and $\epsilon > 0$ according to 11.4 the inverse image $\varphi^{-1}[V_{f;\epsilon}(\delta_x)] = f^{-1}[B_\epsilon(f(x))]$ is **open** in X. Furthermore φ is **open** since owing to 10.3 for every $x \in X$ and every open neighbourhood $x \in U$ exists a $f \in C_c(X;\mathbb{R})$ with $\{x\} \prec f \prec U$ such that $V_{f;1/2}(\delta_x) = \left\{\delta_y \in \varphi[X] : f(y) > \frac{1}{2}\right\} \subset \{\delta_y \in \varphi[X] : y \in U\} = \varphi[U]$.

11.14 Metrizability and completeness of $\mathcal{M}_{i0}^c(X;\mathbb{R}^+)$

A **locally compact** space X is **polish** iff the **convex cone** of the **Radon measures** $\mathcal{M}_{i0}^c(X;\mathbb{R}^+)$ is **polish** with regard to the **vague topology**.

Note:

Due to [13, th. 15.2] a **locally compact** space X is **polish** iff it is **second countable** and according to Urysohns metrization theorem [13, th. 11.14.3] it is **σ-compact** such that in this case **every Radon measure is complete and regular**: $\mathcal{M}_{i0}^c(\mathcal{B}(X);\mathbb{R}^+) = \mathcal{M}_{\sigma 0}(\mathcal{B}(X);\mathbb{R}^+)$.

Proof:

$\Rightarrow$:

Step I. **Definition of the metric**: According to 11.12 exists a **countable dense** set $\mathcal{D} \subset \mathcal{C}_c(X;\mathbb{R})$. Owing to [13, th. 10.6] the space X is **σ-compact** so that we have an increasing sequence $(L_n)_{n \in \mathbb{N}}$ of **compact** L_n with $\bigcup_{n \in \mathbb{N}} L_n = X$ and 10.3 yields another **countable** set $\mathcal{E} \subset \mathcal{C}_c(X;[0;1])$ containing for each L_n exactly one e_n with $L_n \prec e_n \prec B_1(L_n) = \{x \in X : d(x;L_n) < 1\}$. The set of products $\mathcal{D} \cdot \mathcal{E} = \{d \cdot e_n : d \in \mathcal{D}; e_n \in \mathcal{E}\}$ is still **countable**. The map $\rho : \mathcal{M}_{i0}^c(X;\mathbb{R}^+) \times \mathcal{M}_{i0}^c(X;\mathbb{R}^+) \to [0;1]$ defined by $\rho(\mu;\nu) = \sum_{n \geq 1} 2^{-n} \cdot \min\{1; |\int d_n d\mu - \int d_n d\nu|\}$ with $d_n \in \mathcal{D} \cup \mathcal{E} \cup \mathcal{D} \cdot \mathcal{E}$ obviously is **symmetric** and satisfies the **triangle inequality**. Concerning the **positive definiteness** for every $f \in \mathcal{C}_c(X;\mathbb{R})$ and $\epsilon > 0$ there is a $k \geq 1$ with $\mathrm{supp} f \subset L_k$ whence $f = e_k \cdot f$ and a $d \in \mathcal{D}$ with $\|f - d\| \leq \epsilon$. Hence we have $|\int f d\mu - \int d \cdot e_k d\mu| \leq \int |f - d \cdot e_k| d\mu \leq \epsilon \int e_k d\mu$ and analogously $|\int f d\nu - \int d \cdot e_k d\nu| \leq \epsilon \int e_k d\nu$. The hypothesis $\rho(\mu;\nu) = 0$ implies $\int d \cdot e_k d\mu - \int d \cdot e_k d\nu$ for every $n \geq 1$ whence follows $|\int f d\mu - \int f d\nu| \leq 2\epsilon \int e_k d\mu$. This estimate holds for every $\epsilon > 0$ and every $f \in \mathcal{C}_c(X;\mathbb{R})$ whence the **Riesz representation theorem for positive functionals** 10.13 implies $\mu = \nu$.

Step II. The metric determines the vague topology: According to the definition of the vague topology in 11.4 for every $\epsilon > 0$ exists an $m \geq -\frac{\ln \epsilon}{\ln 2}$ such that $\sum_{n>m} 2^{-n} < \frac{\epsilon}{2}$ and consequently $\bigcap_{1 \leq n \leq m} V_{d_n;\epsilon/2}(\mu) = \left\{\nu \in \mathcal{M}_{i0}^c(X;\mathbb{R}^+) : \left|\int d_n d\mu - \int d_n d\nu\right| < \frac{\epsilon}{2} \forall n \leq m\right\} \subset B_\epsilon(\mu)$. Conversely for every $f \in \mathcal{C}_c(X;\mathbb{R})$, $\mu \in \mathcal{M}_{i0}^c(X;\mathbb{R}^+)$ and $\epsilon > 0$ exists a $k \geq 1$ with $\operatorname{supp} f \subset L_k$ and a $d \in \mathcal{D}$ such that $\|f - d\| < \delta = \frac{\epsilon}{2+2\int e_k d\mu} < 1$ whence $|f - d \cdot e_k| \leq \delta \cdot e_k$. As above we obtain $\left|\int f d\mu - \int d \cdot e_k d\mu\right| \leq \delta \int e_k d\mu$ but also $\left|\int f d\nu - \int d \cdot e_k d\nu\right| \leq \delta \int e_k d\nu$ for every other $\nu \in \mathcal{M}_{i0}^c(X;\mathbb{R}^+)$. For $m \geq 1$ large enough so that $\{d \cdot e_k; e_k\} \subset \{d_1; ...; d_m\}$, $\nu \in B_\eta(\mu)$ with $\eta = \delta \cdot 2^{-m}$ and every $j \leq m$ follows $2^{-j} \cdot \min\left\{1; \left|\int d_j d\mu - \int d_j d\nu\right|\right\} \leq \rho(\mu;\nu) < \eta \leq \delta \cdot 2^{-j}$ and consequently $\left|\int d_j d\mu - \int d_j d\nu\right| < \delta$ which implies $\left|\int d \cdot e_k d\mu - \int d \cdot e_k d\nu\right| < \delta$. The triangle equation yields $\left|\int f d\mu - \int f d\nu\right| \leq \delta\left(1 + \int e_k d\mu + \int e_k d\nu\right)$ and since the choice of $m \geq 1$ also implies $\left|\int e_k d\mu - \int e_k d\nu\right| < \delta$ resp. $\int e_k d\nu < \delta + \int e_k d\mu$ we finally obtain $\left|\int f d\mu - \int f d\nu\right| \leq \delta^2 + \delta\left(1 + 2\int e_k d\mu\right) \leq \delta\left(2 + 2\int e_k d\mu\right) = \epsilon$. Hence we have shown that $B_\eta(\mu) \subset V_{f;\epsilon}(\mu)$.

Step III. The metric space $\left(\mathcal{M}_{i0}^c(X;\mathbb{R}^+);\rho\right)$ is complete: For every $f \in \mathcal{C}_c(X;\mathbb{R})$ and every $0 < \delta < 1$ exits a $k \geq 1$ with $\operatorname{supp} f \subset L_k$ and a $d \in \mathcal{D}$ such that $\|f - d\| < \delta$. As above we choose an $m \geq 1$ such that $\{d \cdot e_k; e_k\} \subset \{d_1; ...; d_m\}$, $\nu \in B_\eta(\mu)$ and define $\eta = \delta \cdot 2^{-m}$. Then for a ρ-Cauchy sequence $(\mu_n)_{n \geq 1} \subset \mathcal{M}_{i0}^c(X;\mathbb{R}^+)$ exists an $N \geq 1$ such that $\rho(\mu_r;\mu_s) < \eta$ for every $r; s \geq N$. Following **step II** again we conclude $\left|\int d_j d\mu_r - \int d_j d\mu_s\right| < \delta$ for every $r; s \geq N$ and $j \leq m$ whence $\left|\int d \cdot e_k d\mu_r - \int d \cdot e_k d\mu_s\right| < \delta$. As above we arrive at $\left|\int f d\mu_r - \int f d\mu_s\right| \leq \delta^2 + \delta\left(1 + 2\int e_k d\mu\right)$. The estimate $\left|\int e_k d\mu_r - \int e_k d\mu_s\right| < \delta$ for every $r; s \geq N$ implies the existence of an $M < \infty$ such that $\int e_k d\mu_n < M$ for $n \geq 1$ whence $\left|\int f d\mu_r - \int f d\mu_s\right| \leq \delta^2 + \delta(1 + 2M)$. Since M depends only on the choice of f and $(\mu_n)_{n \geq 1}$ for every $\epsilon > 0$ we find a $\delta = \frac{\epsilon}{2+2M} < 1$ such that $\left|\int f d\mu_r - \int f d\mu_s\right| < \epsilon$ for every $r; s \geq N$. Hence $(\mu_n)_{n \geq 1} \subset \mathcal{M}_{i0}^c(X;\mathbb{R}^+)$ is a **Cauchy sequence** with regard to the **vague topology** whence the Riesz representation theorem for positive functionals 10.13 resp. the **completeness** of $\mathcal{M}_{i0}^c(X;\mathbb{R}^+) = \mathcal{M}_{\sigma 0}(X;\mathbb{R}^+)$ (see **Note**) imply the **vague convergence** of $(\mu_n)_{n \geq 1}$ to a **uniquely determined limit** $\mu \in \mathcal{M}_{\sigma 0}(X;\mathbb{R}^+)$.

Step IV. The metric space $\left(\mathcal{M}_{i0}^c(X;\mathbb{R}^+);\rho\right)$ is second countable: According to 11.7.1 for every $f \in \mathcal{C}_c(X;\mathbb{R})$, $\mu \in \mathcal{M}_{i0}^c(X;\mathbb{R}^+)$ and $\epsilon > 0$ esists a **discrete Radon measure** $\delta = \sum_{1 \leq i \leq k} \alpha_i \delta_{x_i}$ with $\mathbf{x}_i \in X$ and $\alpha_i \geq 0$ for $1 \leq i \leq k$ such that $\left|\int f d\mu - \int f d\delta\right| = \left|\int f d\mu - \sum_{1 \leq i \leq k} \alpha_i f(x_i)\right| < \frac{\epsilon}{3}$. For $1 \leq i \leq k$ we choose $\beta_i \in \mathbb{Q}$ with $|\alpha_i - \beta_i| < \frac{\epsilon}{3k\|f\|}$. According to [13, th. 2.8] the **second countable** set X is **separable** with a countable dense subset $Y \subset X$ such that we can find $y_i \in Y$ with $|f(x_i) - f(y_i)| < \frac{\epsilon}{3k\beta_i}$. Hence we obtain a new discrete Radon measure $\gamma = \sum_{1 \leq i \leq k} \beta_i \delta_{y_i}$ with

$$\begin{aligned}
\left|\int f d\mu - \int f d\gamma\right| &\leq \left|\int f d\mu - \int f d\gamma\right| + \left|\int f d\delta - \int f d\gamma\right| \\
&\leq \frac{\epsilon}{3} + \left|\sum_{1 \leq i \leq k} \alpha_i f(x_i) - \beta_i f(x_i) + \beta_i f(x_i) - \beta_i f(y_i)\right| \\
&\leq \frac{\epsilon}{3} + \sum_{1 \leq i \leq k} |\alpha_i - \beta_i| \cdot \|f\| + \sum_{1 \leq i \leq k} |\beta_i| \cdot |f(x_i) - f(y_i)| \\
&\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\
&= \epsilon
\end{aligned}$$

Hence the **countable** set $\mathcal{D} = \left\{\sum_{1 \leq i \leq k} \beta_i \delta_{y_i} : \beta_i \in \mathbb{Q}; y_i \in Y; 1 \leq i \leq k; k \geq 1\right\}$ of **discrete Radon measures** with **rational coefficients** on points of the **dense countable** subset $Y \subset X$ is **dense** in $\mathcal{M}_{i0}^c(X;\mathbb{R}^+)$ with regard to the vague topology whence again due to [13, th. 2.8] follows that $\mathcal{M}_{i0}^c(X;\mathbb{R}^+)$ is **second countable**.

$\Leftarrow$: Directly follows from the preceding theorem 11.13.

11.15 Convergence of sequences in vaguely bounded sets

Every **vaguely bounded** sequence $(\mu_n)_{n \geq 1} \subset \mathcal{M}_{i0}^c(X;\mathbb{R}^+)$ of **Radon measures** on a **polish** space X has a **vaguely convergent subsequence**.

Proof: Follows directly from 11.10, 11.14 and the Bolzano-Weierstrass theorem [13, th. 10.12].

11.16 Metrizability and completeness of $\mathcal{C}\left(\mathbb{R}^+;X\right)$

For every **polish** space X the vector space of the **continuous paths** $\mathcal{C}\left(\mathbb{R}^+;X\right)$ is **polish** with regard to the **compact open topology**.

Note: Every metric $\rho : X \times X \to \mathbb{R}^+$ can be shrinked to the range $[0;1]$ by transition e.g. to $\rho' = \min\{1;\rho\}$ or $\rho'' = \frac{\rho}{1+\rho}$.

Proof: The function $d : \mathcal{C}\left(\mathbb{R}^+;X\right) \times \mathcal{C}\left(\mathbb{R}^+;X\right) \to [0;1]$ defined by $d(f;g) = \sum_{n\geq 1} 2^{-n} d_n(f;g)$ with $d_n(f;g) = \sup\{\rho(f(x);g(x)) : x \in [0;n]\}$and the metric $\rho : X \times X \to [0;1]$ obviously is again a **metric** with $2^{-n} d_n(f;g) \leq d(f;g) \leq \sum_{1\leq i\leq n} 2^{-i} d_n(f;g) + \sum_{i>n} 2^{-i} \leq d_n(f;g) + 2^{-n}$. Hence the metric d induces the **compact open topology** of the space $\mathcal{C}\left(\mathbb{R}^+;X\right)$ which is **complete** according to [13, p. 18.7.3]. Analogously to the proof of 11.12 we show that $\mathcal{C}\left(\mathbb{R}^+;X\right)$ has a **countable basis**: According to [13, th. 2.8] the **second countable** set X is **separable** with a **countable dense** subset $Y \subset X$ so that $\mathcal{G} = \{B_r(y) : r \in \mathbb{Q};\ y \in Y\}$ is a **countable basis** of the open sets in X. For every $n \geq 1$ the products $G_1 \times ... \times G_n \subset X^n$ with $G_i \in \mathcal{G}$ and $I_1 \times ... \times I_n \subset \mathbb{R}^n$ with $I_i \in \mathcal{R} = \left\{]a;b[\subset \mathbb{R}^+ : 0 < a < b \in \mathbb{Q}\right\}$ for $1 \leq i \leq n$ are **compatible** iff there is at least one **compatibility function** $f \in \mathcal{C}\left(\mathbb{R}^+;X\right)$ with $f[I_i] \subset G_i$. For every compatible product $G_1 \times ... \times G_n \times I_1 \times ... \times I_n$ we choose **one** possible compatible function such that the resulting set $\mathcal{F}$ of these functions is **countable**. For every $g \in \mathcal{C}\left(\mathbb{R}^+;X\right)$; $N \geq 1$, $0 < \epsilon \in \mathbb{Q}$ and $x \in [0;N]$ exists an **open neighbourhood** $x \in U_x \in \mathcal{R}$ with $g[U_x] \subset B_\epsilon(g(x))$ and a **finite subcover** $(U_{x_i})_{1\leq i\leq n}$ with $[0;N] \subset \bigcup_{1\leq i\leq n} U_{x_i}$. Then for $1 \leq i \leq n$ we have $G_i = B_{2\epsilon}(y_i) \in \mathcal{G}$ with some $y_i \in Y \cap B_\epsilon(g(x_i))$ such that for every $x \in U_{x_i}$ follows $\rho(g(x);y_i) \leq \rho(g(x);g(x_i)) + \rho(g(x_i);y_i) < 2\epsilon$ whence $g[U_{x_i}] \subset G_i$. Hence g is a **compatible function** and there exists an $f \in \mathcal{F}$ for the same compatible product $G_1 \times ... \times G_n \times I_1 \times ... \times I_n$ with $f[U_{x_i}] \subset G_i$ for $1 \leq i \leq n$ whence $\rho(f(x);g(x)) < 4\epsilon$ for every $x \in [0;N]$. Hence $\mathcal{F}$ is **dense** in $\mathcal{C}_c(X;\mathbb{R})$.

12 Probability measures on metric spaces

In this chapter without further notice $P \in \mathcal{M}(X;[0;1])$ will always be a **probability measure**s on the **Borel-σ-algebra** $\mathcal{B}(X)$ of a **metric space** $(X;d)$.

12.1 Discontinuities of functions between metric spaces

The set $D_f = \{x \in X : \exists \epsilon > 0 : \forall \delta > 0\, \exists y;z \in B_\delta(x) : D(f(y);f(z)) \geq \epsilon\}$ of **discontinuities** of the (not necessarily measurable) function $f : (X;d) \to (Y;D)$ lies in $\mathcal{B}(X)$.

Notes:

1. In [12, th. 3.1] it is shown that for every **monotone** $f :]a;b[\to \mathbb{R}$ the set D_f of **discontinuities** is **countable** and all discontinuities $c \in D_f$ are **simple**, i.e. $-\infty < \sup\limits_{a<x<c} f(x) = \lim\limits_{n\to\infty} f\left(c - \frac{1}{n}\right) < \lim\limits_{n\to\infty} f\left(c + \frac{1}{n}\right) = \inf\limits_{c<x<b} f(x) < \infty$.
2. In [9, th. 1.2] it is proved that for every **real** $f : \mathbb{R} \to \mathbb{R}$ the set of **jump** and **vertex** points with **existing** but **differing Dini derivatives** $\left\{D_+f = D^+f = D_+^+f \neq D_-^-f = D_-f = D^-f\right\}$ is **countable.**

Proof: The sets $A_{\epsilon;\delta} = \{x \in X : \exists y;z \in B_\delta(x) : D(f(y);f(z)) \geq \epsilon\}$ are **open** because $x \in B_\delta(y) \cap B_\delta(y)$ whence $D_f = \bigcup\limits_{\epsilon\in\mathbb{Q}^+} \bigcap\limits_{\delta\in\mathbb{Q}^+} A_{\epsilon;\delta} \in \mathcal{B}(X)$.

12.2 Regularity on metric spaces

Every **probability measure** $P \in \mathcal{M}(X;[0;1])$ on a **metric space** $(X;d)$ is **weakly regular**, i.e. for every measurable set $A \in \mathcal{B}(X)$ there is a **closed** K and an **open** O such that $K \subset A \subset O$ and $P(O \setminus K) < \epsilon$.

Proof: The family $\mathcal{A}$ of alls sets $A \in \mathcal{B}(X)$ satisfying the hypothesis is a **σ-algebra** since for every sequence $(A_n)_{n\geq 1} \subset \mathcal{A}$ and $\epsilon > 0$ due to the hypothesis we have **closed** K_n with **open** O_n such that $K_n \subset A_n \subset O_n$ and $P(O_n \setminus K_n) < \frac{\epsilon}{2^{n+1}}$ and owing to the **continuity from below** 2.2.2 there is an $m \in \mathbb{N}$ with $P\left(\bigcup\limits_{n\geq 1} K_n \setminus \bigcup\limits_{n=1}^{m} K_n\right) < \frac{\epsilon}{2}$ whence $\bigcup\limits_{n=1}^{m} K_n \subset \bigcup\limits_{n\geq 1} A_n \subset \bigcup\limits_{n\geq 1} O_n$ and $P\left(\bigcup\limits_{n\geq 1} O_n \setminus \bigcup\limits_{n=1}^{m} K_n\right)$ $< P\left(\bigcup\limits_{n\geq 1} O_n \setminus \bigcup\limits_{n\geq 1} K_n\right) + P\left(\bigcup\limits_{n\geq 1} K_n \setminus \bigcup\limits_{n=1}^{m} K_n\right) \leq P\left(\bigcup\limits_{n\geq 1} (O_n \setminus K_n)\right) + \frac{\epsilon}{2} < \epsilon$. The closedness under **complementation** is obvious. Due to the **continuity from above** $\mathcal{A}$ includes every **closed** set A with a suitable **δ-neighbourhood** $A^{\delta} = \{x \in X : d(x;A) < \delta\}$ hence every **open** set whence follows $\sigma(\mathcal{O}) = \mathcal{B}(X) \subset \mathcal{A}$ and the theorem is proved.

12.3 Regularity on Polish spaces

Every **probability measure** $P \in \mathcal{M}(X;[0;1])$ on a **polish space** $(X;d)$ is **regular**, i.e. for every measurable set $A \in \mathcal{B}(X)$ there is a **compact** K and an **open** O such that $K \subset A \subset O$ and $P(O \setminus K) < \epsilon$.

Proof: Since Ω is **separable** there is a **countable and dense subset** $(\omega_n)_{n\geq 1}$ and for every $k \geq 1$ a sequence $(B_{k_i})_{k_i \geq 1}$ of **open balls** $B_{k_i} = B_{\frac{1}{k}}(\omega_{k_i})$ covering Ω. Owing to the **continuity from below** 2 there is an $n_k \geq 1$ such that $P\left(\bigcup_{i\leq n_k} B_{k_i}\right) > 1 - \frac{\epsilon}{2^k}$. Since Ω is complete the **closure** $K = \overline{\bigcap_{k\geq 1} \bigcup_{i\leq n_k} B_{k_i}}$ due to [13, p. 10.12] is **compact**: For any sequence $(x_j)_{j\geq 1} \subset K$ and every $k \geq 1$ exists an open ball B_{k_i} containing infinitely many elements of $(x_j)_{j\geq 1}$ such that the resulting subsequence is **Cauchy** and due to the **completeness** converges to an $x \in K$. Since $P(K) > 1 - \epsilon$ and every intersection between a compact and a closed set is compact again we have shown that every **closed** set A for every $\epsilon > 0$ contains a **compact** set $K \subset A$ with $\mathrm{P}(A \setminus K) < \epsilon$.

Since every closed set $A = \bigcap_{n\geq 1} O_{1/n}$ is the intersection of open $O_{1/n} = \left\{\omega \in \Omega : d(\omega;A) < \frac{1}{n}\right\}$ and P is **continuous from above** we have shown that P is **regular** on all **closed** sets $A \subset \Omega$: For any $\epsilon > 0$ there are **compact** K resp. **open** O with $K \subset A \subset O$ and $P(O \setminus K) < \epsilon$. Since $\mathcal{B}(\Omega)$ is generated by the closed sets it remains to prove that the family $\mathcal{R} \subset \mathcal{P}(\Omega)$ of all sets which satisfy the regularity property is a σ-algebra. To this end for $\epsilon > 0$ and any given sequence $(A_n)_{n\geq 1} \subset \mathcal{R}$ we choose compact K_n and open O_n with $K_n \subset A_n \subset O_n$ and $\mathrm{P}(O_n \setminus K_n) \leq \frac{\epsilon}{2^{n+1}}$. Then we find an $n_{\epsilon} \geq 1$ such that with $P\left(\bigcup_{n\geq 1} K_n \setminus K\right) < \frac{\epsilon}{2}$ for $K = \bigcup\limits_{n=1}^{n_{\epsilon}} K_n$ whence $K \subset \bigcup_{n\geq 1} A_n \subset O = \bigcup_{n\geq 1} O_n$ with $P(O \setminus K) < \epsilon$. Hence $\mathcal{R}$ is closed under **countable unions**. Since it is obviously closed under **intersection** and **complementation** the proof is complete.

12.4 Characterization by bounded continuous functions

Two **probability measures** $P, Q : \mathcal{B}(X) \to [0;1]$ on a **metric space** $(X;d)$ **coincide** iff $\int f dP = \int f dQ$ for every **bounded** and **uniformly continuous** $f : X \to \mathbb{R}$.

Proof: For every **closed** $A \subset X$ the functions $f_n : X \to [0;1]$ with $f_n(x) = (1 - n \cdot d(x;A))^{+}$ are **bounded** and **uniformly continuous** since $|f(x) - f(y)| < n \cdot d(x;y)$. Since we have pointwise everywhere $\chi_A = \lim\limits_{n\to\infty} f_n$ the **dominated convergence theorem** 5.15 yields $P(A) = \int \chi_A dP = \int \lim\limits_{n\to\infty} f_n dP = \lim\limits_{n\to\infty} \int f_n dP = \lim\limits_{n\to\infty} \int f_n dQ = \int \lim\limits_{n\to\infty} f_n dQ = Q(A)$. Since the **closed** sets are a π **-basis** for $\mathcal{B}(X)$ the assertion follows from the **uniqueness theorem** 3.4.

12.5 Tightness on complete and separable metric spaces

Every **probability measure** $P \in \mathcal{M}(X;[0;1])$ on a **complete and separable metric space** $(X;d)$ is **tight**, i.e. for every $\epsilon > 0$ exists a **compact** $K \subset X$ with $P(K) > 1 - \epsilon$.

Proof: According to the **separable** character for every $n \geq 1$ there is a sequence $(x_k)_{k \geq 1} \subset X$ with $\bigcup\limits_{k \geq 1} B_{1/n}(x_k) = X$ and in particular an $n_k \geq 1$ such that $P\left(\bigcup\limits_{k=1}^{n_k} \overline{B_{1/n}(x_k)}\right) > 1 - \frac{\epsilon}{2^n}$. The set $B = \bigcap\limits_{n \geq 1} \bigcup\limits_{k=1}^{n_k} \overline{B_{1/n}(x_k)}$ is **precompact** resp. **totally bounded** whence due to the **complete** character and [13, th. 17.2] it has a **compact closure** $K = \overline{B}$ with $P(K) > 1 - \epsilon$.

12.6 The Portmanteau theorem for metric spaces

A sequence $(P_n)_{n \geq 1} \subset \mathcal{M}(X;[0;1])$ of **probability measures weakly converges** to a probability measure $P \in \mathcal{M}(X;[0;1])$ iff one of the following equivalent conditions is satisfied:

1. $\lim\limits_{n \to \infty} \int f dP_n = \int f dP$ for every **bounded, continuous** $f : X \to \mathbb{R}$.
2. $\lim\limits_{n \to \infty} \int f dP_n = \int f dP$ for every **bounded, uniformly continuous** $f : X \to \mathbb{R}$.
3. $\limsup\limits_{n \to \infty} P_n(K) \leq P(K)$ for every **closed** $K \subset X$.
4. $\liminf\limits_{n \to \infty} P_n(O) \geq P(O)$ for every **open** $O \subset X$.
5. $\lim\limits_{n \to \infty} P_n(A) = P(A)$ for every P**-continuous** $A \subset X$ with $P(\delta A) = 0$.

In a **separable** metric space $(X;d)$ we have the additional equivalent property:

6. There is a **convergence-determining** π**-system** $\mathcal{A}$ such that for every $x \in X$ and $\epsilon > 0$ the subfamily $\delta\mathcal{A}_{x;\epsilon} = \left\{A \in \mathcal{A} : x \in \mathring{A} \subset A \subset B_\epsilon(x)\right\}$ contains a P**-null set** $A \in \delta\mathcal{A}_{x;\epsilon} \subset \mathcal{A}$ with $P(A) = 0$ and $\lim\limits_{n \to \infty} P_n(A) = P(A)$ for every $A \in \mathcal{A}$. Due to 2.2 the former condition is satisfied if $\delta\mathcal{A}_{x;\epsilon}$ contains **uncountably many disjoint sets**.

Note: The **Helly-Bray theorem** [12, th. 3.8] is a **corollary** to the Portmanteau theorem for the case $X = \mathbb{R}$ with an application to **distribution functions**.

Proof:

1. $\Rightarrow$ 2.: trivial

2. $\Rightarrow$ 3.: The separation function $K \prec f \prec K^\epsilon$ defined in the proof of 12.4 by $f(x) = \left(1 - \frac{d(K;x)}{\epsilon}\right)^+$ is **uniformly continuous** with $\limsup\limits_{n \to \infty} P_n(K) \leq \limsup\limits_{n \to \infty} \int f dP_n = \int f dP \leq P(K^\epsilon)$ for every $\epsilon > 0$.

3. $\Rightarrow$ 4.: $\liminf\limits_{n \to \infty} P_n(O) = \liminf\limits_{n \to \infty} (1 - P(X \setminus O)) = 1 - \limsup\limits_{n \to \infty} P_n(X \setminus O) \geq 1 - P(X \setminus O) = P(O)$.

3.&4. $\Rightarrow$ 5.: According to the hypothesis $P\left(\mathring{A}\right) \leq \liminf\limits_{n \to \infty} P_n\left(\mathring{A}\right) \leq \liminf\limits_{n \to \infty} P_n(A) \leq \limsup\limits_{n \to \infty} P_n(A) \leq \limsup\limits_{n \to \infty} P_n\left(\overline{A}\right) \leq P\left(\overline{A}\right)$ and in the case of $P\left(\overline{A}\right) - P\left(\mathring{A}\right) = P(\delta A) = 0$ all terms coincide.

5. $\Rightarrow$ 1.: By the decomposition $f = f^+ - f^-$ resp. the **bounded** character of f and the **linearity** of the integral it suffices to examine the case $f : X \to [0;1]$. The **continuity** of f implies $\delta\{f > t\} \subset \{f = t\}$ for every $t \geq 0$. According to 2.1 we have $P(f = t) > 0$ for at most countably many t whence the sets $\{f > t\}$ are λ**-almost everywhere** P**-continuous**. By [12, th. 1.5] and the **dominated convergence theorem** 5.15 we conclude $\lim\limits_{n \to \infty} \int f dP_n \overset{1.5}{=} \lim\limits_{n \to \infty} \int P_n(f > t)\, dt \overset{5.15}{=} \int \lim\limits_{n \to \infty} P_n(f > t)\, dt \overset{5.}{=} \int P(f > t)\, dt = \int f dP$.

5. $\Rightarrow$ 6.: According to 12.2 every **open** set is P**-continuous.** Since the topology $\mathcal{O}$ is a π**-system** and every $\delta\mathcal{O}_{x;\epsilon}$ contains **uncountably many disjoint sets** we chan choose $\mathcal{A} = \mathcal{O}$.

6. ⇒ 4.: Since $\delta\left(A\cup B\right)\subset\delta A\cup\delta B$ the class $\mathcal{A}_P$ of all P-**continuous** sets in$\mathcal{A}$ is a π-**system**. Since each $\delta\mathcal{A}_{x;\epsilon}$ contains a P-null set $A_x\in\mathcal{A}_P$ with $x\in\mathring{A}_x\subset A_x\subset B_\epsilon\left(x\right)$ and X is **separable** for every **open** $O\subset X$ exists a sequence $\left(A_{x_i}\right)_{i\geq1}\subset\mathcal{A}_P$ with $\bigcup\limits_{i\geq1}A_{x_i}=O$. Hence for every $\eta>0$ there is an $r\in\mathbb{N}$ such that $P\left(\bigcup\limits_{i=1}^{r}A_{x_i}\right)>P\left(O\right)-\eta$. The hypothesis implies that

$$\lim_{n\to\infty}P_n\left(\bigcup_{i=1}^{r}A_{x_i}\right)=\lim_{n\to\infty}\sum_{i=1}^{r}P_n\left(A_{x_i}\right)-\lim_{n\to\infty}\sum_{i;j=1}^{r}P_n\left(A_{x_i}\cap A_{x_j}\right)+\lim_{n\to\infty}\sum_{i;j;k=1}^{r}P_n\left(A_{x_i}\cap A_{x_j}\cap A_{x_k}\right)-\ldots$$

$$=\sum_{i=1}^{r}P\left(A_{x_i}\right)-\sum_{i;j=1}^{r}P\left(A_{x_i}\cap A_{x_j}\right)+\sum_{i;j;k=1}^{r}P\left(A_{x_i}\cap A_{x_j}\cap A_{x_k}\right)-\ldots$$

$$P\left(\bigcup_{i=1}^{r}A_{x_i}\right)$$

whence follows $P\left(O\right)-\eta<P\left(\bigcup\limits_{i=1}^{r}A_{x_i}\right)=\lim\limits_{n\to\infty}P_n\left(\bigcup\limits_{i=1}^{r}A_{x_i}\right)\leq\liminf\limits_{n\to\infty}P_n\left(O\right)$.

12.7 Weak convergence on product spaces

A sequence $\left(P_n\otimes Q_n\right)_{n\geq1}\subset\mathcal{M}\left(X\times Y;[0;1]\right)$ on the product of **separable** metric spaces $(X;d)$ and $(Y;e)$ **weakly converges** to a $P{\otimes}Q\in\mathcal{M}\left(X\times Y;[0;1]\right)$ iff $\lim\limits_{n\to\infty}\left(P_n\otimes Q_n\right)\left(A\times B\right)=\left(P\otimes Q\right)\left(A\times B\right)$ for every P-**continuous** $A\subset X$ and Q-**continuous** $B\subset Y$.

Proof: The family $\mathcal{A}=\left\{A\times B\in\mathcal{B}\left(X\times Y\right)=\mathcal{B}\left(X\right)\otimes\mathcal{B}\left(Y\right):\ P\left(\delta A\right)=Q\left(\delta B\right)=0\right\}$ (cf. [13, th. 4.2] and 7.7) is a π-**system** since $(A\times B)\cap(C\times D)=(A\cap C)\times(B\cap D)$ and $\delta\left(A\cap C\right)\subset\delta A\cup\delta C$. It is $P\otimes Q$-**continuous** since $\delta\left(A\times B\right)\subset\left(\left(\delta A\right)\times Y\right)\cup\left(X\times\left(\delta B\right)\right)$. If we choose the **metric** $\Delta:X\times Y\to\mathbb{R}^+$ defined by $\Delta\left(\left(x;y\right);\left(u;v\right)\right)=\max\left\{d\left(x;u\right);e\left(y;v\right)\right\}$ (cf. [13, th. 1.8.3]) the balls $B_{\Delta<\epsilon}\left(\left(x;y\right)\right)=B_{d<\epsilon}\left(x\right)\times B_{e<\epsilon}\left(y\right)$ have **boundaries** of the form $\delta\left(B_{\Delta<\epsilon}\left(\left(x;y\right)\right)\right)\subset\delta\left(B_{d<\epsilon}\left(x\right)\right)\times Y\cup X\times\delta\left(B_{e<\epsilon}\left(y\right)\right)$. Hence **they all are $P\otimes Q$-null sets and lie in** $\delta\mathcal{A}_{(x;y);\epsilon}\subset\mathcal{A}$ so that we can apply 12.6.6 and the theorem is proved.

12.8 The mapping theorem

For every $\mathcal{B}\left(X\right)$ - $\mathcal{B}\left(Y\right)$ **measurable**, P-**a.e. continuous** $f:(X;d)\to(Y;D)$ and a sequence $\left(P_n\right)_{n\geq1}\subset\mathcal{M}\left(X;[0;1]\right)$ **weakly converging** to $P\in\mathcal{M}\left(X;[0;1]\right)$ the images $\left(f\circ P_n\right)_{n\geq1}\subset\mathcal{M}\left(Y;[0;1]\right)$ **weakly converge** to $f\circ P\in\mathcal{M}\left(Y;[0;1]\right)$.

Proof: For every **closed** $K\in\mathcal{B}\left(Y\right)$ and $x\in\overline{f^{-1}\left[K\right]}\setminus D_f$ with the set $D_f\in\mathcal{B}\left(X\right)$ of the **discontinuities** of f according to 12.1 there is a sequence $\left(x_n\right)_{n\geq1}\subset X$ with $\lim\limits_{n\to\infty}x_n=x$ and $\left(f\left(x_n\right)\right)_{n\geq1}\subset K$ whence follows $f\left(x\right)\in K$ since f is **continuous** in x and K is **closed**. Therefore $\overline{f^{-1}\left[K\right]}\setminus D_f\subset f^{-1}\left[K\right]$ and from the hypothesis $P\left(D_f\right)=0$ follows $\limsup\limits_{n\to\infty}\left(f\circ P_n\right)\left(K\right)=\limsup\limits_{n\to\infty}P_n\left(f^{-1}\left[K\right]\right)=\limsup\limits_{n\to\infty}P_n\left(\overline{f^{-1}\left[K\right]}\right)\leq P\left(\overline{f^{-1}\left[K\right]}\right)=P\left(\overline{f^{-1}\left[K\right]}\setminus D_f\right)\leq P\left(f^{-1}\left[K\right]\right)=\left(f\circ P\right)\left(K\right)$ which by3 proves the theorem.

12.9 The diagonal principle

For every **real double sequence** $\left(x_{i,j}\right)_{i,j\geq1}\subset\mathbb{R}$ with **bounded rows** $\left(x_{i,j}\right)_{j\geq1}$ there is a sequence $\left(i_k\right)_{k\geq1}$ such that **in each row** $i\geq1$ the **limit** $\lim\limits_{k\to\infty}x_{i,j_k}\in\mathbb{R}$ exists.

Proof: According to the **Heine-Borel theorem** [13, p. 9.10] we can find a subsequence $\left(j_{1,n}\right)_{n\geq1}\subset\mathbb{N}$ such that $\lim\limits_{n\to\infty}x_{1,j_{1,n}}\in\mathbb{R}$ exists. Given a subsequence $\left(j_{k,n}\right)_{n\geq1}$ such that $\lim\limits_{n\to\infty}x_{k,j_{k,n}}\in\mathbb{R}$ exists we change into the next row and by the same argument from the preceding subsequence choose a further subsequence $(j_{k+1,n})_{n\geq 1} \subset (j_{k,n})_{n\geq 1}$ such that $\lim\limits_{n\to\infty} x_{k+1,j_{k+1,n}} \in \mathbb{R}$ exists. Since the subsequences $\left((j_{k,n})_{n\geq 1}\right)_{k\geq 1}$ form a **decreasing** family of sets with $\lim\limits_{n\to\infty} x_{k,j_{k,n}} \in \mathbb{R}$ for every row $k \geq 1$ the **diagonal sequence** $(j_k)_{k\geq 1}$ with $j_k = j_{k,k}$ is **increasing** and $\lim\limits_{k\to\infty} x_{i,j_k} \in \mathbb{R}$ for every row $i \geq 1$.

12.10 Tight families of measures and distribution functions

Weak limits $P = \lim\limits_{n\to\infty} P_n$ of probability measures P_n on measurable spaces $(\Omega;\mathcal{A})$ may exist in the form of finite measures $\mu \in \mathcal{M}_0(\mathcal{A};\mathbb{R}^+)$ but in order to guarantee the condition $\mu(\Omega) = 1$ resp. $\lim\limits_{m\to\infty} F(m) = 1$ and $\lim\limits_{m\to\infty} F(-m) = 0$ in terms of the **distribution function** $F : \mathbb{R} \to [0;1]$ defined by $F(x) = P(]-\infty;x])$ in the case of $\Omega = \mathbb{R}$ we have to avoid the **"loss of mass"** as in the two following examples:

1. The sequence $(X_n)_{n\geq 1}$ with $X_n = n$ resp. $P_{X_n} = \delta_n$ and $F_n = \chi_{[n;\infty[}$ has $\lim\limits_{n\to\infty} F_n = 0$ since the mass "escapes to infinity" .
2. The sequence $(Y_n)_{n\geq 1}$ with $P(|Y_n| \leq n) = \frac{1}{2n}\lambda$ such that $F_n(t) = \left(\frac{1}{2} + \frac{t}{n}\right) \cdot \chi_{[-n;n]}$ again has $\lim\limits_{n\to\infty} F_n = 0$ since in this case the uniformly over the interval $[-n;n]$ distributed part of the mass "evaporates" .

Thus we define that the family Π of probability measures on the **Borel σ-algebra $\mathcal{B}(X)$** of a **metric space** $(X;d)$ is **tight** iff for every $\epsilon > 0$ exists a compact $K_e \subset X$ such that $P(K_\epsilon) > 1 - \epsilon$ for every $p \in \Pi$. In the case of $X = \mathbb{R}$ this definition extends to the corresponding family $\Phi = \{F_P : P \in \Pi\}$ of **distribution functions** which is **tight** iff for every $\epsilon > 0$ exist real numbers $a_\epsilon < b_\epsilon \in \mathbb{R}$ such that $P([a;b]) \geq P(]a;b]) = F_P(b) - F_P(a) \geq P([a+\epsilon;b]) > 1 - \epsilon$ for every $P \in \Pi$.

12.11 Prohorov's theorem

Every family Π of probability measures on the **Borel σ-algebra $\mathcal{B}(X)$** of a **separable and complete metric space** $(X;d)$ is **tight** iff it is **sequentially compact** with regard to **weak convergence**.

Notes:

1. The set $\mathcal{P}$ of all probability measures on a **separable and complete metric space** $(X;\mathcal{B}(X))$ according to [2, th. 6.8] by the **Skorohod metric** π becomes itself a **separable and complete metric space** with **pointwise π-convergence** being equivalent to **weak convergence**. Hence due to [13, th. 10.12] the spaces X resp. $\mathcal{P}$ are **second countable** whence the properties of being **compact**, **countably compact** and **sequentially compact** are **equivalent**.
2. **Helly's selection theorem** [12, th. 3.9] is a **corollary** to Prohorov's theorem for the case $X = \mathbb{R}$ and applied to **distribution functions**.

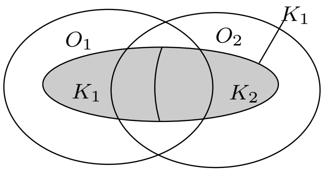
Proof:

$\Rightarrow$: Since X is **second countable** there is an increasing sequence of **open** sets G_n with $\bigcup_{n\geq 1} G_n = X$. Then for every $\epsilon > 0$ there is an $n \geq 1$ such that $P(G_n) > 1 - \epsilon$ for every $P \in \Pi$ since otherwise we had a sequence $(P_n)_{n\geq 1} \subset \Pi$ with $P_n(G_n) < 1 - \epsilon$ and by the hypothesis a subsequence $(P_{n_k})_{k\geq 1}$ with a weak limit $P = \lim\limits_{k\to\infty} P_{n_k} \in \Pi$ whence 12.6.4 implied $P(G_n) \leq \liminf\limits_{k\to\infty} P_{n_k}(G_n) \leq \liminf\limits_{k\to\infty} P_{n_k}(G_{n_k}) \leq 1 - \epsilon$ and by 2.2.2 followed $P(X) = \lim\limits_{n\to\infty} P(G_n) \leq 1 - \epsilon$. With this result we can proceed as in the proof of 12.5: According to the **separable** character for every $n \geq 1$ there is a sequence $(x_k)_{k\geq 1} \subset X$ with $\bigcup\limits_{k\geq 1} B_{1/n}(x_k) = X$ and an $n_k \geq 1$ such that $P\left(\bigcup\limits_{k=1}^{n_k} \overline{B_{1/n}(x_k)}\right) > 1 - \frac{\epsilon}{2^n}$ for all $P \in \Pi$. The set $B = \bigcap\limits_{n\geq 1} \bigcup\limits_{k=1}^{n_k} \overline{B_{1/n}(x_k)}$ is **precompact** resp. **totally bounded** whence due to the **complete** character and [13, th. 17.2] it has a **compact closure** $K = \overline{B}$ with $P(K) > 1 - \epsilon$ for all $P \in \Pi$.

$\Leftarrow$: According to the hypothesis for any given sequence $(P_n)_{n\geq 1} \subset \Pi$ there is an increasing sequence $\mathcal{K} = (K_u)_{u\geq 1}$of **compact** sets such that $P_n(K_u) \geq 1 - \frac{1}{u}$ for all $n; u \geq 1$. Since $K = \bigcup_{u\geq 1} K_u$ is **separable** there exists a countable family $\mathcal{B} = B_{k;n} = \left(B_{1/n}(x_k)\right)_{k;n\geq 1}$ such that for every **open** $O \subset X$ and $x \in O \cap K$ there is an $B_{k;n} \in \mathcal{B}$ with $x \in B_{k;n} \subset \overline{B}_{k;n} \subset O$. Let $\mathcal{H}$ be the **countable** class containing $\emptyset$ and every **finite union** of sets $\overline{B}_{k;n} \cap K_u$ with $B_{k;n} \in \mathcal{B}$ and $K_u \in \mathcal{K}$. According to the **diagonal principle** 12.9 there is a subsequence $(P_{n_i})_{i\geq 1}$ such that for every $H \in \mathcal{H}$ exists the limit $\alpha(H) = \lim\limits_{i\to\infty} P_{n_i}(H)$. It is **monotone** with $\alpha(H_1) \leq \alpha(H_2)$ if $H_1 \subset H_2$, **subadditive** with $\alpha(H_1 \cup H_2) \leq \alpha(H_1) + \alpha(H_2)$ with equality if $H_1 \cap H_2 = \emptyset$ and obviously $\alpha(\emptyset) = 0$. The set function $\beta : \mathcal{O} \to [0;1]$ defined by $\beta(O) = \sup\limits_{H\subset O} \alpha(H)$ for every open $O \in \mathcal{O}$ is still **monotone** and satisfies $\beta(\emptyset) = 0$. In the following six steps we show that $\gamma : \mathcal{P}(X) \to [0;1]$ defined by $\gamma(A) = \inf\limits_{A\subset O} \beta(O)$ for every $A \subset X$ is an **outer measure**:

Step I: For every **closed** $K \subset O \cap H$ with $O \in \mathcal{O}$ and $H \in \mathcal{H}$ exists a $H_0 \in \mathcal{H}$ with $K \subset H_0 \subset O$: Due to **Heine-Borel** [13, th. 9.10] the set $K \subset H$ is **compact** whence there is an $u \geq 1$ with $K \subset K_u$ and a **finite subcover** $(B_{x_i})_{1\leq i\leq k} \subset \mathcal{B}$ with $B_{x_i} \subset O\,\forall 1 \leq i \leq k$ and $K \subset \bigcup_{1\leq i\leq k} B_{x_i}$ such that we can choose $H_0 = \bigcup_{1\leq i\leq k} \overline{B}_{x_i} \cap K_u$.

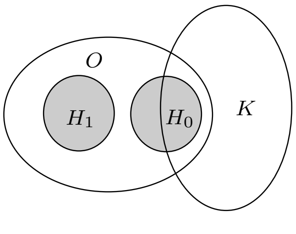
Step II: β is **subadditive** on the **open** sets with $\beta(O_1 \cup O_2) \leq \beta(O_1) + \beta(O_2)$ for every $O_1; O_2 \in \mathcal{O}$: For every $H \in \mathcal{H}$ and open $O_1; O_2$ with $H \subset O_1 \cup O_2$ define $K_1 = \{x \in H : d(x; X \setminus O_1) \geq d(x; X \setminus O_2)\}$ and $K_2 = \{x \in H : d(x; X \setminus O_2) \geq d(x; X \setminus O_1)\}$. Since $X \setminus O_2$ is **closed** for every $x \in K_1 \cap X \setminus O_1$ follows the contradiction $d(x; X \setminus O_1) = 0 < d(x; X \setminus O_2)$ so that we infer $K_1 \subset O_1$and analogously $K_2 \subset O_2$. Since $K_1 \subset H \in \mathcal{H}$ by **step I** exist $H_1; H_2 \in \mathcal{H}$ with $K_1 \subset H_1 \subset O_1$ resp. $K_2 \subset H_2 \subset O_2$. By the **monotonicity** resp. **subadditivity** of α follows $\alpha(H) \leq \alpha(H_1 \cup H_2) \leq \alpha(H_1) + \alpha(H_2) \leq \beta(O_1) + \beta(O_2)$. Since we can find an increasing sequence $(H_j)_{j\geq 1} \subset \mathcal{H}$ with $\bigcup_{j\geq 1} H_j = O_1 \cup O_2$ the assertion follows.



Step III: β is **σ-subadditive** on the **open** sets with $\beta\left(\bigcup_{n\geq 1} O_n\right) \leq \sum_{n\geq 1} \beta(O_n)$ for every sequence $(O_n)_{n\geq 1} \subset \mathcal{O}$: For every $H \in \mathcal{H}$ with $H \subset \bigcup_{n\geq 1} O_n$ its **compact** character implies the existence of an $m \geq 1$ with $H \subset \bigcup_{1\leq n\leq m} O_n$ and by **step II** we have $\alpha(H) \leq \beta\left(\bigcup_{1\leq n\leq m} O_n\right) \leq \sum_{1\leq n\leq m} \beta(O_n) \leq \sum_{n\geq 1} \beta(O_n)$. Since this estimate holds for **every** $H \subset \bigcup_{n\geq 1} O_n$ we can infer its validity for the **supremum** of all such H whence follows the proposition.

Step IV: γ is an **outer measure**: Since γ is obviously **monotone** with $\gamma(\emptyset) = 0$ we only have to prove the **σ-subadditivity**: For every $\epsilon > 0$ and a sequence $(A_n)_{n\geq 1}$ of **arbitrary** subsets $A_n \subset X$ there are **open** $O_n \supset A_n$ with $\beta(O_n) < \gamma(A_n) + \frac{\epsilon}{2^n}$ and by **step III** follows $\gamma\left(\bigcup_{n\geq 1} A_n\right) \leq \beta\left(\bigcup_{n\geq 1} A_n\right) \leq \sum_{n\geq 1} \beta(A_n) \leq \sum_{n\geq 1} \gamma(A_n) + \epsilon$ whence $\gamma\left(\bigcup_{n\geq 1} A_n\right) \leq \sum_{n\geq 1} \gamma(A_n)$ since ϵ was arbitrary.

Step V: For every **closed** $K \subset X$ and **open** $O \subset X$ holds $\beta(O) \geq \gamma(O \cap K) + \gamma(O \setminus K)$: For every $\epsilon > 0$ there is an $H_1 \in \mathcal{H}$ with $H_1 \subset O \setminus K$ and $\alpha(H_1) > \beta(O \setminus K) - \epsilon$. Now choose an $H_0 \in \mathcal{H}$ with $H_0 \subset O \setminus H_1$ and $\alpha(H_0) > \beta(O \setminus H_1) - \epsilon$. Since $H_0 \cap H_1 = \emptyset$ and $H_0 \cup H_1 \subset O$ by the **additivity** of α follows $\beta(O) \geq \alpha(H_0 \cup H_1) = \alpha(H_0) + \alpha(H_1) > \beta(O \setminus H_1) + \beta(O \setminus K) - 2\epsilon \geq \gamma(O \setminus K) + \gamma(O \setminus K) - 2\epsilon$.



Step VI: Every **closed** set K is **γ-measurable**: For every **arbitrary** $A \subset X$ and **open** $O \supset A$ **step V** and the **montonicity** of γ imply $\beta(O) \geq \gamma(O \cap K) + \gamma(O \setminus K) \geq \gamma(A \cap K) + \gamma(A \setminus K)$. Taking the **infimum** over all such A we obtain $\gamma(A) \geq \gamma(A \cap K) + \gamma(A \setminus K)$ and the **subadditivity** of γ yields the desired equality according to the definition 3.2.4.

According to **Carathéodory's theorem** 3.3 the restriction $P = \gamma|_{\mathcal{A}}$ of the **outer measure** γ to the **σ-algebra** $\mathcal{A}$ of all **γ-measurable** sets is a **measure** and since every **closed** set is γ-measurable we have $\mathcal{B}(X) \subset \mathcal{A}$ and in particular $X \in \mathcal{A}$. For every open set $O \in \mathcal{O} \subset \mathcal{B}(X) \subset \mathcal{A}$ follows $P(O) = \gamma(O) = \beta(O)$. Owing to their **compact** character all K_u lie in $\mathcal{H}$ such that $1 \geq P(X) = \beta(X) \geq \sup\limits_{u \geq 1} \alpha(K_u) \geq \sup\limits_{u \geq 1} 1 - \frac{1}{u} = 1$ whence P is a **probability measure**. For every $H \in \mathcal{H}$ with $H \subset O$ follows $\alpha(H) = \lim\limits_{i \to \infty} P_{n_i}(H) \leq \liminf\limits_{i \to \infty} P_{n_i}(O)$ and in particular $P(O) = \gamma(O) = \alpha(O) \leq \liminf\limits_{i \to \infty} P_{n_i}(O)$ which by the **Portmanteau theorem** 12.6.4 completes the proof.

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