Topology

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Preface

Topology is the branch of mathematics concerned with generalized concepts of **distance** which are the foundation for the study of **convergence**, **continuity** and **differentiation** of **functions** in **complex and functional analysis**, **differential geometry** as well as **probability theory**. There is a close relationship between topology and **measure theory**, which is dedicated to the understanding of **volume** in mathematics leading to the theory of **integration** in the above mentioned fields..

Apart from minor alterations and reorganisations this text follows the classical expositions by those **von Querenburg [10]** and **J. Kelley** [3]. The required basic facts of set theory can be found in e.g. [19] and are not included in this text. Results from measure theory and complex analysis which are needed in the latter parts are dealt with in [18] resp. [15] or alternatively in the standard books [13] resp. [12] by **W. Rudin** and [6] resp. [4] by **S. Lang**. The introductory first chapter revises topological facts in metric spaces which are familiar from calculus. Since different metrics result in equivalent properties concerning convergence and continuity these fundamental qualities of functions obviously depend on simpler concepts. After this motivation chapters 2 - 5 develop these fundametal concepts of topology: **Open sets, neighborhoods, axioms of countability, continuous** and **open functions,** construction of topological spaces on **subsets, products, quotients and sums of topologicals spaces** as well as **connectedness**. During the early 20th century **E. H. Moore** introduced **nets** resp. **directed sets** to generalize the concept of **convergence** from well ordered sequences to less structured sets. Nets were further developd by american mathematicians a.o. **J. W. Tukey** and **G. Birkhoff**; **J. Kelley** bases his classical work [3] on nets. In chapter 6 of this text, however, the theory of convergence is instead built on the concept of a **filter** developed in the 1940's by french mathematicians around **H. Cartan** and **J. Dieudonné** under the pseudonym of **N. Bourbaki** [2]. It is more abstract than nets but leads to the same results via very short and elegant proofs. The subsequent parts 7 - 8 deal with **separation axioms**, standard theorems of **Urysohn** and **Tietze** about the extension of continuous functions on normal spaces and **partitions of unity**. In 9 - 14 the next steps follow in canonical order: **compactness**, **uniformization, metrization** and **completion** of topological spaces. The basic characteristics of **Polish** and **Baire spaces** are developed in chapters 15 and 16. Chapter 17 presents the **Stone-Čech-compactification**. Further results about **function spaces** required in **complex and functional analysis** are presented in the next chapters: In chapter 18 the **topology of compact convergence** leads to the **Stone-Weierstrass-theorem** while in chapter 19 the concept of **equicontinuity** leads to the theorem of **Ascoli** with applications in **differential equations** and **complex analysis**. The last section returns to the geometrical origins of topology in the form of **topological manifolds**, i.e. sets which locally can be described as topological vector spaces. The classification of their global structures leads the notions of **homotopy** and **homology**; a very general construction method for manifolds is given by **cell complexes** leading to the characterisation of topological properties in the language of **algebraic topology** and **combinatorics**. Theses topics are drawn from [7] and as far as possible adapted to infinite-dimensional Banach spaces following [5] and [11]. The necessary prerequites from functional analysis can be found in [16].

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1.1 Index notation

The **axiom of choice** (cf. [19, p. 14.2.1]) postulates the existence of a **choice function** x with $x(i) \in i$ for every set i. Thus for every set I there is a **product** $\prod I := \{x : I \to \bigcup I : x(i) \in i \forall i \in I\}$ where x to each element i of the **index set** I assigns an element $x(i)$ of the **set** i. The **big union** $\bigcup I$ is the set of all elements $x(i)$ of elements i from I. To emphasize the set character of the indices i in this rather terse notation we usually write $X_i := i$ and arrive at the common **index notation** $\prod_{i \in I} X_i := \{x : I \to \bigcup_{i \in I} X_i : x(i) := x_i \in X_i\}$.

1.2 Metrics

The essential characteristics of our notion of distance on a set X are given by the **metric** $d : X^2 \to [0; \infty[$ with the following conditions:

1. $d(x; y) = 0 \Leftrightarrow x = y$ (**positive definiteness**)
2. $d(x; y) = d(y; x)$ for all $x, y \in X$ (**symmetry**)
3. $d(x; y) + d(y; z) \leq d(x; z)$ for all $x, y, z \in X$ (**triangle inequality**)

One of the principal interests of topology consists in the research for necessary and sufficient conditions for the existence of a metric. The study of these conditions (cf. section 12) has revealed that the **separability** of two points as expressed by positive definiteness 1.2.1 is quite independent of the problem of measuring of distances in general. Consequently the necessary minimum requirement for a quantification of distance is the **pseudometric** which differs from the metric by reducing 1.2.1 to 1.2.1': $d(x, x) = 0$. The separability of points is postulated independently by the **separation axioms** T_1 resp. T_2 (ch.7.1). In most cases of practical interest these axioms are satisfied anyway or can at least be achieved by identifying inseparable objects as **eqivalence classes** (cf. 1.3) thereby restoring positive definiteness. For this reason and the sake of simplicity we will presume the **Hausdorff property** T_2 almost throughout the text with the exception of an exemplary consideration of pseudometrization of non-Haussdorff-spaces in section 13. The ordered pair $(X; d)$ is called a **metric space**.

1.3 Normed vector spaces

A function $\|\| : X \to [0; \infty[$ on the **vector space** X over the **field** $K \in \{\mathbb{R}; \mathbb{C}\}$ is a **norm** iff it satisfies the following conditions:

1. $\|x\| = 0 \Leftrightarrow x = 0$ (**positive definiteness**)
2. $\|\lambda \cdot x\| = |\lambda| \cdot \|x\|$ for all $\lambda \in K$ (**linearity** with respect to multiplication)
3. $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$ (**triangle inequality**).

Via $d(x; y) := \|x - y\|$ the norm induces a **metric** on X.

Examples:

1. The **Euclidean norm** $\|(x_1; ...; x_n)\| := \sqrt{\sum_{1 \leq i \leq n} x_i \cdot \overline{x}_i}$ on a vector space $X = \mathbb{C}^n$.
2. The **Supremum norm** $\|f\| := \sup\{|f(x)| : a \leq x \leq b\}$ on the vector space of all **continuous complex-valued functions** $X = C([a; b]; \mathbb{C})$ on a **closed interval** $[a; b]$**.**
3. The L^p**-norm** $\|f\|_p := (\int |f|^p \, d\mu)^{\frac{1}{p}}$ on the vector space $L^p = \mathcal{L}^p / R$ of all **p-integrable functions** on a **measure space** $(X; \mu)$ with $\mathcal{L}^p = \left\{f : (X; \mu) \to (\mathbb{R}; \lambda) : \|f\|_p < \infty\right\}$ and $fRg \Leftrightarrow f = g$ almost everywhere (i.e. everywhere except on a set with measure zero).

The last item provides an important example for a **seminorm** $\|f\|_p$ on $\mathcal{L}^p$ with weakened condition 1.3.1': $\|x\| = 0$ which is extended to a full norm with restored positive definiteness by reducing $\mathcal{L}^p$to the quotient space $L^p = \mathcal{L}^p/R$, i.e. by identifying formerly inseparable functions as common equivalence classes.

1.4 Topological concepts on metric spaces

On a metric space $(X; d)$ a **neighborhood** of x is defined as a set including an **open ball** $B_r(x) := \{y \in X : d(x; y) < r\}$ with center x and radius r. A subset $A \subset X$ is **open** iff it is a neighborhood for each of its elements. A is **closed** iff its **complement** $X \setminus A$ is open. A function $f : (X; d) \to (X'; d')$ is **continuous** iff the **inverse images** $f^{-1}[O]$ of open sets $O \subset X'$ are again open in X. For every $\epsilon > 0$ there is a $\delta > 0$ such that $f[B_\delta(x)] \subset B'_\epsilon(f(x))$. A **sequence** $(x_n)_{n\in\mathbb{N}}$ on a metric space X **converges** to a **limit point** $x = \lim_{n\to\infty} x_n \in X$ iff every neighborhood of x contains **almost all** members: $\forall\epsilon > 0 \exists n(\epsilon) \in \mathbb{N} : \forall n \geq n(\epsilon) : x_n \in B_\epsilon(x)$. A point $y \in X$ is an **accumulation point** iff every neighborhood of y contains **infinitely many** members: $\forall\epsilon > 0 \forall n \in \mathbb{N} \exists m \geq n : x_m \in B_\epsilon(y)$. The sequence is a **Cauchy sequence** iff for each $\epsilon > 0$ **almost all** members have a distance less than ϵ to each other: $\forall\epsilon > 0 \exists k \in \mathbb{N} : \forall n, m \geq k : d(x_n; x_m) < \epsilon$. A Cauchy sequence converges to all of its accumulation points. X is **complete** iff every Cauchy sequence converges to a limit point in X. $\mathbb{Q}$ is not complete since e.g. $(x_n)_{n\in\mathbb{N}}$ with $x_0 := 1$ and $x_{n+1} := \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)$ is a Cauchy sequence converging to $\sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$. A function $f : (X; d) \to (X'; d')$ is **continuous** iff for every sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n \overset{n\to\infty}{\to} x \in X$ the **image sequence** $f(x_n) \overset{n\to\infty}{\to} f(x) \in X'$. The function sequence $(f_n)_{n\in\mathbb{N}} : \mathbb{N} \to (X'; d')$ converges **uniformly** resp. with reference to the **supremum norm** to $f : X \to X'$ **iff every neighborhood of f contains almost all** members: $\forall\epsilon > 0 \exists m \in \mathbb{N} : \forall n \geq m \forall x \in X : f_n(x) \in B_\epsilon(f(x))$. The following well-known theorem from analysis provides a good example for the usefulness of these concepts and will appear in a generalized form as theorem 18.3 :

1.5 Uniform convergence of continuous functions

If all $f_n : (X; d) \to (X'; d')$ are **continuous** and **converge uniformly** to f then f is also **continuous**.

Proof: $\forall x \in X \land \epsilon > 0 \exists \delta > 0 \land m \in \mathbb{N} : \forall n \geq m \land y \in B_\delta(x) : f(y) \in B_{\epsilon/3}(f_n(y))$ for **a δ independently** of the chosen y on account of the **uniform** convergence of thef_n, furthermore $f_n(y) \in B_{\varepsilon/3}(f_n(x))$ because f_n is **continuous** in $x \in X$ and finally $f_n(x) \in B_{\epsilon/3}(f(x))$ due to the **convergence** of the f_n at the point x. The **triangle inequality** yields $f(y) \in B_\varepsilon(f(x))$.

Example: The **continuous** parabolae $f_n(x) = x^n$ converge in $[0; 1]$ **pointwise but not uniformly** to f with $f(x) = 0$ for $0 \leq x < 1$ and $f(1) = 1$. Obviously f is **discontinuous** in $x = 1$.

1.6 Equivalent metrics

On a metric space (X, d) the metric $d' := \frac{d}{1+d} < 1$ is **equivalent** to d: A set $O \subset X$ is open with reference to d iff it is open with reference to d'.

Proof: The positive definiteness and symmetry of d' are a direct result of the corresponding properties of d. To check the triangle equality let $a := d(x; z)$, $b := d(x; y)$ and $c := d(y; z)$. Then due to the the premise $a \leq b + c$ and because of $x \to \frac{x}{1+x}$ being monotone for $x \geq 0$ we can infer $\frac{a}{1+a} \leq \frac{b+c+bc}{1+b+c+bc} \leq \frac{b+c+2bc}{1+b+c+bc} = \frac{b}{1+b} + \frac{c}{1+c}$. From $\frac{d'}{2+d} < d' < d$ follows $B_\epsilon \subset B'_\epsilon \subset B_{\delta(\epsilon)}$ with $\delta(\epsilon) = \frac{2\epsilon}{1-\epsilon}$ and $\epsilon < 1$ and thus the equivalencce of the topologies induced bei the neighborhood systems B_ϵ und B'_ϵ.

1.7 Product metrics

On a **finite product** $\prod_{i\in I} X_i$ of metric spaces $(X_i; d_i)$ the following three metrics are equivalent:

1. $d'(x;y) := \sum_{i\in I} d_i(x_i;y_i)$
2. $d''(x;y) := \sqrt{\sum_{i\in I} d_i^2(x_i;y_i)}$
3. $d'''(x;y) := max_{i\in I} d_i(x_i;y_i)$

Proof: From $0 \le (d_i - d_j)^2 \Leftrightarrow 2d_id_j \ge 2d_id_j$ follows $\left(\sum_{1\le i\le n} d_i\right)^2 = \sum_{1\le i,j\le n} d_id_j \le \frac{1}{2}\sum_{1\le i,j\le n}(d_i^2 + d_j^2) = n\sum_{1\le i\le n} d_i^2$. This yields the estimate $d'' \le d' \le \sqrt{n}\cdot d'' \Leftrightarrow B'_r(x) \subset B''_r(x) \subset B'_{\sqrt{n}\cdot r}(x)$. Obviously we also have $d''' \le d' \le n \cdot d''' \Leftrightarrow B'_r(x) \subset B'''_r(x) \subset B'_{n\cdot r}(x)$.

1.8 Countable products of metric spaces

On a **countably infinite product** $\prod_{n\in\mathbb{N}} X_n$ of metric spaces $(X_n; d_n)$ with $d_n < 1 \forall n \in \mathbb{N}$ the expression $d(x;y) := \sum_{n\in\mathbb{N}} \frac{d_n(x_n;y_n)}{2^{n+1}}$ defines a metric.

Proof: Due to the **absolute convergence** of the **geometric series** the properties 1.2.1 - 3 transfer from the single summands resp. the partial sums to the limit.

The existence of different equivalent metrics for a common topology leads to the insight that the metric is not fundamental for the concepts of space and distance. Consequetly the follwowing parts develop the concepts of open sets, neighborhoods and continuity independently of a metric. After having established a general theory for these ideas we will return to the metric and determine necessary and sufficient conditions for its existence.

2 Topological Spaces

2.1 Open and closed sets

A set family $\mathcal{O} \subset 2^X$ on a space X is a **topology** iff it is closed under **arbitrary unions** and **finite intersections**. Every topology contains the sets $\emptyset = \bigcup\emptyset$ and $X := \bigcup\mathcal{O}$. The ordered pair $(X;\mathcal{O})$ is a **topological space**. The elements $O \in \mathcal{O}$ are the **open sets** and their complements $X \setminus O$ are the **closed sets**. On a given set X the **indiscrete topology** $\{\emptyset; X\}$ is the minimal and the **discrete topology** 2^X the maximal topology. For two topologies $\mathcal{O}_1 \subset \mathcal{O}_2$ means $\mathcal{O}_2$ **is stronger** than $\mathcal{O}_1$ und $\mathcal{O}_1$ **weaker** than $\mathcal{O}_2$.

2.2 Bases and subbases

A subfamily $\mathcal{B} \subset \mathcal{O}$ is a **basis** of the topology $\mathcal{O}$ iff for each $O \in \mathcal{O}$ and $x \in O$ exists a $B_x \in \mathcal{B}$ with $x \in B_x \subset O$. A subfamily $\mathcal{B} \subset 2^X$ is a basis for a unique topology $\mathcal{O} = \tau(\mathcal{B})$ iff $\bigcup\mathcal{B} = X$ and every open set $O = \bigcup_{x\in O}\{B_x : x \in B_x \in \mathcal{B}\}$ is a union of basis sets i.e. iff the intersection of every nonempty finite subfamily from $\mathcal{B}$ is equal to the union of elements from $\mathcal{B}$. In a **metric space** $(X;d)$ the **open balls** $B_r(x)$ for $r > 0$ and $x \in X$ constitute a basis for the open sets. The **natural topology** $\tau(d)$ on $\mathbb{R}^n$ is generated by the **euclidean norm** resp. the euclidean metric according to 1.3.

A family $\mathcal{S} \subset 2^X$ is a **subbasis** of the topology $\mathcal{O}$ iff the system of all intersections from finite subfamilies from $\mathcal{S}$ constitutes a basis $\mathcal{B}$ of the topology $\mathcal{O}$. For example the intervals $]-\infty; a[$ and $]a;\infty[$ for $a \in \mathbb{R}$ are a subbasis of the **natural topology** on $\mathbb{R}$.

2.3 Neighborhoods

A family $\mathcal{U}(x)$ is a **neighborhood system** of a point $x \in X$ and the sets $U \in \mathcal{U}(x)$ are **neighborhoods** of x iff the following conditions are satisfied:

1. Every $U \in \mathcal{U}(x)$ contains x and another neighborhood $V \in \mathcal{U}(x)$ such that $U \in \mathcal{U}(y)$ for all $y \in V \subset U$.
2. With U every overlying set $U' \supset U$ belongs to $\mathcal{U}(x)$.
3. With finitely many sets $U_1; ...; U_n$ their intersection $\bigcap_{1 \leq i \leq n} U_i$ also belongs to $\mathcal{U}(x)$.

A subfamily $\mathcal{B}(x)$ of a neighborhood system $\mathcal{U}(x)$ is a **neighborhood basis** iff for every $U \in \mathcal{U}(x)$ there is a $B \in \mathcal{B}(x)$ with $B \subset U$. The following theorem desribing the generation of a uniquely determined topology by a neighborhood system will reappear in a simplified and generalized form as Th. 12.3 relating to **uniform structures**

2.4 Equivalence of neighborhood systems and topologies

1. For a given set of **neighborhood systems** $\mathcal{U}$(x) for each point x of a set X the family $\mathcal{O}$ of all sets which are a neighborhood for each of their points constitutes a **topology** on X.
2. For a given **topology** $\mathcal{O}$ on a set X the families $\mathcal{U}$(x) of sets $U \in X$ which contain an open set $O \in \mathcal{O}$ with $x \in O \subset U$ constitute for each $x \in X$ a **neighborhood system**.
3. The neighborhood systems $\mathcal{U}$(x) of 2. are uniquely determined by condition 1.
4. The topology $\mathcal{O}$ of 1. is uniquely determined by condition 2.: $\mathcal{U}\,(x) \underset{1.}{\overset{2.}{\rightleftarrows}} \mathcal{O}$

Proof:

1. Due to 2.3.1 arbitrary unions of subsets and the whole set X belong to $\mathcal{O}$ and on account of 2.3.3 this is also true for finite intersections. The empty set $\emptyset$ belongs to $\mathcal{O}$ because it contains no point and the condition is satisfied in a trivial way.
2. The conditions 2.3.1. - 3. are trivially satisfied with $V = O$.
3. We have to show: The construction $\mathcal{U}\,(x) \longrightarrow \mathcal{O} \longrightarrow \mathcal{U}'\,(x)$ leads back to the same neighborhood system $\mathcal{U}'\,(x) = \mathcal{U}\,(x)$ it started with. Assume $\mathcal{U}'$(x) for each $x \in X$ is another neighborhood system satisfying condition 1. For $x \in X$ and $U' \in \mathcal{U}'$(x) it follows from 2.3.1 that $\overset{o}{U} := \{y \in U' : U' \in \mathcal{U}'(y)\}$ contains at least one element. Again with 2.3.1 there is a $V' \in \mathcal{U}'$(y) for a $y \in \overset{o}{U}$ with $U' \in \mathcal{U}'(z)$ for all $z \in V' \subset U'$. But this implies $V' \subset \overset{o}{U}$ and with 2.3.2 we get $\overset{o}{U} \in \mathcal{U}'$(y). Consequently $\overset{o}{U}$ is a neighourhood for each of its points and according to 1. it is the desired open set over x in U'.
4. We have to show: The construction $\mathcal{O} \longrightarrow \mathcal{U}\,(x) \longrightarrow \mathcal{O}'$ leads back to the same topology $\mathcal{O}' = \mathcal{O}$ it started with. Assume $\mathcal{O}'$ is a second topology on X satisfying condition 2. and choose a set $O' \in \mathcal{O}'$. But then O' is a neighborhood for each of its points and hence belongs to $\mathcal{O}$ according to 1.

2.5 Axioms of countability

A topological space $(X; \mathcal{O})$ is **first countable** resp. satisfies the **first axiom of countability** iff every point has a countable neighborhood basis. Every metric space $(X; d)$ is first countable with the open balls $B_{1/n}(x)$ for $n \in \mathbb{N}^*$ and $x \in X$. A topological space $(X; \mathcal{O})$ is **second countable** resp. satisfies the **second axiom of countability** iff $\mathcal{O}$ has a countable basis. According to 2.4.2 and 2.4.3 the second axiom of countability includes the first.

2.6 Closure and interior

Let A be a subset of the topological space $(X;\mathcal{O})$. A point $x \in X$ is an **accumulation point** or **limit point** of A iff every neighborhood of x intersects A. The set of all accumulation points of A is the **closure** $\overline{A}$ of A. If a subset A of a topological space X is **closed** then it contains every accumulation point of every sequence in A. The converse is only true if X is first countable. The closure $\overline{A} = \bigcap\{F \supset A : X \setminus F \in \mathcal{O}\}$ is the minimal closed set including A. A point $x \in X$ is an **interior point** of A iff A is a neighborhood of x. The set of all interior points of A is the **interior** $\overset{o}{A}$ of A. The interior $\overset{o}{A} = \bigcup\{O \subset A : O \in \mathcal{O}\}$ is the maximal open set included in A. We have $\overline{\bigcup_{i\in I} A_i} \supset \bigcup_{i\in I} \overline{A_i}$ and $\overline{\bigcap_{i\in I} A_i} \subset \bigcap_{i\in I} \overline{A_i}$ resp. $\overset{\circ}{\bigcup_{i\in I} A_i} \supset \bigcup_{i\in I} \mathring{A}_i$ and $\overset{\circ}{\bigcap_{i\in I} A_i} \subset \bigcap_{i\in I} \mathring{A}_i$ with **equality** for **finite** index sets I.

2.7 Boundary points, dense and nowhere dense sets

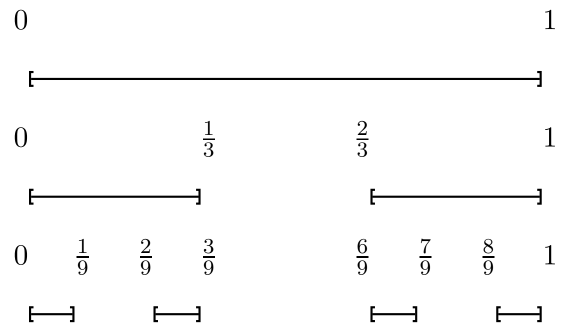
A point $x \in X$ is a **boundary point** of A iff x is a limit point of A as well as of the complement $X \setminus A$. The set of all boundary points is the **boundary** $\partial A = \overline{A} \setminus \overset{o}{A}$ of A. Since $X \setminus \partial A = \overset{o}{A} \cup X \setminus \overline{A}$ is open the boundary is always **closed**. A is **dense** in X iff every point in X is an accumulation point of A: $\overline{A} = X$. A is **nowhere dense** in X iff $\overset{o}{\overline{A}} = \emptyset$. In this case $\overline{A}$ **has no interior points** and the **complement** $X \setminus \overline{A}$ is **dense**. Conversely the **complement** of a dense set need not be nowhere dense: Both $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are **dense** in $\mathbb{R}$. For a **dense** set A its **interior** $\mathring{A}$ is dense too: $\overline{A} = \overline{\mathring{A}} = X$; for every **nowhere dense** set A its **closure** $\overline{A}$ obviously is **nowhere dense**. Note that **in general** $\overset{o}{\overline{A}} \neq \mathring{A}$ and $\overline{A} \neq \overline{\mathring{A}}$: E.g. $\mathbb{R} = \overset{o}{\overline{\mathbb{Q}}} \neq \mathring{\mathbb{Q}} = \emptyset$ and $\mathbb{R} = \overline{\mathbb{Q}} \neq \overline{\mathring{\mathbb{Q}}} = \emptyset$. The equality $\overset{o}{\overline{A}} = \mathring{A}$ holds if the **interior is nonempty**. In that case for any $x \in \overset{o}{\overline{A}}$ there is an open O with $x \in O \subset \overline{A}$ and since $x \in \partial A$ would yield the contradiction $O \cap X \setminus \overline{A} \neq \emptyset$ we infer $x \in \mathring{A}$, i.e. $\overset{o}{\overline{A}} \subset \mathring{A}$. For open A the boundary ∂A lies outwards of A and for closed A it lies inside A. In this case ∂A is nowhere dense in X. A topological space is **separable** iff it contains a **countable dense** subset. Obviously every **second countable** space is **separable** and every **separable** and **metric** space is **second countable**. The following paragraph presents a space which is **first countable** and **separable** but not **second countable.**

2.8 Natural topologies and the Sorgenfrey line

Obviously $\mathbb{N}$ is closed and nowhere dense in $\mathbb{R}$. $\mathbb{Q}$ is neither open nor closed in $\mathbb{R}$ since between two rational numbers $\frac{n}{m}$ and $\frac{n+1}{m}$ assuming w.l.o.g. $n < m$ we find an irrational number $\frac{n}{r}\sqrt{2}$ with $r < n$ such that $\frac{r}{m} < \sqrt{2} < \frac{r+1}{m}$. The rational numbers have no interior points and are dense in $\mathbb{R}$: $\overset{o}{\mathbb{Q}} = \emptyset$ and $\partial\mathbb{Q} = \overline{\mathbb{Q}} = \mathbb{R}$. The same is true for the irrational numbers $\mathbb{R} \setminus \mathbb{Q}$. The **natural topology** on $\mathbb{R}^n$ satisfies the second axiom of countability since $\mathbb{Q}^n$ is a countable dense subset and the open balls $B_{1/n}(x)$ with $n \in \mathbb{N}^*$ for $x \in \mathbb{Q}^n$ are a basis for the open sets. The half open intervals $[a;b[$ with $a < b \in \mathbb{R}$ generate the topology $\mathcal{O}_\tau$ of the **Sorgenfrey line** $\mathcal{R} := (\mathbb{R};\mathcal{O}_\tau)$ (cf. [14, example 51]) which on account of $]a;b[= \bigcup_{n\in\mathbb{N}} \left[a + \frac{1}{n}; b\right[$ is **stronger** than the **euclidean topology** but **weaker** than the **discrete** topology since atoms $\{a\}$ contain no half open intervals. The rational numbers $\mathbb{Q}$ are a **countable and dense** subset since every basis set contains a rational number. The sets $\left[a; a + \frac{1}{n}\right[$ form a **countable neighborhood basis** for every $a \in \mathbb{R}$. $\mathcal{R}$ is **separable** and **first countable**, but not **second countable**, since for every countable family $\mathcal{B} := ([a_n;b_n[)_{n\in\mathbb{N}}$ there is an $a \in \mathbb{R}$ with $a \notin \{a_n;b_n\}\ \forall n \in \mathbb{N}$ such that no interval $[a;b[$ with $b > a$ can be a union of basis sets.

2.9 The Cantor set

Let $f : \{0;2\}^{\mathbb{N}} \to [0;1]$ be defined by $f(x) = \sum_{n\geq 1} \frac{x_n}{3^n}$ for every sequence $x = (x_n)_{n\geq 1}$ with $x_n \in \{0;2\}$. Then f is **injective** and the set $T := f\left[\{0;2\}^{\mathbb{N}}\right]$ is **non countable , closed** and **nowhere dense** in $[0;1]$.



Proof: Since on the one hand $\sum_{1\leq n<m} \frac{x_n}{3^n} + 0 \leq \sum_{n\geq 1} \frac{x_n}{3^n} \leq \sum_{1\leq n<m} \frac{x_n}{3^n} + \frac{0}{3^m} + \sum_{n>m} \frac{2}{3^n} = \sum_{1\leq n<m} \frac{x_n}{3^n} + \frac{1}{3^m}$ and on the other hand $\sum_{1\leq n<m} \frac{x_n}{3^n} + \frac{2}{3^m} = \sum_{1\leq n<m} \frac{x_n}{3^n} + \frac{2}{3^m} + \sum_{n>m} \frac{0}{3^n} \leq \sum_{n\geq 1} \frac{x_n}{3^n} \leq \sum_{1\leq n<m} \frac{x_n}{3^n} + \sum_{n\geq m} \frac{2}{3^n} = \sum_{1\leq n<m} \frac{x_n}{3^n} + \frac{1}{3^{m-1}}$ the set $T = \bigcap_{n\geq 0} \bigcup_{0\leq m<\frac{1}{2}(3^n-1)} \left[\frac{2m}{3^n}; \frac{2m+1}{3^n}\right]$ is generated starting with $[0;1]$ by subsequently removing the middle third $\left]\sum_{1\leq n<m} \frac{x_n}{3^n} + \frac{1}{3^m}; \sum_{1\leq n<m} \frac{x_n}{3^n} + \frac{2}{3^m}\right[$ from the intervals $\left[\sum_{1\leq n<m} \frac{x_n}{3^n} + 0; \sum_{1\leq n<m} \frac{x_n}{3^n} + \frac{1}{3^{m-1}}\right]$. From this construction we infer that T is **closed**. For all $x \in \{0;2\}^{\mathbb{N}}$ and $m \in \mathbb{N}$ there is an $a = \sum_{n\geq 1} \frac{a_n}{3^n} \in [0;1] \setminus T$ with $|f(x) - a| = \frac{1}{3^m}$. Simply choose $a_n = x_n$ for $n \neq m$ and $a_m = 1$. Consequently T has **no interior points** and the closed character implies being nowhere dense in $[0;1]$. Since f is injective and $\{0;2\}^{\mathbb{N}}$ is not countable (cf. [19, Satz 17.9]) this is also true for the image T.

3 Continuous functions

3.1 Continuous functions

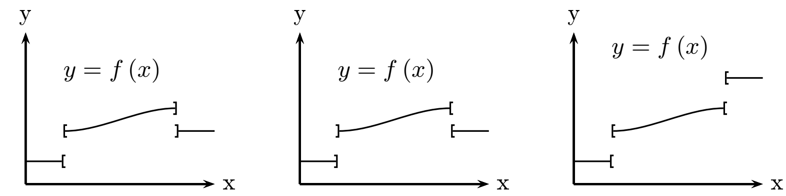
$f : (X;\mathcal{O}_X) \to (Y;\mathcal{O}_Y)$ is **continuous** iff the **inverse images** of **open** sets in $(Y;O_Y)$ under f are **open** in $(X;O_X)$: $\forall O \in \mathcal{O}_Y : f^{-1}(O) \in \mathcal{O}_X$. Since $f^{-1}(Y \setminus O) = X \setminus f^{-1}(O)$ the function f is continuous iff the inverse images of **closed** sets in $(Y;O_Y)$ under f are **closed** in $(X;O_X)$. Since $f^{-1}[\bigcap_{i\in I} A_i] = \bigcap_{i\in I} f^{-1}[A_i]$ and $f^{-1}[\bigcup_{i\in I} A_i] = \bigcup_{i\in I} f^{-1}[A_i]$ the function f is continuous iff the inverse images of a **subbasis** $\mathcal{S}$ of $\mathcal{O}_Y$ are open in $(X;\mathcal{O}_X)$. $(X;\mathcal{O}_X)$ carries the discrete topology and $(Y;\mathcal{O}_Y)$ carries the indiscrete topology iff every function $f : (X;\mathcal{O}_X) \to (Y;\mathcal{O}_Y)$ is continuous. $\mathcal{O}_1$ is **stronger** than $\mathcal{O}_2$ iff the **identity** $\mathrm{id} : (X;\mathcal{O}_1) \to (X;\mathcal{O}_2)$ is continuous. For two continuous mappings $f : (X;\mathcal{O}_X) \to (Y;\mathcal{O}_Y)$ and $g : (Y;\mathcal{O}_Y) \to (Z;\mathcal{O}_Z)$ the **composition** $g \circ f : (X;\mathcal{O}_X) \to (Z;\mathcal{O}_Z)$ is also continuous. Hence for every continuous function $f : X \to \mathbb{C}$ the **reciprocal value** $\frac{1}{f}$ for $f(x) \neq 0$, the **absolute value** $|f|$ and the **multiple** $\alpha \cdot f$ with $\alpha \in \mathbb{C}$ are again continuous.

3.2 Continuity at a point

$f : (X;\mathcal{O}_X) \to (Y;\mathcal{O}_Y)$ **is continuous at** $x \in X$ iff the **inverse images of neighborhoods** of $f(x)$ under f are **neighborhoods** of x: $\forall U \in \mathcal{U}(f(x)) : f^{-1}(U) \in \mathcal{U}(x)$. Due to 2.2 the function f is continuous at $x \in X$ iff the **inverse images of a neighborhood basis** $\mathcal{B}(f(x))$ are included in the **neighborhood system** $\mathcal{B}(x)$: $\forall B \in \mathcal{B}(f(x)) : f^{-1}(B) \in \mathcal{B}(x)$. f is continuous on X iff f is continuous at every $x \in X$. On a **first countable** space X the function f is continuous at $x \in X$ iff for every convergent sequence $(x_n)_{n\in\mathbb{N}} \subset X$ with $\lim\limits_{n\to\infty} x_n = x$ follows $\lim\limits_{n\to\infty} f(x_n) = f(x)$

3.3 Semicontinuity

$f : (X; \mathcal{O}) \to \mathbb{R}$ is **upper semicontinuous** at a point $x \in X$ iff $\forall \epsilon > 0$ $\exists U \in \mathcal{U}(x)$: $U \subset \{f < f(x) + \epsilon\}$ and **upper semicontinuous** on X iff it is continuous relative to the topology $\mathcal{O}^+ = \{]-\infty; a[: a \in \mathbb{R}\} \cup \{\emptyset; \mathbb{R}\}$. Correspondingly f is **lower semicontinuous** at $x \in X$ iff $\forall \epsilon > 0$ $\exists U \in \mathcal{U}(x)$: $U \subset \{f > f(x) - \epsilon\}$ resp. on X iff it is continuous with regard to $\mathcal{O}^- = \{]a; \infty[: a \in \mathbb{R}\} \cup \{\emptyset; \mathbb{R}\}$. A special case of an upper semicontinuous function is a cumulative **distribution function** $f : \mathbb{R} \to [0; 1]$ with $f(x) = p(X \le x)$ for a **random variable** $X : \Omega \to \mathbb{R}$ on a **probability space** $(\Omega; p)$. Such a function is **strictly increasing** with existing **left-limits** $\lim_{x \to a^-} f(x)$ and **right-limits** $\lim_{x \to a^+} f(x) = f(a)$ for $a \in \mathbb{R}$ or in short **càdlàg** (*continue à droite, limite à gauche*).



Upper Semicontinuity | Lower Semicontinuity | right continuous left limits

3.4 Homeomorphisms

$f : (X; \mathcal{O}_X) \to (Y; \mathcal{O}_Y)$ is **open** resp. **closed** iff the images of open resp. closed sets from $\mathcal{O}_X$ are again open resp. closed in $\mathcal{O}_Y$. Since $f\left[\bigcup_{i \in I} A_i\right] = \bigcup_{i \in I} f[A_i]$ (cf. [19, p. 9.2.1]) f is **open** iff the images of a **basis** $\mathcal{B}$ of $\mathcal{O}_X$ are **open** in $\mathcal{O}_Y$. Corresponding to 3.2 the mappimg f is open iff for all $x \in X$ the **images of neighborhoods of** x under f are again **neighborhoods of** $f(x)$ $\Leftrightarrow$ $\forall x \in X \forall U \in \mathcal{U}(x) : f[U] \in \mathcal{U}(f(x))$. f is a **homeomorphism** iff it is continuous, open and bijective. A continuous function $f : (X; \mathcal{O}_X) \to (Y; \mathcal{O}_Y)$ is a homeomorphism iff the **inverse** $f^{-1} : (Y; \mathcal{O}_Y) \to (X; \mathcal{O}_X)$ exists and is continuous. The topological spaces $(X; \mathcal{O}_X)$ and $(Y; \mathcal{O}_Y)$ then are **homeomorphi**c to each other. For example the **open ball** $B_1(\mathbf{0}) \subset X$ in every **Banach space** X is homeomorphic to X by means of $p : B_1(\mathbf{0}) \to X$ given by $p(\boldsymbol{x}) = \frac{\boldsymbol{x}}{1 - \|\boldsymbol{x}\|}$ and $p^{-1}(\boldsymbol{y}) = \frac{\boldsymbol{y}}{1 + \|\boldsymbol{y}\|}$.

3.5 Continuous and open functions

For $f : (X; \mathcal{O}_X) \to (Y; \mathcal{O}_Y)$ the following statements hold:

1. f is **continuous** $\Leftrightarrow f\left[\overline{A}\right] \subset \overline{f[A]} \, \forall A \subset X \Leftrightarrow f^{-1}\left[\overline{B}\right] \supset \overline{f^{-1}[B]} \, \forall B \subset Y$.
2. f is **open** $\Leftrightarrow f\left[\overline{A}\right] \supset \overline{f[A]} \, \forall A \subset X \Leftrightarrow f^{-1}\left[\overline{B}\right] \subset \overline{f^{-1}[B]} \, \forall B \subset Y$.

Proof:

1. $\Rightarrow$: Assume there is an $x \in \overline{A}$ with $f(x) \in Y \setminus \overline{f[A]}$. Since $Y \setminus \overline{f[A]}$ is **open** there is an $U \in \mathcal{U}(f(x))$ with $U \subset Y \setminus \overline{f[A]}$. Since f is **continuous** we have $f^{-1}[U] \in \mathcal{U}(x)$. Since x is an **accumulation point** of A we have $\emptyset \ne f^{-1}[U] \cap A$ and hence $\emptyset \ne f\left[f^{-1}[U] \cap A\right] \subset U \cap f[A]$ contrary to $U \subset X \setminus \overline{f[A]}$. $\Leftarrow$: Assuming f **is not continuous** there is a $x \in X$ and an $U \in \mathcal{U}(f(x))$ with $\emptyset \ne V \cap (X \setminus f^{-1}[U]) \, \forall V \in \mathcal{U}(x)$, i.e. $x \in \overline{A}$ resp. $f(x) \in f\left[\overline{A}\right]$ with $A := X \setminus f^{-1}[U]$. On the other hand because of $U \in \mathcal{U}(f(x))$ **open** and [19, Satz 9.2.3] $f(x) \notin \overline{Y \setminus U} = \overline{f[X] \setminus f[f^{-1}[U]]} \supset \overline{f[X \setminus f^{-1}[U]]} = \overline{f[A]}$, i.e. $f\left[\overline{A}\right] \not\subset \overline{f[A]}$. The second equivalence follows with $A := f^{-1}[B]$ resp. $B := f[A]$ and $A \subset f^{-1}[f[A]]$.
2. $\Rightarrow$: Assume there is a $x \in X \setminus \overline{f^{-1}[B]}$ with $f(x) \in \overline{B}$. Since $X \setminus \overline{f^{-1}[B]}$ is **open** there is an $U \in \mathcal{U}(x)$ with $U \subset X \setminus \overline{f^{-1}[B]}$. On account of f being **open** we have $f[U] \in \mathcal{U}(f(x))$. Since $f(x)$ is an **accumulation point** of B we have $f[U] \cap B \ne \emptyset$ and thus $f^{-1}[f[U] \cap B] = U \cap f^{-1}[B]$ contrary to $U \subset X \setminus \overline{f^{-1}[B]}$. $\Leftarrow$: Assuming f is **not open** there is an $x \in X$ and an $U \in \mathcal{U}(x)$ with $f[U] \notin \mathcal{U}(f(x)$, i.e. $\emptyset \ne V \cap Y \setminus f[U] = V \cap f[A] \ne \emptyset \; \forall V \in \mathcal{U}(f(x))$ with $B := Y \setminus f[U]$ and so $f(x) \in \overline{B}$ resp. $x \in f^{-1}\left[\overline{B}\right]$. On the other hand because of $U \in \mathcal{U}(x)$ is open we have $x \notin \overline{X \setminus U} = \overline{f^{-1}[Y] \setminus f^{-1}[f[U]]} = \overline{f^{-1}[B]}$. The first equivalence follows as above with $A := f^{-1}[B]$ resp. $B := f[A]$ and $A \subset f^{-1}[f[A]]$.

3.6 Closed maps

Let $f : (X;\mathcal{O}_X) \to (Y;\mathcal{O}_Y)$ be a **closed** map. Then for each $B \subset Y$ and **open** $U \subset X$ with $f^{-1}[B] \subset U$ there is an **open** $V \supset B$ with $f^{-1}[V] \subset U$.

Proof: $V := Y \setminus f[X \setminus U]$ is open and so from $X \setminus U \subset f^{-1}[f[X \setminus U]]$ follows $f^{-1}[V] = X \setminus f^{-1}[f[X \setminus U]] \subset U$. Furthermore we have $f^{-1}[B] \subset U \Rightarrow B \subset f[U] \Rightarrow Y \setminus B \supset Y \setminus f[U] \supset f[X \setminus U]$ $\Rightarrow B \subset Y \setminus f[X \setminus U] = V$.

4 Initial and final topologies

4.1 The initial topology

The **initial** or **weak topology** $\tau(f_i : i \in I)$ on a set X with reference to the functions $f_i : U_i \to Y_i$ from subsets $U_i \subset X$ covering $X = \bigcup_{i \in I} U_i$ into the topological spaces $(Y_i;\mathcal{O}_i)$ with $i \in I$ is the **minimal** resp. **weakest** topology on X such that all f_i are **continuous**. On account of 3.1 the inverse images $f_i^{-1}[O_i]$ of open sets $O_i \in \mathcal{O}_i$ form a **subbasis** for the initial topology. The initial topology is uniquely determined by the following **universal property**: any map $g : Z \to X$ is continuous iff all compositions $f_i \circ g : g^{-1}[U_i] \to Y_i$ are continuous.

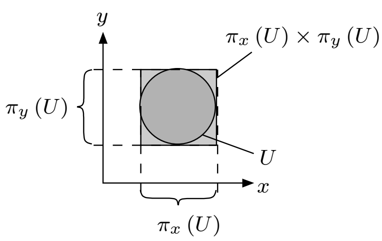
$$\begin{tikzcd} & (Y_i;\mathcal{O}_i)_{i\in I} \\ Z \arrow[ur, "f_i \circ g"] \arrow[r, "g"'] & (X;\tau) \arrow[u, "(f_i)_{i\in I}"'] \end{tikzcd}$$

Proof: For every element $f_i^{-1}[O_i]$ of the subbasis with open $O_i \in \mathcal{O}_i$ we have $g^{-1}\left[f_i^{-1}[O_i]\right] = (f_i \circ g)^{-1}[O_i]$ whence by 3.1 follows the universal property. Conversely the case $g = \mathrm{id} : (X;\tau) \to (X;\tau)$ shows that the universal property implies the continuity of all f_i and in particular that τ must contain all $f_i^{-1}[O_i]$ with $O_i \in \mathcal{O}_i$. For any other topology σ on X satisfying the universal property the case $\mathrm{id} : (X;\sigma) \to (X;\tau)$ shows that $\tau \subset \sigma$ while $\mathrm{id} : (X;\tau) \to (X;\sigma)$ implies $\sigma \subset \tau$ whence follows that τ is uniquely determined by the universal property.

4.2 The product topology

The **product topology** $\bigotimes_{i \in I} \mathcal{O}_i = \tau(\pi_i : i \in I)$ on the product space $\prod_{i \in I} X_i$ of the topological spaces $(X_i;\mathcal{O}_i)_{i \in I}$ is the initial topology of the **projections** $\pi_i : \prod_{i \in I} X_i \to X_i$. On account of 3.4 and $\pi_j\left(\bigcap_{i \in I} \pi_i^{-1}[O_i]\right) = O_j$ for $j \in I$ resp. $= X$ for $j \notin J$ the projections π_i are **open**. Owing to 3.1 the mapping $f : Y \to \prod_{i \in I} X_i$ is **continuous** iff the inverse images $f^{-1}\left[\pi_i^{-1}[O_i]\right] = (\pi_i \circ f)^{-1}[O_i]$ of subbasis sets are open in $(Y;O)$.

Hence f is continuous iff all **components** $\pi_i \circ f : (Y;\mathcal{O}) \to (X_i;\mathcal{O}_i)$ are continuous. The open sets of the product topology are unions of finite intersections of **subbasis** sets, i.e. the **basis** sets have the structure of **cylinder sets** $O = O_T \times \prod_{i \in I \setminus T} X_i$ with $O_T \in \bigotimes_{i \in T} \mathcal{O}_i$ and **finite** $T \subset I$. The drawing on the right hand shows that an open set in the product topology (the circle) is included in the product of its components (the square) but may be much smaller. Conversely it can be shown that infinite products of open sets need not be open in the product algebra.



1. Due to $\prod_{1 \le k \le n} B_{\epsilon/\sqrt{n}}(x_k) \subset B_\epsilon^n(x) \subset \prod_{1 \le k \le n} B_{\epsilon \cdot \sqrt{n}}(x_k)$ with $x = (x_1, ..., x_n)$ the **euclidean norm** $\|x\| = \sqrt{x_1^2 + ... + x_n^2}$ generates a topology on $\mathbb{R}^n$ which is identical to the product of the topologies on $\mathbb{R}$ induced by the **absolute values** $|x_k|$.
2. Regarding the complex numbers $\mathbb{C} \simeq \mathbb{R}^2$ as a product of $\mathbb{R}$ we infer from 3.1 that the **real part** $\mathrm{Re} f = p_{Re} \circ f$ and the **imaginary part** $\mathrm{Im} f = p_{Im} \circ f$ of a **continuous** mapping $f : \mathbb{C} \to \mathbb{C}$ are again **continuous**.

3. Due to $\left\{(u+v) : u \in B_{\epsilon/3}(x) \wedge v \in B_{\epsilon/3}(y)\right\} \subset B_\epsilon(x+y)$ and $\{(u \cdot v) : u \in B_\delta(x) \wedge v \in B_\delta(y)\} \subset B_\epsilon(x \cdot y)$ with $\delta = \frac{\epsilon}{3\max\{\|x\|;\|y\|;1\}}$ the **addition** $+ : \mathbb{C}^{2n} \to \mathbb{C}^n$ and **multiplication** $: \mathbb{C}^{2n} \to \mathbb{C}^n$ are continuous mappings such that the **sum** $f+g$ resp. the **product** $f \cdot g$ of continuous $f, g : \mathbb{C}^n \to \mathbb{C}^n$ are continuous again.

4. Owing to $d(x;y) \leq d(x;u) + d(u;v) + d(v;y)$ the **metric** $d : X \times X \to [0;\infty[$ is an **uniformly continuous** mapping on the product $\mathcal{O}_1(d) \otimes \mathcal{O}_2(d)$ generated by the **squares** $B_{1/n}(x) \times B_{1/n}(y)$ with $n \in \mathbb{N}^*$ since for any $(u;v) \in B_\delta(x) \times B_\delta(y)$ we have $|d(x;y) - d(u;v)| \leq d(x;u) + d(y;v) \leq 2\delta$ and consequently $d(u;v) \in B_{2\delta}(d(x;y))$. It is also uniformly continuous on the **initial topology** $\tau(d) \subset \mathcal{O}_1(d) \otimes \mathcal{O}_2(d)$ induced by the **diagonal strips** $d(x;y) < \frac{1}{n}$ with $n \in \mathbb{N}^*$**.** Due to 1.7 the product topology is identical to the **Euclidean topology** $\mathcal{O}''$ defined by the **product metric** $d''((x;y);(u;v)) = \sqrt{d^2(x;y) + d^2(u;v)}$ via the **open discs** $B''_{1/n}(x;y)$ with $n \in \mathbb{N}^*$. (cf. 11.13)

4.3 The subspace topology

The **subspace or trace topology** $\mathcal{O}_A = \tau(i_A)$ on a **subset** A of the topological space $(X;\mathcal{O}_X)$ is the initial topology with reference to the **canonical injection** $\iota_A : A \to X$ and consists of the intersections $\iota_A^{-1}[O] = O \cap A$ with open sets $O \in \mathcal{O}$. Hence a map $g : (Y;\mathcal{O}_Y) \to (A;\mathcal{O}_A)$ is continuous iff $\iota_A \circ g : (Y;\mathcal{O}_Y) \to (X;\mathcal{O}_X)$ is continuous.

1. Due to 3.1 for every continuous $f : (X;\mathcal{O}_X) \to (Y;\mathcal{O}_Y)$ its **restriction** $f \circ i_A := f|_A : (A;\mathcal{O}_A) \to (Y;\mathcal{O}_Y)$ is continuous too.

2. The converse holds only if the subsets $(A_i)_{i\in I}$ on which the restrictions $f|_{Ai} : (A_i;\mathcal{O}_{Ai}) \to (Y;\mathcal{O}_Y)$ are continuous form a **cover** of X so that every open set $O \in \mathcal{O}_Y$ can be represented as a union $f^{-1}[O] = f^{-1}[O] \cap \bigcup_{i\in I} A_i = \bigcup_{i\in I} f^{-1}[O] \cap A_i = \bigcup_{i\in I} f|_{Ai}^{-1}[O]$.

3. For all A_i **open** the covering condition is already sufficient since in that case every set $O \cap A_i$ open in $\mathcal{O}_{A_i}$ is also open in $\mathcal{O}_X$.

4. For all A_i **closed** the covering has to be at least **locally finite** i.e. for every $x \in X$ there is a neighborhood $U \in \mathcal{U}(x)$ and a finite subset $J \in I$ with $U \subset \bigcup_{i\in J} A_i$. We can assume that $x \in A_i \forall i \in J$ because in the case of $x \notin A_j$ for a $j \in J$ one may exclude the A_j from the local covering by reducing the neighborhood from U to $U \cap X \setminus A_j$. For a given $V \in \mathcal{U}(f(x))$ there is for each $i \in J$ an $U_i \in \mathcal{U}(x)$ with $f|_{Ai}[U_i] \subset V$ and $W := \bigcap_{i\in J} U_i \cap U \in \mathcal{U}(x)$ is the desired neighborhood with $f[W] \subset V$ since for each $y \in W$ there is an $i \in J$ with $y \in U_i \cap A_i$ and therefore $f(y) = f|_{Ai}(y) \in V$.

4.4 Topological embeddings

A map $f : X \to Y$ is an **embedding** of $(X;\mathcal{O}_X)$ into $(Y;\mathcal{O}_Y)$ iff f is a **homeomorphism** from X **onto** the subspace $f[X]$. For example the parametrization $f : I \to I \times \{0\} \subset \mathbb{R}^2$ of a **line** $I = [0;2\pi[$ given by $f(t) = (t;0)$ is a toplogical embedding. In particular it is open with regard to the trace topology on $I \times \{0\} \subset \mathbb{R}^2$ whereas the same map $f : I \to \mathbb{R}^2$ is still continuous but **not open** any more since $f[I]$ is not open in $\mathbb{R}^2$. The corresponding movement on the **unit circle** $g : I \to \mathbb{S}^1 \subset \mathbb{R}^2$ with $g(t) = (\cos t;\sin t)$ is not a topological embedding because the image of the (locally) open time interval $[0;a[\subset I$ with $a < 2\pi$ is neither open nor closed in $\mathbb{S}^1$. The **starting time** $\{0\} \in I$ as well as its image $f(0) = (0;0) \in I \times \{0\}$ on the **line** are **boundary points** but the corresponding image $g(0) = (1;0) \in \mathbb{S}^1$ on the **unit circle** is an **interior point**. The parametrization of the **open spiral** $h : \mathbb{R} \to \mathbb{C}$ with $h(t) = e^{it+t}$ is a topological embedding.

4.5 The final topology

The **final topology** on a set Y with respect to a family $(f_i)_{i\in I}$ of maps $f_i : X_i \to Y$ is the **maximal** or **strongest** topology so that all f_i are **continuous**. Hence a set $O \subset Y$ is **open** resp. **closed** with regard to the final toplogy iff **all** inverse images $f_i^{-1}[O]$ are **open** resp. **closed** in X_i. In general a topology on Y is the final topology with respect to the f_i iff it satisfies the **universal property**: Any map $g : Y \to Z$ **is continuous iff all $g \circ f_i : X_i \to Z$ are continuous for $i \in I$.**

$$\begin{tikzcd} Y \arrow[r, "g"] & Z \\ X_i \arrow[u, "f_i"] \arrow[ur, "g\circ f_i"'] & \end{tikzcd}$$

Proof: The final topology satisfies the universal property since due to 3.1 every map $g : Y \to Z$ is continuous iff the inverse images $f_i^{-1}[g^{-1}[O]] = (g \circ f_i)^{-1}[O]$ of open sets $O \subset Z$ are open in X_i. Then $\Rightarrow$ follows since all f_i are continuous and $\Leftarrow$ is a consequence of the definition since if all $f_i^{-1}[g^{-1}[O]]$ are open in X_i their image $g^{-1}[O]$ is open in Y. Conversely the universal property uniquely determines the final topology since $\Leftarrow$ implies the **continuity** of all f_i and therefore that all $f_i^{-1}[O]$ for open $O \subset Y$ be open in X_i whereas the direction $\Rightarrow$ demands that no other sets be included.

4.6 The coherent topology

The **coherent topology** on a set $X = \bigcup_{j\in J} X_j$ with a covering $(X_j)_{j\in J}$ is the **final topology** with regard to the **injections** $i_j : X_j \to X$. Hence a set $O \subset X$ is **open** resp. **closed** with regard to the coherent toplogy iff all intersections $O \cap X_i$ are **open** resp. **closed** in the subspace topology of X_i.

4.7 The quotient topology

Let $(X; \mathcal{O}_X)$ a topological space and R an **equivalence relation** on X with the **equivalence classes** $\overline{x} = \{y \in X : xRy\}$ forming the quotient set $X/R = \{\overline{x} : x \in X\}$. The **quotient topology** is the **final topology** with regard to the **canonical projection** $\pi : X \to X/R$ defined by $\pi(x) = \overline{x}$. It comprises exactly the images $O = (\pi \circ \pi^{-1})[O]$ of all **open saturated** sets $\pi^{-1}[O]$ but since there may be other open resp. closed sets in $\mathcal{O}_X$ apart from the **saturated** ones the **canonical projection** $\pi : X \to X/R$ in general is **not an open map**.

4.8 Quotient maps

A map $f : X \to Y$ may be decomposed into the **surjective projection** $\pi_f : X \to X/R_f$, the **canonical bijection** $\overline{f} := f \circ \pi_f^{-1} : X/R_f \to f[X]$ and the **injection** $\iota_f : f[X] \to Y$ by means of the equivalence relation $R_f \subset X \times X$ defined by $xR_f y \Leftrightarrow f(x) = f(y)$. Note that π_f^{-1} is only a **relation** but the composition $f \circ \pi_f^{-1}$ is a **function**. Since a set $\pi_f[O]$ is open resp. closed in X/R iff $\left(\pi_f^{-1} \circ \pi_f\right)[O] = (f^{-1} \circ f)[O]$ is open resp. closed in X the quotient topology consists exactly of the projections $\pi_f[O]$ of **saturated open** sets of the form $O = (f^{-1} \circ f)[O]$ and its **closed** sets are the projections of their complements, i.e. projections of **saturated closed** sets. According to 4.3 resp. 4.7 the **continuity** of f extends to all three components with regard to the **quotient topology** on X/R_f and the **trace topology** on $f[X] \subset Y$. A **map** $f : X \to Y$ is a **quotient map** iff one of the following three equivalent conditions is satisfied:

1. f is **continuous** and its canonical bijection $\overline{f}$ is **open** and therefore a **homeomorphism** between X/R_f and $f[X] \subset Y$.
2. f is **continuous** and every **open** resp. **closed saturated set** $O = (f^{-1} \circ f)[O]$ has an image $f[O]$ which is **open** resp. **closed** in $f[X]$.
3. The **trace topology** $\mathcal{O}_Y \cap f[X]$ on $f[X] \subset Y$ coincides with its **final topology** $\mathcal{O}_f$ which is then called the **identification topology** .

Proof:

1. $\Rightarrow$ 2.: Directly follows from the definition of the quotient topology in **??**. Due to the **surjectivity** of f with regard to $f[X]$ the conditions fort open and closed sets are **equivalent**.

2. $\Rightarrow$ 3.: Due to the **continuity** the trace topology on $f[X]$ contains **at most** those $O \subset f[X]$ for which $f^{-1}[O]$ is open while the **open** character of the **canonical bijection** $\bar{f}$ implies that it contains **all** of these sets.

3. $\Rightarrow$ 1.: The **continuity** of f is equivalent to $\mathcal{O}_Y \cap f[X] \subset \mathcal{O}_f$ while $\mathcal{O}_f \subset \mathcal{O}_Y \cap f[X]$ implies that every $\left(\bar{f} \circ \pi_f\right)[O] = f[O] \subset f[X]$ with **saturated open** $O = (f^{-1} \circ f)[O] \subset X$ is open in $f[X]$ whence $\bar{f} : X/R_f \to f[X]$ is an **open map**.

Examples:

1. $f : [0; 2\pi] \to \mathbb{S}^1 \subset \mathbb{R}^2$ with $f(x) = (\cos x; \sin x)$ is **continuous**, **surjective** and **closed** but not open since $f[\,]x; 2\pi]\,]$ is not open in $\mathbb{S}^1$ for $0 \leq x < 2\pi$. But due to its bijective character $\overline{f}$ is open and consequently the loop $[0; 2\pi]/R_f$ with regard to $xR_fy \Leftrightarrow f(x) = f(y)$ is homeomorphic to the **unit circle** (cf 4.4).
2. $g : \mathbb{R}^3_* = \mathbb{R}^3 \setminus \{\mathbf{0}\} \to \mathbb{S}^2$ with $g(\boldsymbol{x}) = \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}$ is **continuous**, **surjective** and **open** such that the space $\mathbb{R}^3_*/R_g$ of all **open rays** $\pi_g^{-1}(\boldsymbol{x}) = \{t\boldsymbol{x} : t > 0\}$ defined by the equivalence relation $\boldsymbol{x}R_g\boldsymbol{y}$ $\Leftrightarrow g(\boldsymbol{x}) = g(\boldsymbol{y})$ is homeomorphic to the **unit sphere**.

4.9 The restriction of a quotient map

The restriction $f|_U : U \to f[U]$ of the quotient map $f : X \to Y$ to a **saturated open** or **closed** set $U \subset X$ is again a quotient map and the trace topology $\mathcal{O}_Y \cap f[U]$ coincides with the induced final topology $\mathcal{O}_{f|_U}$ on $f[U]$

Proof: For **every saturated** $U \subset X$ and **open** $O \subset Y$ the inverse image $f|_U^{-1}[f[U] \cap O] = U \cap f^{-1}[O]$ is open in U whence $f[U] \cap O$ is open in the final topology such that we have shown $\mathcal{O}_Y \cap f[U] \subset \mathcal{O}_{f|_U}$. For an **open saturated** $U \subset X$ and every $O \in \mathcal{O}_{f|_U}$ with **open** inverse image $f|_U^{-1}[O] = f^{-1}[O] \subset U$ this inverse image is also open and saturated in X whence $O = f\left[f|_U^{-1}[O]\right] \subset f[U]$ is open in Y and consequently $O \in \mathcal{O}_Y \cap f[U]$. In the case of a **closed saturated** $U \subset X$ there is an open $f^{-1}[O] \subset V_0 \subset X$ with $f|_U^{-1}[O] = f^{-1}[O] = U \cap V_0$ and the enlarged set $V = V_0 \cup X \setminus U$ is open as well as saturated with $f|_U^{-1}[O] = U \cap V$ whence $f[V]$ is open in Y and consequently $O = f[V \cap U]$ $= f[V] \cap f[U] \in \mathcal{O}_Y \cap f[U]$ such that we have shown $\mathcal{O}_{f|_U} \subset \mathcal{O}_Y \cap f[U]$.

Note: If the restrictions of a the canonical projction to the subset $U \subset X$ is **injective** every open set $O \cap U = f|_U^{-1}[f|_U[O \cap U]]$ is **saturated** such that $f|_U$ is a **homeomorphism** and the subset U can be **identified** with its image $f[U]$.

4.10 The topological sum

The **topological sum** $\bigsqcup_{i\in I} X_i$ is defined as the union $\mathring{\bigcup}_{i\in I} X_i$ of the **disjoint** topological spaces $(X_i, \mathcal{O}_i)_{i\in I}$ endowed with the **final topology** of the **injections** $\iota_i : X_i \to \bigcup_{i\in I} X_i$, i.e. the strongest topology such that all ι_i are continuous. Owing to $(\bigcup_{i\in I} X_i) \setminus O = \bigcup_{i\in I}(X_i \setminus O)$ a set $O \subset \bigsqcup_{i\in I} X_i$ is **open** resp. **closed** iff all $O \cap X_i$ are **open** resp. **closed** in $\mathcal{O}_\mathrm{i}$ for every $i \in I$. In particular the injections are **open** as well as **closed** maps, hence **topological embeddings**. Due to the **universal property** 4.5 a map $f : \bigsqcup_{i\in I} X_i \to Y$ is **continuous** iff every **restriction** $f|_{X_i} = f \circ \iota_i$ is continuous. In the case of the X_i not being disjoint they are placed in separate dimensions via **indexing** and are treated in the form $(X_i \times \{i\}, \{\bigcup_{x\in O}(x; i) : O \in \mathcal{O}_i\})_{i\in I}$ such that the **trace topology** on $X_i \subset \bigsqcup_{i\in I} X_i$ coincides with the original topology $\mathcal{O}_\mathrm{i}$.

4.11 The attaching lemma

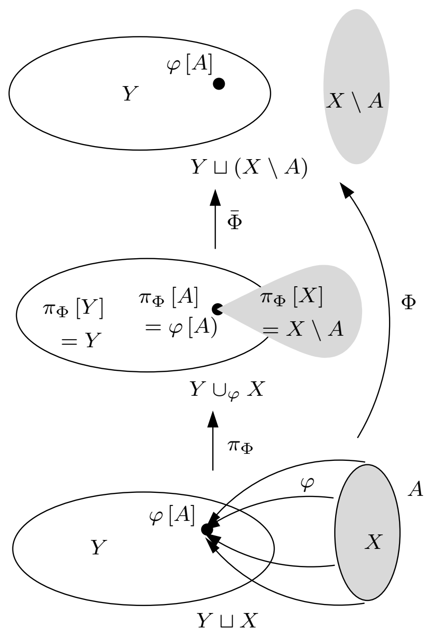
For **continuous** maps $f_i : A_i \to Y$ coinciding on the intersections $f_i|_{A_i \cap A_j} = f_j|_{A_i \cap A_j}$ with $i; j \in I$ of either an **open** or a **finite closed cover** $(A_i)_{i\in I}$ of a common space $X \subset \bigcup_{i\in I} A_i$ exists a **unique continuous extension** $f : X \to Y$ with $f|_{A_i} = f_i$.

Note: The attaching lemma is tacitly applied in the construction of the **Möbius strip** 20.12, the **torus** 20.16 and related manifolds by attaching suitable neighborhoods with corresponding parametrizations.

Proof: f is well defined by $f(x) = f_i(x)$ for $x \in A_i$ and for open $O \subset Y$ and open A_i all $f_i^{-1}[O]$ are open in A_i, hence open in X and so is the the preimage $f^{-1}[O] = \bigcup_{i\in I} f_i^{-1}[O]$. In the case of closed A_i we consider a closed $A \in Y$ with $f_i^{-1}[A]$ closed in all A_i, hence closed in X and so is the finite union $f^{-1}[A] = \bigcup_{i\in I} f_i^{-1}[A]$.

4.12 Adjunction spaces

Two **disjoint** topological spaces X and Y are **attached** by means of a **continuous** map $\varphi : A \to Y$ on a **closed** set $A \subset X$ in the form of the **adjunction space** $X \cup_\varphi Y := (X \sqcup Y) / R_\Phi$ with regard to the **canonical extension** $\Phi : X \sqcup Y \to (X \setminus A) \sqcup Y$ defined by $\Phi|_A = \varphi$ and $\Phi|_{X\setminus A \sqcup Y} = \mathrm{id}$. Hence according to 4.10 a set $\pi_\Phi[O]$ is open resp. closed in $X \cup_\varphi Y$ iff $X \cap O$ is open in X and $Y \cap O$ is open in Y. Then



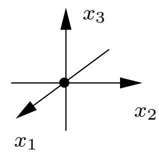
1. The restriction $\pi_\Phi|_Y : Y \to X \cup_\varphi Y$ is an **embedding** and $\pi_\Phi[Y]$ is **closed** in $X \cup_\varphi Y$.
2. The restriction $\pi_\Phi|_{X\setminus A} : X \setminus A \to X \cup_\varphi Y$ is an **embedding** and $\pi_\Phi[X \setminus A]$ is **open** in $X \cup_\varphi Y$.
3. The adjunction space $X \cup_\varphi Y$ is the **disjoint union** of $\pi_\Phi[Y]$ and $\pi_\Phi[X \setminus A]$ and it is **homeomorphic** to the **topological sum** $(X \setminus A) \sqcup Y$. In particular the **sum topology** coincides with the **final topology** of the canonical extension Φ.

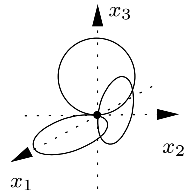
Proof:

1. The restriction of the canonical projection $\pi_\Phi|_Y$ is obviously **continuous**, **injective** and it is also a **closed** map since for every closed $B \subset Y$ the intersection $\pi_\Phi^{-1}[\pi_\Phi[B]] \cap Y = B$ is closed in the closed subset Y while the intersection $\pi_\Phi^{-1}[\pi_\Phi[B]] \cap X = \varphi^{-1}[B]$ is closed in the closed subset X whence according to 4.8 the saturated set $\pi_\Phi^{-1}[\pi_\Phi[B]]$ is closed in the disjoint union $X \sqcup Y$ such that according to 4.8 $\pi_\Phi[B]$ is closed in the quotient topology of $X \cup_\varphi Y$.
2. Follows from $\pi_\Phi^{-1}[\pi_\Phi[C]] = C$ for every $C \subset X \setminus A$.
3. Obvious since $\pi_\Phi[A] \subset \pi_\Phi[X]$. The homeomorphy is a consequence of 1. and 2.

4.13 The wedge sum

The **wedge sum** $\bigvee_{i \in I} X_i = \left(\bigsqcup_{i \in I} X_i\right) \cup_g \{p\}$ of the **disjoint** topological spaces $(X_i, \mathcal{O}_i)_{i \in I}$ with regard to the **base points** $p_i \in X_i$ is the **adjunction space** of the **topological sum** $\bigsqcup_{i \in I} X_i$ and the **base point** p attached to each other by $g : (p_i)_{i \in I} \to \{p\}$ with $g(p_i) = p$ for every $i \in I$. All open sets $O_i \subset X$ with $p_i \notin O_i$ are still open in $\bigvee_{i \in I} X_i$ but the neighborhood system of the base point consists of all **unions** $\bigcup_{i \in I} O_i$ of open sets containing $p_i \in O_i$. For example the wedge sums $\bigvee_{1 \le i \le n} \mathbb{R}_i$ resp. $\bigvee_{1 \le i \le n} \mathbb{S}^1$ are homeomorphic to the union of the **coordinate axes** $\bigcup_{1 \le i \le n} \{x_j = 0 \, \forall j \neq i\} \subset \mathbb{R}^n$ resp. to a subset of the **bouquet of circles** $\bigcup_{1 \le i \neq j \le n} S_{i;j} \subset \mathbb{R}^n$ with $S_{i;j} = \partial \bar{B}_1^n(\boldsymbol{e}_i) \cap \{x_k = 0 \, \forall k \neq i \vee k \neq j\}$ each furnished with the **trace topology** in $\mathbb{R}^n$.





5 Connected spaces

5.1 Connectedness

A topological space $(X; \mathcal{O})$ is **connected** iff it cannot be decomposed into two **disjoint open** sets. Hence it is connected iff there are no other sets being **open as well as closed** apart from $\emptyset$ and X. It is **totally disconnected** iff every set in X is open as well as closed. $(X; \mathcal{O})$ is connected iff there is a surjective mapping on a discrete space containing at least two points. The **continuous image** $f[X]$ of a connected space X obviously stays connected. A connected set A stays connected if boundary points are added: any B satisfying $A \subset B \subset \overline{A}$ is still connected. If a connected set A contains interior and exterior points of a set B it also must contain boundary points of B since otherwise $\overset{o}{B}$ and $X \setminus \overset{o}{B}$would constitute an open disjoint covering of A. The union $A \cup B$ of two connected sets A und B is connected iff $A \cap B \neq \emptyset$.

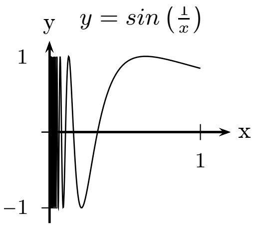
5.2 Intervals and the intermediate value theorem

Every **interval** $I \subset \mathbb{R}$ is **connected** with reference to the natural topology. In particular every continuous function $f : X \to \mathbb{R}$ on a connected set X with $s, t \in f[X]$ assumes every value between s and t.

Proof: Assume I is open. Let O_1 and O_2 be open disjoint sets in $\mathbb{R}$ both meeting I with $I \subset O_1 \cup O_2$. Let $u \in O_1 \cap I$, $v \in O_2 \cap I$ with $u < v$ and $s := \sup\{w \in I : [u; w] \subset O_1\}$. Hence $u < s < v$ and due to the definition of an interval we have $s \in I$ so either $x \in O_1$ or $x \in O_2$. Since I is open there is an $\epsilon > 0$ with either $B_{2\varepsilon}(s) \subset I \cap O_1$or O_1 or in $B_{2\varepsilon}(s) \subset I \cap O_2$. In the first case we have $[u; s + \epsilon] \subset O_1$ and s is not an upper **bound**. In the second case it follows that $[u; s - \epsilon] \nsubseteq O_1$ and s is not the **least** upper bound. On account of 5.1 the connectedness extends to arbitrary intervals.

5.3 Connected graphs

In set theory a function is defined as a set of ordered pairs such that the second coordinate $y = f(x) \in Y$ is uniquely determined by the first coordinate $x \in X$ (cf [19, p. 9.1]). In analysis a (not necessarely existing) **algebraic expression** $f(x)$ determining the value of the second coordinate from a given value of the first coordinate is usually being referred to as the **function** in contrast to its **graph** $G_f = \{(x; f(x)) : x \in X\} \subset X \times Y$ illustrating the **geometric aspect** of f. Although set theory does not distinguish between graph and function this text will follow the analytical fashion:. Hence the "graph" $G_f = \{(x; f(x)) : x \in I\} \subset \mathbb{R}^2$ of a continuous real-valued function $f : I \to \mathbb{R}$ on the **interval** $I \subset \mathbb{R}$ is itself connected in $\mathbb{R}^2$ since it is the continuous image of the **trajectory** $k : I \to \mathbb{R}^2$ with $k(t) = (t; f(t))$



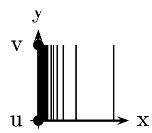
which in turn is continuous according to 4.2. An interesting case is $I =]0;1]$ and $f(x) = \sin\left(\frac{1}{x}\right)$. Due to 5.1 the **closure** $\overline{G_f} = G_f \cup \{0\} \times [-1;1]$ is also connected but for $I = [-1;1] \setminus \{0\}$ this is not true any more because the **right half plane** $H_+ = \{(x;y) \in \mathbb{R}^2 : x > 0\}$ and the **left half plane** $H_- = \{(x;y) \in \mathbb{R}^2 : x < 0\}$ form an open disjoint covering of G_f. By adding an arbitrary **boundary point** $(0;a)$ with $-1 \le a \le 1$ the connectedness can be restored. The **Sorgenfrey line** from 2.8 is **totally disconnected** since all basis sets $[a,b[= \mathbb{R} \setminus (]-\infty;a[\cup [b;\infty[)$ are closed on account of $]-\infty;a[= \bigcup_{n\in\mathbb{N}} [a-n;a[$ resp. $[a,\infty[= \bigcup_{n\in\mathbb{N}} [a;a+n[$.

5.4 Connected components

A **simple chain** between two points a and b in a topological space X is a **finite sequence of open sets** $U_1, ..., U_n$, so that only the first set contains a, only the last set contains b and each set intersects only the directly adjacent sets. a and b are **connected** iff **every open covering** $\mathcal{U}$ of X contains a simple chain between a and b. Since this definition refers to **every possible** open covering the elements of a simple chain must be connected and since the union of two open connected sets is again connected the union of a chain is an open connected set. For a given open covering $\mathcal{U}$ the union $K_{\mathcal{U}}(a)$ of all chains in $\mathcal{U}$ containing a is an open connected set and consequently if a and b are connected there is an **open connected set** containing both points. The converse is also true for in the case of an open covering $\mathcal{U}$ such that $b \notin K_{\mathcal{U}}(a)$ and vice versa we have a partition of X into disjoint open sets $K_{\mathcal{U}}(a)$, $K_{\mathcal{U}}(b)$ and possibly $X \setminus (K_{\mathcal{U}}(a) \cup K_{\mathcal{U}}(b))$ such that **every** open set O containing both a and b admits an open partition of the corresponding intersections. The connectedness between two points is an equivalence relation and the equivalence class of a point x is the **connected component** $K(x)$ i.e. the union of all connected open sets containing x. The connected components form a **disjoint covering** of X. Every $K(x)$is closed since $y \notin K(x)$ must have a neighborhood $U \in \mathcal{U}(y)$ contained in $X \setminus K(x)$. Hence if the number of connected components is finite, they are open as well as closed. In general every open as well as closed set containing x also contains $K(x)$. Consequently $K(x)$ lies in the intersection of alle open as well as closed sets containig x. **But $K(x)$is not equal to this intersection:**

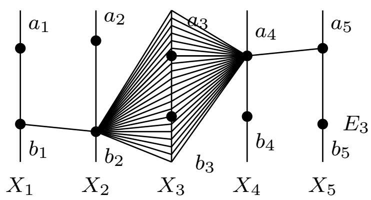
5.5 Connected components in the plane

The set $X \subset \mathbb{R}^2$ contains the points $u = (0;0)$, $v = (0;1)$ and the lines $s_n = \left\{\frac{1}{n}\right\} \times [0;1]$ for $n \in \mathbb{N}^\star$ with the trace topology in $\mathbb{R}^2$. Since all s_n are disjoint, connected, open and closed sets it follows that $K(u) = \{u\}$ and $K(v) = \{v\}$. But any open and closed set M containing u meets infinitely many s_n and must contain all of these lines since each of them is connected. Hence v is a boundary point of M and so belongs to M. $K(u) = \{u\}$ and $K(v) = \{v\}$ are closed but not open.



5.6 Connected products

$X = \prod_{i\in I} X_i$ is connected iff all X_i are connected



Proof: $\Rightarrow$ follows from the continuity of the components resp. 5.1. Concerning $\Leftarrow$ we show that the connected component $K(a)$ of an arbitrary point $a = (a_i)_{i\in I} \in X$ is **dense** in X and apply 5.1. Let $U = \bigcap_{k\in K} p_k^{-1}(U_k)$ with U_k open in X_k and finite $K \subset \mathbb{N}$ an arbitrary basis set of the product topology on X. W.l.o.g. let $K = \{1;...;n\}$, choose $b_k \in U_k$ for $k \in K$ and define $E_1 = \{x \in X : x_1 \in X_1 \text{ and } x_i = a_i \text{ else}\}$, $E_2 = \{x \in X : x_1 = b_1, x_2 \in X_2 \text{ and } x_i = a_i \text{ else}\}$,..., $E_n = \{x \in X : x_1 = b_1, ..., x_{n-1} = b_{n-1}, x_n \in X_n \text{ and } x_i = a_i \text{ else}\}$. The E_i are homeomorph to the X_i and connected due to 5.1. Also we have $E_i \cap E_{i+1} \neq \emptyset$ and again due to 5.1 the set $A = \bigcup_{1\le i\le n} E_i$ is connected. On account of $a \in E_1 \subset A$ we have $A \subset K(a)$ and hence $\emptyset \neq U \cap E_n \subset U \cap A \subset U \cap K(a)$. Consequently $K(a)$ intersects every basis set U and so is dense in X.

5.7 Connected components of products

For every $x = (x_i)_{i \in I} \in \prod_{i \in I} X_i$ is $K(x) = \prod_{i \in I} K(x_i)$.

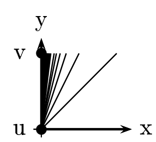
Proof: Due to 5.6 the product $\prod_{i \in I} K(x_i)$ is connected and contains x so that $\prod_{i \in I} K(x_i) \subset K(x)$. On the other hand with $K(x)$ every $p_i(K(x))$ is connected too and contains x_i. Hence $p_i(K(x)) \subset K(x_i)$ for all $i \in I$ and hence $K(x) \subset \prod_{i \in I} K(x_i)$.

5.8 Path connectedness

A continuous map $f : I \to X$ on the closed interval $I = [0;1] \subset \mathbb{R}$ into the topological space X is a **path** and X is **path connected** iff for any two points $x, y \in X$ there is a path f with $f(0) = x$ and $f(1) = y$. Since with I the continuous image $f[I]$ is connected **every path connected space is connected**. The converse is not true since for example the closure $\overline{f}$ of 5.3 is connected but not path connected. A map $g : [0;1] \to \overline{f}$ with $g(0) = (0;0)$ and $g(0) \neq g(1) \in \overline{f}$ cannot be continuous in $x = 0$ since for any $\delta > 0$ there is a $x < \delta$ with $\|g(x) - g(0)\| = \|g(x)\| > 1$. This example also shows that the **closure** of a path connected space is not necessarily path connected. **Connected open subsets** $O \subset X$ **of metric spaces** $(X;d)$ **are path connected**, since in that case the **path connected open neighborhoods** $B_\epsilon(x)$ form a **neighborhood basis** for every $x \in X$ whence the **path component** $P(x)$ of all points being path connected to x is **open** such that the **path components constitute a disjoint open partition** of O.

5.9 Local path connectedness

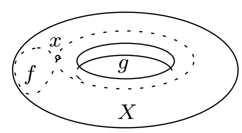
A topological space X is **locally path connected** iff for every $x \in X$ and every neighborhood U of x there is a subordinate path connected neighborhood $V \subset U$ of x. A connected as well as locally path connected X is also (globally) path connected since from the open cover of X by the path connected neightbourhoods of all points $x, y \in X$ we can choose a finite chain of path connected sets whose union is again path connected and contains x as well as y.



For example the set $X = \{0\} \times [0;1] \cup \bigcup_{n \in \mathbb{N}^\star} \left\{(x; nx) \in \mathbb{R}^2 : 0 \leq x \leq \frac{1}{n}\right\} \subset \mathbb{R}^2$ is **connected** and **path connected** (cf. 5.8) but not **locally path connected** since every neighborhood of a point $(0;t)$ with $0 \leq t < 1$ on the vertical line meets infinitely many of the skewed lines and thus can only be path connected if it contains the **node** $(0;0)$ of these lines.

5.10 Simple connectivity

A **loop** in a topological space X based at $x \in X$ is a **closed path** f with $f(0) = f(1) = x$ and it is **contractible** iff there is a continuous **contraction** $F : I^2 \to X$ with $I = [0;1]$ such that $F(t;0) = f(t)$ and $F(t;1) = x$ for every $t \in I$. The space X is **simply connected** iff it is **connected** and every loop in X is **contractible**. Thus simple connectivity indicates **holes** in **manifolds** as e.g. the **torus** X which is **connected** as well as **locally path connected**. The loop f on its side is contractible while the loop ag round its hole is not.



6 Filters and convergence

6.1 Filter

A **Filter** $\mathcal{F}$ on a set X is a family of subsets of X containing for every member F_1 its **nonempty intersection** $F_1 \cap F_2$ with any other member F_2 as well as every **overlying set** $F_3 \supset F_1$ and especially X. A subfamily $\mathcal{B} \subset \mathcal{F}$ is a **filter basis** of $\mathcal{F}$ iff every element of $\mathcal{F}$ includes an element of $\mathcal{B}$. Hence a family of subsets is a filter basis iff every nonempty intersection of two members of $\mathcal{B}$ includes a member of $\mathcal{B}$. For every nonempty subset $A \subset X$ the family of all overlying sets F with $A \subset F \subset X$ is a filter. For an **atom** $A = \{x\}$ this filter includes the neighborhood system $\mathcal{U}(x)$ which is itself a filter, namely the **neighborhood filter**. For a **sequence** $(x_i)_{i \in \mathbb{N}} \subset X$ the **tails** $B_k := \{x_i : i \geq n\}$ for $k \in \mathbb{N}$ form a basis for the filter induced by the sequence.

6.2 Ultrafilter

A filter $\mathcal{F}_1$ is **included** in a filter $\mathcal{F}_2$ resp. $\mathcal{F}_1 \subset \mathcal{F}_2$ iff every member of $\mathcal{F}_1$ **contains** a member of $\mathcal{F}_2$. We say that $\mathcal{F}_1$ is **weaker** than $\mathcal{F}_2$ and $\mathcal{F}_2$ is **stronger** than $\mathcal{F}_1$. A filter is an **ultrafilter** iff there is no **stronger** filter on X. Every filter is included in an ultrafilter since every **linearly ordered** subfamily Φ_0 of the family Φ of all filters containing the given filter $\mathcal{F}$ has the upper bound $\bigcup \Phi_0 \in \Phi$ whence Φ is **inductive** with regard to the inclusion $\subset$. By applying **Zorn's lemma** [19, p. 14.2.4] we obtain a **maximal element** $\mathcal{G}$ of Φ which is the desired ultrafilter.

6.3 Characterization of ultrafilters

$\mathcal{F}$ is an ultrafilter on X iff for any subset $A \subset X$ either $A \in \mathcal{F}$ or $X \setminus A \in \mathcal{F}$.

Proof: Due to $A \cap X \setminus A = \emptyset$ the filter $\mathcal{F}$ cannot contain two sets $F_1 \subset A$ and $F_2 \subset X \setminus A$: All elements of $\mathcal{F}$ intersect either A or $X \setminus A$. W.l.o.g. assuming the first case $\{F \cap A : F \in \mathcal{F}\}$ is a basis for a filter $\mathcal{G}$ which is stronger than $\mathcal{F}$ and contains A. Since $\mathcal{F}$ is an ultrafilter it follows $\mathcal{F} = \mathcal{G}$ and consequently $A \in \mathcal{F}$. Conversely let $\mathcal{F}$ be a filter containing every subset of X or its complement. Then for any filter $\mathcal{G} \supsetneq \mathcal{F}$ there exists a set $A \in \mathcal{G} \setminus \mathcal{F}$ but applying the hypothesis we also have $X \setminus A \in \mathcal{F} \subset \mathcal{G}$ so $\mathcal{G}$ cannot be a filter.

6.4 Free and principal filters

The **principal filter** of a given set A is the family of **all** sets including A. A filter $\mathcal{F}$ is **free** iff $\bigcap \mathcal{F} = \emptyset$. Hence every free filter is nonprincipal but the converse is not true since a nonprincipal filter may be a proper subset of a principal filter. A filter $\mathcal{F}$ is a **principal ultrafilter** iff $\mathcal{F} = \{F \subset X : x \in F\}$ for a $x \in X$.

6.5 Convergence

A filter $\mathcal{F} \to x$ **converges** to the **limit point** $x \in X$ iff it includes the neighborhood filter of x. The element $x \in X$ is an **accumulation point** of $\mathcal{F}$ iff x is an accumulation point for every element $F \in \mathcal{F}$. The set of accumulation points of $\mathcal{F}$ is $\bigcap_{F \in \mathcal{F}} \overline{F}$. (cf. 2.6)

Examples:

The **Fréchet filter** induced by $\mathcal{B}$=$\{]a;\infty[: a \in \mathbb{R}\}$ is **free** and does not have any accumulation points. A point is an **accumulation point** or **cluster point** of a **sequence** iff it is an accumulation point of the filter induced by the sequence. The closure $\overline{A}$ a of a nonempty set A is the set of all accumulation points of its **principal filter** $\mathcal{F} = \{F \subset X : A \subset F\}$. A point x is an accumulation point of a filter $\mathcal{F}$ iff there is a **stronger filter** $\mathcal{G}$ converging to x. $\mathcal{G}$ is generated by the basis $\mathcal{B} = \{F \cap U : F \in \mathcal{G} \wedge U \in \mathcal{U}(x)\}$. Hence an **ultrafilter** converges to its accumulation points.

6.6 Continuity

The **image** $f(\mathcal{F})$ of a filter $\mathcal{F}$ under the mapping $f : X \to Y$ is the filter generated by $\mathcal{B} := \{f[F] : F \in \mathcal{F}\}$ on Y. Due to $f[F] \cap f[G] \supset f[F \cap G]$ the family $\mathcal{B}$ is a filter basis. The image $f(\mathcal{F})$ of an **ultrafilter** $\mathcal{F}$ on X is again an ultrafilter on Y since for every set $A \subset Y$ holds either $f^{-1}(A) \in \mathcal{F} \Rightarrow A = i(i^{-1}(A)) \in i(\mathcal{F})$ or $X \setminus f^{-1}(A) \in \mathcal{F} \Rightarrow f(X \setminus f^{-1}A) = f(X) \setminus A \subset Y \setminus A \in f(\mathcal{F})$. The following three statements are equivalent:

1. $f : X \to Y$ is continuous in $x \in X$.
2. $\mathcal{U}(f(x)) \subset f(\mathcal{U}(x))$.
3. $\mathcal{F} \to x \Rightarrow f(\mathcal{F}) \to f(x)$.

6.7 Convergence on initial topologies

A filter $\mathcal{F}$ on the space X with the initial topology $\mathcal{O}$ with reference to the mappings $f_i : X \to (Y_i; \mathcal{O}_i)$ converges to $x = \bigcap_{i \in I} f_i^{-1}(x_i) \in X$ iff the **image filter** $f_i(\mathcal{F})$ converges to x_i for all $i \in I$.

Proof: $\Rightarrow$ follows from f_i being continuous and 6.6. In order to show $\Leftarrow$ we choose for every $i \in I$ and $U_i \in \mathcal{U}(x_i)$ a $F_i \in \mathcal{F}$ with $f_i[F_i] \subset U_i$ and for a basis set $U = \bigcap_{i \in E} f_i^{-1}[U_i] \in \mathcal{U}(x)$ with finite E follows that $F = \bigcap_{i \in E} F_i \in \mathcal{F}$ with $F \subset U$ hence $\mathcal{U}(\text{x}) \subset \mathcal{F}$ resp. $\mathcal{F} \to x$.

6.8 Trace filter

The **trace** $\mathcal{F} \cap A = \{F \cap A : F \in \mathcal{F}\}$ of a filter $\mathcal{F}$ on a nonempty set $A \subset X$ is a **trace filter** on A iff A intersects every filter set F. For an **ultrafilter** the trace $\mathcal{F} \cap A$ is a filter on A iff $A \in \mathcal{F}$ and in this case $\mathcal{F} \cap A$ is an ultrafilter on A. The following three statements are equivalent:

1. $x \in \overline{A}$.
2. The trace $\mathcal{U}(x) \cap A$ is a filter.
3. There is a filter on A whose image under the injection $j : A \to X$ converges to x.

7 Separation axioms

7.1 Separation axioms (*Trennungseigenschaften*)

A topological space X is a

- $\mathbf{T_1}$-space iff two distinct points in X have neighborhoods which do not meet the respective other point.
- $\mathbf{T_2}$- or **Hausdorff space** iff two distinct points in X have disjoint neighborhoods .
- $\mathbf{T_3}$**-space** iff every closed set $A \subset X$ and every $x \in X \setminus A$ have disjoint neighborhoods and **regular** iff $\mathbf{T_1}$ holds as well.
- $\mathbf{T_{3a}}$**-space** iff for every closed set $A \subset X$ and for every $x \in X \setminus A$ there is a continuous function $f : X \to [0; 1]$ with $f[A] = \{0\}$ as well as $f(x) = \{1\}$ and **completely regular** iff $\mathbf{T_1}$ is satisfied as well .
- $\mathbf{T_4}$**-space** iff any two disjoint closed sets have disjoint overlying open sets and **normal** iff it complies with $\mathbf{T_1}$ as well.

7.2 Separation axioms in metric spaces

In metric spaces all separation axioms are valid and furthermore two disjoint closed sets A and B can be separated by a continuous function $f : X \to [0;1]$ with $f[A] = \{0\}$ and $f[B] = \{1\}$.

Proof: We only need to show T_4: Let A and B be **closed** and **disjoint**. For every $x \in A$ there is an $\epsilon(x) > 0$ with $B_{2\varepsilon(x)}(x) \cap B = \emptyset$ and for each $x \in B$ an $\epsilon(x) > 0$ such that $B_{2\varepsilon(x)}(x) \cap A = \emptyset$. The sets $\bigcup_{x\in A} B_\varepsilon(x)$ resp. $\bigcup_{x\in B} B_\varepsilon(x)$ separate A and B. For every (not necessarily closed) $A \subset X$ we have $d_A : X \to \mathbb{R}^+$ with $d_A(x) := \inf\{d(x;y) : y \in A\}$ continuous since for $x_0 \in d_A^{-1}[\,]\mathrm{a};\mathrm{b}[\,]$ there is an $\epsilon > 0$ with $\mathrm{a} + 2\varepsilon < a + 2\epsilon < d_A(x_0) < b - 2\epsilon$ and consequently $B_\varepsilon(x_0) \subset d_A^{-1}[\,]a;b[\,]$, i.e. $d_A^{-1}[\,]a;b[\,]$ is open in X. Due to 3.3 the function $f(x) := \frac{d_A(x)}{d_A(x)+d_B(x)}$ is still continuous with $f[A] = \{0\}$ and $f[B] = \{1\}$. On account of $d_A(x) = 0 \Leftrightarrow x \in \overline{A}$ the sets A resp. B need not be closed but they cannot have a common boundary point.

7.3 Separation axioms in subspaces

All separation axioms with exception of T_4 are inherited by arbitrary subspaces. T_4 extends to **closed** subspaces only.

Proof: The validity of T_1 and T_2 is trivial since every neighborhood in the subspace $X \subset Y$ is the intersection of a neighborhood in Y with X. In order to show T_3 let A be closed in $X \subset Y$ and $x \in X \setminus A$. There is a neighborhood U of x in the superordinate space Y which does not meet A i.e. x is not a boundary point of A in Y either. Hence the closure $\overline{A}$ in Y can be separated from x by open sets in Y and the intersections of these open sets with X separate A and x in X. The proof of $\mathrm{T}_{3\mathrm{a}}$ is analogous: There is a continuous function $f : Y \to [0;1]$ with $f\left[\overline{A}\right] = \{0\}$ and $f(x) = \{1\}$. Due to 4.3.1 its **restriction** $f|_X$ is still continuous on X with $f|_X[A] \subset \{0\}$ as well as $f|_X(x) = 1$. Concerning T_4 we note the fact that any set A being closed with reference to a closed subspace $X \subset Y$ is still closed in the superordinate space Y.

7.4 $\mathbf{T_1}$-spaces

The following statements are equivalent:

1. X is a $\mathbf{T_1}$**-space.**
2. Every **atom** $\{x\}$ is a closed set.
3. Every set is the intersection of all of its neighborhoods.

7.5 $\mathbf{T_2}$-spaces

The following statements are equivalent:

1. X is a $\mathbf{T_2}$**-space**.
2. Every convergent filter on X has exactly one limit point.
3. Every point on X is the intersection of all of its closed neighborhoods.
4. The diagonal Δ is closed in X^2.

Proof:

1. $\Rightarrow$ 2. : Two limit points would have two disjoint neighborhoods which both would belong to $\mathcal{F}$.

2. $\Rightarrow$ 3. Assuming the intersection of all closed neighborhoods contains another point y this point would also be a boundary point of the neighborhood filter of x and according to 6.6 there would exist a stronger Filter converging to x as well as to y.

3. $\Rightarrow$ 4. : Assuming Δ is not closed there would exist two points $x \neq y$ so that $(x;y) \notin \Delta$ were a boundary point of Δ. Consequently every neighborhood $U \times V$ with $U \in \mathcal{U}(x)$ and $V \in \mathcal{U}(y)$ meets Δ i.e. every neighborhood of x meets every neighborhood of y. But then y is a boundary point of all neighborhoods of x and thus it lies in the intersection of all closed neighborhoods, i.e. $x = y$.

4. $\Rightarrow$ 1. Assuming there are two points $x \neq y$ with every neighborhood of x meeting every neighborhood of y then $(x;y) \notin \Delta$ is a boundary point of the diagonal Δ.

7.6 The cofinite topology

Every $\mathrm{T_2}$-space is $\mathrm{T_1}$ but the converse is not true: Given an infinite set X with the **cofinite topology** consisting of all complements of finite sets for two points $x, y \in X$ the set $X \setminus \{x;y\}$ is a neighborhood of x as well as of y not meeting the corresponding other point. Since in this topology no complement of an open set contains another open set there are no open disjoint sets separating x and y. Hence the cofinite topology satisfies $\mathrm{T_1}$ but not $\mathrm{T_2}$.

7.7 $\mathrm{T_3}$-spaces

A topological space X satisfies $\mathrm{T_3}$ iff the **closed** neighborhoods of every point $x \in X$ form a neighborhood basis. A look at the **indiscrete topology** shows that a $\mathrm{T_3}$-space needs neither be $\mathrm{T_1}$ nor $\mathrm{T_2}$. **Regular** spaces are **Hausdorff** on account of 7.4.2. In $\mathrm{T_3}$-spaces the **closure** $\overline{A} = \bigcap \{O \supset A : O \in \mathcal{O}\}$ of a set A is equal to the intersection of all open sets containing A and the **interior** $\overset{o}{A} = \bigcup \{F \subset A : X \setminus F \in \mathcal{O}\}$ is the union of all open sets being included in A. (cf. 2.6)

7.8 $\mathrm{T_{3a}}$-spaces

For a topological space $(X;\mathcal{O})$ with the family $C(X;[0;1])$ of all continuous functions $f : X \to [0;1]$ the following statements are equivalent:

1. $(X;\mathcal{O})$ is a $\mathbf{T_{3a}}$-space.
2. $\{f^{-1}[O] : O \in \mathcal{O}; f \in C(X;[0;1])\}$ is a basis of the topology in X.
3. Every closed set A has the representation $A = \{f^{-1}(0) : f \in C(X;[0;1])\}$.

Proof:

1. $\Rightarrow$ 2. : Any open set can be represented as the union $\bigcup_{x \in O} f_x^{-1}\,]0;1]$ of open sets produced by the continuous functions $f_x : X \to [0;1]$ with $f_x[X \setminus A] \subset \{0\}$ and $f(x) = 1$.

2. $\Rightarrow$ 3. Let A be closed in X. According to the hypothesis there is a point $x \in X \setminus A$, an open set $U_x \subset [0;1]$ and a $f_x \in C(X;[0;1])$ with $x \in f_x^{-1}[U_x] \subset X \setminus A$. Since $\mathbb{R}$ is **completely regular** there exists a $g_x \in C(\mathbb{R};[0;1])$ with $g_x[\mathbb{R} \setminus U_x] \subset \{0\}$ and $g_x(x) = 1$. Thus we get $A \subset X \setminus f_x^{-1}[U_x] = f_x^{-1}[\mathbb{R} \setminus U_x] \subset f_x^{-1}(g_x^{-1}(0)) = (g_x \circ f_x)^{-1}(0)$. Consequently $A = \bigcap_{x \in X \setminus A} (g_x \circ f_x)^{-1}(0)$.

3. $\Rightarrow$ 1. : Let A be closed in X and $x_0 \in X \setminus A$. Accoording to the hypothesis there is a $g \in C(X;[0;1])$ with $g[A] = \{0\}$ and $g(x_0) \neq 0$. Take $f(x) := \frac{g(x)}{g(x_0)}$.

The following theorem 7.9 is fundamental for **Urysohn's metrization theorems** 11.12 as well as for the **Stone-Čech-compactification 17.7:**

7.9 Embedding of a $\mathrm{T_{3a}}$- space

A $\mathbf{T_{3a}}$**-** space can be embedded in a **product** $\prod_{\varphi\in C^*(X)} I_\varphi$ of **real intervals**.

Proof: The image $\varphi[X]$ of every $\varphi : X \to \mathbb{R}$ taken from the set $\Phi := C^*(X)$ of **real continuous bounded** functions on X lies in a **minimal closed** interval $I_\varphi \subset \mathbb{R}$. The mapping $e : X \to \prod_{\varphi\in C^*(X)} I_\varphi$ with $e(x) := (\varphi(x))_{\varphi\in C^*(X)}$ is **injective** since for $x \neq y$ there is a $\varphi \in C^*(X)$ with $\varphi(x) \neq \varphi(y)$ and consequently $e(x) \neq e(y)$. The components $p_\varphi \circ e : X \to I_\varphi$ with $(p_\varphi \circ e)(x) = \varphi(x)$ are **continuous** and so is e due to 4.1. The mapping e is **open** since the inverse images $(p_\varphi \circ e)^{-1}[U] = \varphi^{-1}[U]$ with U open in I_φ constitute a basis for the topology on X owing to 7.8.2 and their images $e\left[(p_\varphi \circ e)^{-1}[U]\right] = e[X] \cap p_\varphi^{-1}[U]$ are open in $e[X]$.

7.10 Separation axioms in product spaces

All separation axioms with the exception of T_4 transfer to the product space.

Proof: T_1 and T_2 follow from the fact that $\bigcap_{i\in K} p_i^{-1}[U_i]$ with $U_i \in \mathcal{U}(x_i)$ and finite K form a neighborhood basis for every $x = (x_i)_{i\in I} \in \prod_{i\in I} X_i$. For the proof of T_3 we apply 7.7 two times: Since the components satisfy T_3 for every basis set $\bigcap_{i\in K} p_i^{-1}[U_i]$ with $U_i \in \mathcal{U}(x_i)$ we have closed A_i and open $V_i \in \mathcal{U}(x_i)$ with $V_i \subset A_i \subset U_i$. Hence $\bigcap_{i\in K} p_i^{-1}[A_i] \subset \bigcap_{i\in K} p_i^{-1}[U_i]$ is closed and includes $\bigcap_{i\in K} p_i^{-1}[V_i] \in \mathcal{U}(x)$ i.e. it is a closed neighborhood of x. Concerning $\mathrm{T_{3a}}$ let $\bigcap_{i\in K} p_i^{-1}[U_i]$ be a neighborhood of x which does not intersect A. The continuous $f_i : X_i \to [0;1]$ with $f_i[X_i \setminus U_i] \subset \{0\}$ and $f_i(x_i) = 1$ can be combined e.g. via $f(y) := \min\{(f_i \circ p_i)(y) : i \in K\}$ to form a continuous $f : \prod_{i\in I} X_i \to [0;1]$ with $f[A] \subset \{0\}$ and $f(x) = 1$. According to 3.3 the composition of$f$ needs only to maintain the continuity and the values 0 resp. 1. Hence other possible compositions include the maximum or the mean value of thef_i.

7.11 Separation axioms in quotient spaces

Let R be an equivalence relation on X and $\pi : X \to X/R$ the canonical projection.

1. X/R is a T_1-space iff the equivalence classes $\pi^{-1}(\pi(x))$ are **closed** in X for every $x \in X$.
2. X/R is a T_2-space if π is **open** and R is **closed** in X^2.
3. X/R is a T_2-space if π is **open** as well as **closed** and X is **regular**.
4. X/R is a T_4-space resp. **normal** if π is **closed** and X is a T_4-space resp. **normal**.

Proof:

1. $\Rightarrow$ follows from the continuity of the projection π and $\Leftarrow$ is obvious from 7.4.2.
2. For $\pi(x) \neq \pi(y) \in X/R$ we have $\pi^{-1}(\pi(x)) \times \pi^{-1}(\pi(x)) \subset X^2 \setminus R$ and due to R closed in X^2 there are open $U, V \subset X$ with $\pi^{-1}(\pi(x)) \times \pi^{-1}(\pi(x)) \subset U \times V \subset X^2 \setminus R$. Since π is open the images $\pi(U)$ resp. $\pi(V)$ are open disjoint neighborhoods of $\pi(x)$ resp. $\pi(y)$.
3. For $(x; y) \in X^2 \setminus R$ we have $x \notin \pi^{-1}(\pi(y))$. On account of T_1 the point x is closed in X and since π is continuous as well as closed its image $\pi(y)$ is closed and so is the inverse image $\pi^{-1}(\pi(y))$. Due toT_3 there are open and disjoint sets U and V with $x \in U$ and $\pi^{-1}(\pi(y)) \subset V$. According to 3.6 we have an open neighborhood W of $\pi(y)$ with $\pi^{-1}(\pi(y)) \subset \pi^{-1}[W] \subset V$. Thus $U \times \pi^{-1}[W]$ is a neighborhood of $(x; y)$ which does not meet R. Consequently R ist closed in X^2 and the proposition follows from 2.
4. For closed disjoint sets A and B in X/R the inverse images $\pi^{-1}[A]$ and $\pi^{-1}[B]$ are closed and disjoint in X due to the continuity of π. Owing to the hypothesis there are open disjoint U_A and U_B in X with $\pi^{-1}[A] \subset U_A$ and $\pi^{-1}[B] \subset U_B$. On account of 3.6 there are open neighborhoods V_A of A with $\pi^{-1}[V_A] \subset U_A$ and V_B of B with $\pi^{-1}[V_B] \subset U_B$. Since V_A and V_B are disjoint the proposition then follows. In the case of X satisfying T_1 every point $x \in X$ is closed and since π is closed every equivalence class $\pi(x) \in X/R$ is closed too so that the proposition follows from 7.4.2.

7.12 Continuous functions into Hausdorff spaces

The **graph** $\{(x; y) \in X \times Y : y = f(x)\}$ of a continuous $f : X \to Y$ into a Hausdorff-space Y is **closed** in $X \times Y$ since it is the inverse image of the closed diagonal $\Delta \subset Y^2$ (cf. 7.5.4) under the continuous mapping $(f; id) : X \times Y \to Y^2$ (cf. 4.2). For **injectiv**e f the domain X is Hausdorff too since it is the continuous inverse image of the Hausdorff-space $f[X] \subset Y$ (cf. 7.3). This precondition can be forced upon X by substituting it with the **quotient space** X/R under $xRy \Leftrightarrow f(x) = f(y)$.

7.13 Extension of continuous functions in Hausdorff spaces

For two continuous functions $f, g : X \to Y$ into a Hausdorff space Y the set $\{x \in X : f(x) = g(x)\}$ is **closed** in X. Especially f and g are **identical** iff they coincide on a **dense subset** of X.

7.14 Extension of continuous functions in regular spaces

A continuous mapping $f : D \to Y$ on a **dense** set $D \subset X$ into the **regular** space Y can be extended to a continuous mapping $\overline{f} : X \to Y$ iff for every $x \in X$ the image$f(\mathcal{U}(x) \cap D)$ of the **neighborhood filter** converges in Y.

Proof: Let $\overline{f}(x)$ be the uniquely determined (cf. 7.5.2 and 7.7) limit point of $f(\mathcal{U}(x) \cap D)$. $\overline{f}$ coincides with f on D since for any $x \in D$ the image filter $\overline{f}(\mathcal{U}(x) \cap D) = f(\mathcal{U}(x) \cap D)$ converges to $f(x)$ due to 6.6. We show that $\overline{f}$ is continuous: For $x \in X$ and $U \in \mathcal{U}\left(\overline{f}(x)\right) \subset f(\mathcal{U}(x) \cap D)$ there is an open $V \in \mathcal{U}(x)$ with $f[V \cap D] \subset U$. On account of 7.7 we can assume U closed so that even $\overline{f[V \cap D]} \subset U$. For every $y \in V$ holds $V \in \mathcal{U}(y)$ and so $f[V \cap D] \in f(\mathcal{U}(y) \cap D)$. Since the limit point $\overline{f}(y)$ is an accumulation point it follows that $\overline{f}(y) \in \overline{f[V \cap D]} \subset U$ hence $\overline{f}[V] \subset U$, i.e. $\overline{f}$ is continuous.

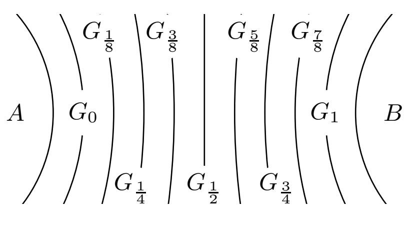
8 Normal spaces

8.1 Urysohn's lemma

A topological space X is a T_4-space iff any pair of disjoint closed sets A and B can be separated by a continuous $f : X \to [0; 1]$ with $f[A] = \{0\}$ and $f[B] = \{1\}$.

Corollaries:

1. For every **closed** set $A \subset O \subset X$ in an **open** set O in a **normal** space X exists a **continuous bump function** $f : X \to [0; 1]$ **on** A **supported in** O with $f^{-1}\{1\} = A$ and $\overline{\{f \neq 0\}} \subset O$.
2. A **normal** space is **completely regular**.



Proof: In a T_4-space for any open set G_j and an underlying closed set $\overline{G_i} \subset G_j$ exists an open set $G_{(i+j)/2}$ lying with its closure between them: $\overline{G_i} \subset G_{(i+j)/2} \subset \overline{G}_{(i+j)/2} \subset G_j$. We begin with $A \subset X \setminus B$ and apply this nesting two times to obtain G_0 and G_1 with $A \subset G_0 \subset \overline{G}_0 \subset G_1 \subset \overline{G}_1 \subset X \setminus B$. Subsequently we proceed as above for any pair $\overline{G_i} \subset G_j$ and obtain for each $i; j \in \left\{\sum_{1 \leq m \leq n} \frac{z(m)}{2^m} : z(m) \in \{0; 1\} \wedge n \in \mathbb{N}\right\}$ an open set G_i with $\overline{G}_j \subset G_i \Leftrightarrow j < i$. We define $f : X \to [0; 1]$ by $f(x) = \inf\{t \in \mathbb{R} : x \in G_t\}$ if $x \notin G_1$ and $f(x) = 1$ if $x \in G_1$. Hence we obtain the desired properties $f[A] = \{0\}$, $f[B] = \{1\}$ and $f(x) \leq i \Leftrightarrow x \in G_i$. f is continuous in $x \in X$ since for any $\epsilon > 0$ we have $f\left[G_{f(x)+\delta} \setminus \overline{G}_{f(x)-\delta}\right] \subset B_\varepsilon(f(x))$ and $G_{f(x)+\delta} \setminus \overline{G}_{f(x)-\delta} \in \mathcal{U}(x)$ for $0 < \delta < \epsilon$. Conversely if such an f exists the corresponding open disjoint sets are given in the form of $A \subset f^{-1}[B_\varepsilon(0)]$ resp. $B \subset f^{-1}[B_\varepsilon(1)]$ with $\epsilon < \frac{1}{2}$.

8.2 G_δ- and F_σ-sets

A subset A of a topological space X is a **G_δ-set** iff it is the countable intersection $A = \bigcap_{n\in\mathbb{N}} G_n$ of open sets G_n and it is a F_σ**-set** iff it is the countable union $A = \bigcup_{n\in\mathbb{N}} F_n$ of closed sets F_n. (G = *Gebiet; F = fermé ; σ = Summe ; δ = Durchschnitt*).

A nonempty **closed** set A in a T_4-space X is a **kernel** $A = f^{-1}\{0\}$ of a **continuous** $f : X \to [0;1]$ iff it is a G_δ-set. Correspondingly the closure of a nonempty open set $O = \{f \neq 0\}$ is the support of a **continuous** $f : X \to [0;1]$ iff it is a F_σ-set.

Proof:

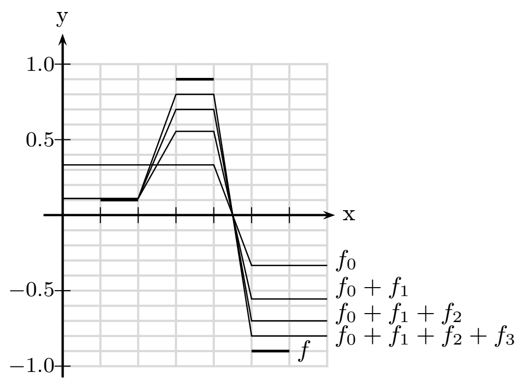
$\Rightarrow$: Assuming the open sets G_n from the preceding definition and the **separating functions** $f_n : X \to [0;1]$ with $f_n[A] = \{0\}$ and $f_n[X \setminus G_n] = \{1\}$ according to 8.1 we obtain the desired function by $f(x) = \sum_{n\in\mathbb{N}} \frac{f_n(x)}{2^n}$ which is continuous due to the continuity of the partial sums (1.6).

$\Leftarrow$: The given f provides the desired open sets by means of $G_n = f^{-1}\left[B_{1/n}(0)\right]$.

8.3 Tietze's extension theorem

A topological space X is a T_4-space iff every real-valued continuous function on a closed subset $A \subset X$ can be extended to X.

Proof:



$\Leftarrow$: Let A be closed in X and since $\mathbb{R}$ is homeomorph to $]-1;1[$ (cf. 3.4) we assume a continuous $f : A \to]-1;1[$. According to 8.1 there is a continuous $f_0 : X \to \left[-\frac{1}{3};\frac{1}{3}\right]$ with $f_0\left[\left\{f \leq -\frac{1}{3}\right\}\right] = \left\{-\frac{1}{3}\right\}$ and $f_0\left[\left\{f \geq \frac{1}{3}\right\}\right] = \left\{\frac{1}{3}\right\}$. This approximation of order $n = 0$ satisfies $|f(x) - f_0(x)| \leq \frac{2}{3}$. Applying 8.1 again we improve this approximation for every $n \geq 1$ by means of $f_n : X \to \left[-\frac{1}{3}\left(\frac{2}{3}\right)^n;\frac{1}{3}\left(\frac{2}{3}\right)^n\right]$ with $f_n\left[\left\{f - \sum_{0\leq i\leq n-1} f_i \leq -\frac{1}{3}\left(\frac{2}{3}\right)^n\right\}\right] = \left\{-\frac{1}{3}\left(\frac{2}{3}\right)^n\right\}$ and $f_n\left[\left\{f - \sum_{0\leq i\leq n-1} f_i \geq \frac{1}{3}\left(\frac{2}{3}\right)^n\right\}\right] = \left\{\frac{1}{3}\left(\frac{2}{3}\right)^n\right\}$. Due to 1.6 the limit $\overline{f} := \sum_{n\in\mathbb{N}} f_n$ is continuous with $\left|\overline{f}(x)\right| \leq \sum_{n\in\mathbb{N}} \frac{1}{3}\left(\frac{2}{3}\right)^n = 1$ and coincides with $f$on A. Finally we remove the remainig values ± 1 by means of a 8.1 yet again: Let $g : X \to [0;1]$ be continuous with $g\left[\left\{\left|\overline{f}\right| = 1\right\}\right] \subset \{0\}$ and $g[A] = 1$. Then $g \circ \overline{f} : X \to]-1;1[$ is the desired continuous extension of f.

$\Rightarrow$: For disjoint closed sets A and B the function $f : A \cup B \to]1;1[$ with $f[A] = \left\{-\frac{1}{2}\right\}$ and $f[B] = \left\{\frac{1}{2}\right\}$ is continuous on the closed set $A \cup B$ since the connected components A and B are both open as well as closed on $A \cup B$. Hence it can be extended to a continuous $\overline{f} : X \to]-1;1[$. The sets $\overline{f}^{-1}[]-1;0[]$ und $\overline{f}^{-1}[]0;1[]$ are open and disjoint sets separating A and B.

8.4 Open covers

A set family $(U_i)_{i\in I}$ on a topological space X is **open** resp. **closed** iff the corresponding property is assumed by all U_i and **finite** resp. **countable** referring to the index set I. It is **point finite** resp. **locally finite** iff every point $x \in X$ meets only finitely many U_i resp. has a **neighborhood** which meets only finitely many U_i.

For every closed set A in a **normal** space X **and a point finite open cover** $(U_i)_{i\in I}$ of A there is a further open cover $(O_i)_{i\in I}$ of A with $\overline{O}_i \subset U_i$ for all $i \in I$.

Proof: Let $\mathcal{M}$ be the family of all open covers of A of the form $(O_k)_{k\in K} \cup (U_l)_{l\in L}$ with $K \cup L = I$, $K \cap L = \emptyset$ and $\overline{O}_k \subset U_k$ for $k \in K$. For two covers $\mathcal{C} = (O_k)_{k\in K} \cup (U_l)_{l\in L} \in \mathcal{M}$ and $\mathcal{C}' = (O'_k)_{k\in K'} \cup (U_l)_{l\in L'} \in \mathcal{M}$ let $\mathcal{C} \leq \mathcal{C}'$ iff $K \subset K'$ and $O_k = O'_k$ for all $k \in K$. For a linearly ordered subfamily $(\mathcal{C}^s)_{s\in S} = (O^s_k)_{k\in K^s} \cup (U_l)_{l\in L^s} \subset \mathcal{M}$ let $K := \bigcup_{s\in S} K^s$, $L := \bigcap_{s\in S} L^s$ and $\mathcal{C} := (O^s_k)_{k\in K} \cup (U_l)_{l\in L}$. Since $K \cup L = I$ and $K \cap L = \emptyset$ the family $\mathcal{C}$ is well defined. It is also a cover: Assume $x \in A$ and $P(x) := \{i \in I : x \in U_i\}$. In the case of $P(x) \cap L = \emptyset$ and since $(U_i)_{i\in I}$ is a cover there is an i with $x \in U_i \in \mathcal{C}$. If $P(x) \subset K$ and since $P(x)$ is finite there is an s with $P(x) \subset K^s$ and due to the **linear order** of $(\mathcal{C}^s)_{s\in S}$ and the covering property of the $\mathcal{C}^s$ there is an $i \in K$ with $x \in O_i \in \mathcal{C}$. Hence $\mathcal{C}$ is a cover of A and consequently the upper bound of $(\mathcal{C}^s)_{s\in S}$. **Zorn's lemma** (cf. [19, Satz 14.2.4]) then delivers a maximal element $\mathcal{C}^* = (O_k)_{k\in K^*} \cup (U_l)_{l\in L^*}$. Assuming $L^* \neq \emptyset$ for an $i \in L^*$ the set $B := A \setminus \left(\bigcup_{k\in K^*} O_k \cup \bigcup_{l\in L^*\setminus\{i\}} U_l\right)$ must be closed and is included in an open set U_i. Since X is normal there is an open O_i with $B \subset O_i \subset \overline{O}_i \subset U_i$ and substituting U_i with O_i results in $\mathcal{C}^{**} = (O_k)_{k\in K^*\cup\{i\}} \cup (U_l)_{l\in L^*\setminus\{i\}} \in M$ with $\mathcal{C}^* < \mathcal{C}^{**}$ contrary to the maximality of $\mathcal{C}^*$. Hence we conclude $L^* = \emptyset$ which completes the proof.

8.5 Partitions of unity

The **support** of a continuous function $f : X \to \mathbb{R}$ is the **closure** $\overline{A}$ of the set $A = \{x \in X : f(x) \neq 0\}$. A system $(f_i)_{i\in I}$ of **continuous functions** $f_i : X \to \mathbb{R}^*$ is a **partition of unity subordinate to the open cover** $(U_i)_{i\in I}$ iff the supports of the f_i form a locally finite system, are contained in the U_i for all $i \in I$ and satisfy $\sum_{i\in I} f_i(x) = 1$ for all $x \in X$. Since the supports are locally finite the sum $\sum_{i\in I} f_i(x)$ is well defined and continuous even without the latter condition.

In a normal space X every locally finite cover $\mathcal{U} = (U_i)_{i\in I}$ has a subordinate partition of unity.

Proof: Due to 8.4 there is an open cover $\mathcal{O} = (O_i)_{i\in I}$ with $\overline{O}_i \subset U_i$ for all $i \in I$. Since X is normal we can find open sets C_i with $\overline{O}_i \subset C_i \subset \overline{C}_i \subset U_i$ and according to 8.1 continuous functions $g_i : X \to [0; 1]$ with $g_i\,[X \setminus C_i] = \{0\}$ and $g_i\left[\overline{O}_i\right] = \{1\}$. The supports of the g_i lie in $\overline{C}_i$ and therefore in U_i. Since $\mathcal{U}$ is locally finite the function $g(x) := \sum_{i\in I} g_i(x)$ is well defined and continuous. Because $\mathcal{O}$ is a cover of X we have $g(x) \geq 1$. The functions $f_i(x) := \frac{g_i(x)}{g(x)}$ are continuous again and form the desired partition of unity subordinate to $\mathcal{U}$.

9 Compact spaces

9.1 Definitions

A topological space X is **quasi-compact** iff every open cover $(U_i)_{i\in I}$ of X has a finite subcover of X. The space X is **compact** iff it is also **Hausdorff**. A subset $A \subset X$ is (quasi)compact iff this property holds for the **subspace** A. The set $A \subset X$ is **precompact** iff the **closure** $\overline{A}$ is compact.

9.2 Properties of compact spaces

The following statements are equivalent:

1. X is quasi-compact.
2. Every family of closed sets with empty intersection has a finite subfamily with empty intersection.
3. Every filter on X has an **accumulation point**. (**Bolzano-Weierstrass**)
4. Every **ultrafilter** on X is convergent.

Proof:

1. $\Rightarrow$ 2.: by taking the complements.

2. $\Rightarrow$ 3.: with 6.5.

3. $\Rightarrow$ 4.: with 6.2.

4. $\Rightarrow$ 1.: If an open cover of X had no finite subcover every finite subfamily of the closed complements would have a nonempty intersection and consequently these intersections would form the basis of an ultrafilter converging to an $x \in X$. But then the ultrafilter would include every neighborhood of x and consquently an element of the open cover as well as its complement.

9.3 Sequences on quasi-compact spaces

According to 9.2.3 every sequence on a quasi-compact space has an accumulation point. The converse is not always true: Let $X = \{0;1\}^{\mathbb{R}}$ the set of all functions $f : \mathbb{R} \to \{0;1\}$ with the product topology and A the subspace of all functions $f \in X$ with countably many zeroes. Fo every sequence $(f_n)_{n\in\mathbb{N}} \subset A$ with countable zero sets I_n the function $f \in A$ with $f(x) = 0$ for $x \in \bigcup_{n\in\mathbb{N}} I_n$ (a countable union of countable sets is again countable, cf. [19, p. 17.6]) and $f(x) = 1$ else is an accumulation point since the sets $\{0\}^I \times \{1\}^J \times \{0;1\}^{\mathbb{R}\setminus(I\cup J)}$ with finite $I \subset \bigcup_{n\in\mathbb{N}} I_n$ and $J \subset \mathbb{R} \setminus \bigcup_{n\in\mathbb{N}} I_n$ form a a neighborhood basis for f in A and contain every f_n. But A is not compact since the open sets $\{0\}_x \times \{0;1\}^{\mathbb{R}\setminus\{x\}}$ for $x \in \mathbb{R}$ and $\{1\}_0 \times \{0;1\}^{\mathbb{R}\setminus\{0\}}$ cover X and especially A but no finite subfamily covers A.

9.4 Compact subsets

Every compact subset $K \subset X$ in a **Hausdorff** space X can be separated from any point $x \in X \setminus K$ by disjoint open neighborhoods. Hence every **compact** subset of a **Hausdorff space** is **closed** and since every **closed** subset of a **quasi-compact space** is obviously **quasi-compact** we conclude that compact spaces are **regular**.

Proof: Every point $y \in K$ has a neighborhood $U(y)$ disjoint from the open neighborhood $U_y(x)$. These $U(y)$ cover K and the union of the finite subcover is an open set containing K which does not meet the finite and therefore open intersection of the corresponding $U_y(x)$.

9.5 Compact spaces are normal

Proof: Due to 9.4 the closed disjoint sets A and B are compact and on account of X being regular every $y \in B$ has a neighborhood $U(y)$ disjoint from an open neighborhood $U_y(A)$ of A. The $U(y)$ cover B and the union of the finite subcover is an open set containing B which does not meet the finite and therefore open intersection of the corresponding $U_y(A)$.

9.6 Alexander's theorem

X is **quasi-compact** iff every cover of X by sets from a **subbasis** $\mathcal{S}$ has a finite subcover.

Proof: Assuming there is an **ultrafilter** $\mathcal{F}$ which does not converge. Then for each $x \in X$ there is a neighborhood $U_x \in \mathcal{U}(x)$ not included in $\mathcal{F}$. Due to 2.2 there is a basis set $B_x \subset U_x$ with $B_x = \bigcap_{k\in K_x} \{S_{xk}\}$ and w.l.o.g. $x \in S_{xk} \in \mathcal{S}$ for finite K_x and $k \in K_x$. There is a $k_x \in K_x$ with $S_{xk_x} \notin \mathcal{F}$ since otherwise with all S_{xk} their finite intersection B_x and especially $U_x \supset B_x$ were included in $\mathcal{F}$ contrary to the hypothesis. Again under the hypothesis there is a finite subcover $(S_{xk_x})_{x\in X}$ with the corresponding complements belonging to $\mathcal{F}$. Their intersection is empty in contradiction to 6.1.

9.7 The Heine-Borel theorem for one dimension

Every **closed** and **bounded** subset of $\mathbb{R}$ is **compact**.

Proof: On account of 9.4 and 9.6 we only have to show that every cover $\mathcal{U}$ of a closed interval $[a;b] \subset \mathbb{R}$ by half open intervals $[a;c[$ and $]d;b]$ possesses a finite subcover. For $c' := \sup\{c \in \mathbb{R} : [a;c[\in \mathcal{U}\} > b$ there is a $c'' < b$ with $[a;c''[\in \mathcal{U}$ which already covers $[a;b[$. In the case of $c \leq b$ there is a $d' < c'$ with $]d';b] \in \mathcal{U}$ since otherwise c' cannot be covered. Furthermore there is a c'' with $d' < c'' < c'$ and $[a;c''[\in \mathcal{U}$ such that $[a;b] \subset [a;c''[\cup]d';b]$.

9.8 The closed map lemma

For every map $f : X \to Y$ from a **compact** space X into a **Hausdorff** space Y we have the following inclusions:

1. f is **continuous** $\Rightarrow$ f is a **closed** map.
2. f is **continuous and surjective** $\Rightarrow$ f is a **quotient** map.
3. f is **continuous and injective** $\Rightarrow$ f is an **open** map onto $f[X]$ and hence a **topological embedding**.
4. f is **continuous and bijective** $\Rightarrow$ f is a **homeomorophism.**

Corollary:

Every **curve** $f\left[\bar{I}\right] \subset \mathbb{R}^n$ parametrized by a **continuous** f **without intersections** on a **closed** interval $\bar{I} \subset \mathbb{R}$ is homeomorphic to this interval.

Proof:

1. Due to 9.4 every closed subset A of a compact space X is compact and its continuous image $f[A]$ is obviously compact, hence closed in the Hausdorff space Y.
2. Due to enu:4.8.2.
3. Due to [19] th. 9.2.5 and enu:4.8.2.
4. obvious.

9.9 Tychonov's theorem

A nonempty **product space** $X = \prod_{i \in I} X_i$ is **quasi-compact** iff all **components** X_i are **quasi-compact**.

Proof:

$\Rightarrow$: follows from 9.8 and from the **projections** $\pi_i : X \to X_i$ being continuous.

$\Leftarrow$: On account of 6.3 the image $\pi_i(\mathcal{F})$ of an **ultrafilter** $\mathcal{F}$ is again an ultrafilter since for every $A_i \subset X_i$ either the $\pi_i^{-1}[A_i]$ or $\pi_i^{-1}[X \setminus A_i]$ are included in $\mathcal{F}$ and consequently either $A_i = \pi_i\left[\pi_i^{-1}[A_i]\right]$ or $X \setminus A_i = \pi_i\left[\pi_i^{-1}[X \setminus A_i]\right]$ are part of $p_i(\mathcal{F})$. Due to 9.2.4 the $\pi_i(\mathcal{F})$ converge to a $x_i \in X_i$ and on account of 6.7 the filter $\mathcal{F}$ converges to $(x_i)_{i \in I} \in X$.

9.10 The Heine-Borel theorem

A subset of $\mathbb{R}^n$ is **compact** iff it is **bounded** and **closed**.

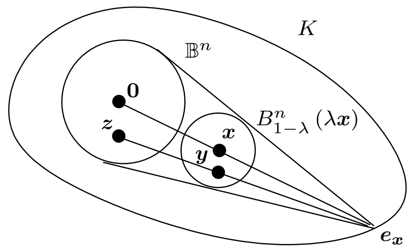
Proof:

$\Rightarrow$: Due to 9.4 a compact subset of the Hausdorff-space $\mathbb{R}^n$ is closed. The open cover $\{B_n(0) : n \in \mathbb{N}\}$ shows that the set is bounded.

$\Leftarrow$: On account of 9.7 and 9.9 every bounded and closed subset of $\mathbb{R}^n$ is compact since it is the subset of the compact cube $[-m;m]^n$.

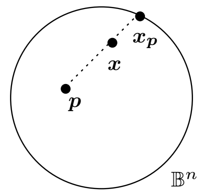
9.11 Open and closed cells

For every $\boldsymbol{p} \in \mathring{K}$ in the **nonempty interior** of a **compact convex** set $K \subset \mathbb{R}^n$ exists a homeomorphism $\varphi : \bar{\mathbb{B}}^n \to D$ with $\varphi(\boldsymbol{0}) = p$ and $\varphi[\mathbb{B}^n] = \mathring{K}$ with regard to the **unit ball** $\mathbb{B}^n = B_1^n(\boldsymbol{0})$. In particular K is a **closed** n**-cell**, i.e. homeomorphic to $\bar{\mathbb{B}}^n$ and its interior $\mathring{K}$ is an **open** n**-cell**, i.e. homeomorphic to $\mathbb{B}^n$.



Proof: The hypothesis implies the existence of an $\epsilon > 0$ such that $B_\epsilon^n(\boldsymbol{p}) \subset \mathring{K}$ and by referring to the homeomorphisms $\boldsymbol{x} \mapsto \boldsymbol{x} - \boldsymbol{p}$ resp. $\boldsymbol{x} \mapsto \frac{1}{\epsilon}\boldsymbol{x}$ we may assume $\boldsymbol{p} = \boldsymbol{0}$ and $\mathbb{B}^n \subset \mathring{K}$. Due to the preceding **closed-map lemma** 9.8 resp. the **Heine-Borel-theorem** 9.10 for every $\boldsymbol{x} \in \mathbb{B}^n$ the **compact ray** $R_{\boldsymbol{x}} = \{t\boldsymbol{x} : t \geq 0\} \cap K$ has a uniquely determined endpoint $\boldsymbol{e_x} \in R_{\boldsymbol{x}}$ with $\|\boldsymbol{e_x}\| = \max\{\|\boldsymbol{y}\| : \boldsymbol{y} \in R_{\boldsymbol{x}}\}$ since $\boldsymbol{y} \mapsto \|\boldsymbol{y}\|$ is **continuous** such that according to 5.1 the set of all norms of ray vectors is **bounded**, **closed** and **connected**, i.e. it is a **closed interval** $\{\|\boldsymbol{y}\| : \boldsymbol{y} \in R_{\boldsymbol{x}}\} = [0; \|\boldsymbol{e_x}\|] \subset \mathbb{R}$. Obviously we have $\boldsymbol{e_x} \in \partial K$ and for every $\boldsymbol{y} \in B_{1-\lambda}^n(\lambda\boldsymbol{e_x})$ exists a $\boldsymbol{z} \in K$ with $\boldsymbol{y} - \boldsymbol{z} = \lambda(\boldsymbol{e_x} - \boldsymbol{z})$ resp. $\boldsymbol{z} = \frac{1}{1-\lambda}(\boldsymbol{y} - \lambda\boldsymbol{e_x})$ whence follows $\|\boldsymbol{z}\| < 1$ resp. $\boldsymbol{z} \in \mathbb{B}^n \subset \mathring{K}$. By the **convexity** of K and $\boldsymbol{e_x} \in K$ we conclude $\boldsymbol{y} \in \mathring{K}$ whence follows $B_{1-\lambda}^n(\lambda\boldsymbol{e_x}) \subset \mathring{K}$ and in particular $\lambda\boldsymbol{e_x} \in \mathring{K}$ for every $0 \leq \lambda < 1$. Hence the map $\varphi : \mathbb{B}^n \to \mathring{K}$ is well defined by $\varphi(\boldsymbol{x}) = \|\boldsymbol{x}\|\,\boldsymbol{e_x}$. Moreover the preceding construction shows that it is **injective**, **surjective** and also **continuous** whence by the **closed-map lemma** follows the assumption.

9.12 Extension of continuous maps on the unit ball



For every $\boldsymbol{p} \in \mathbb{B}$, $c \in \mathbb{R}$ and every **real continuous map** $f : \partial\mathbb{B} \to \mathbb{R}$ on the **compact boundary of the unit ball** $\mathbb{B} = B_1(\boldsymbol{0}) \subset \mathbb{R}^n$ exists a **continuous extension** $f_{\boldsymbol{p}} : \mathbb{B} \to \mathbb{R}$ with $f_{\boldsymbol{p}}(\boldsymbol{p}) = c$ and $f_{\boldsymbol{p}}(\boldsymbol{x}) = c + \beta_{\boldsymbol{xp}} \cdot (f(\boldsymbol{x_p}) - c)$ for $\boldsymbol{x} \in \mathbb{B} \setminus \{\boldsymbol{p}\}$ with $\gamma_{\boldsymbol{xp}} = \sqrt{\frac{1-\|\boldsymbol{p}\|^2}{\|\boldsymbol{x}-\boldsymbol{p}\|^2}}$ and $\boldsymbol{x_p} = \boldsymbol{p} + \gamma_{\boldsymbol{xp}} \cdot (\boldsymbol{x} - \boldsymbol{p}) \in \partial\mathbb{B}$. According to 5.2, 5.8 and 9.10 the image $f[\partial\mathbb{B}] = [a; b]$ is a **closed interval** on the real line such that in the case of $a \leq c \leq b$ the above construction satisfies $f_{\boldsymbol{p}}[\mathbb{B}] = [a; b]$ while for $c \leq a$ follows $f_{\boldsymbol{p}}^{-1}(c) = \{\boldsymbol{p}\}$ and $f_{\boldsymbol{p}}[\mathbb{B}] = [c; b]$.

9.13 Kronecker's approximation theorem

For every **irrational** $\gamma \in [0; 1[$ let $f : \mathbb{N} \to [0; 1]$ be defined by $f(n) := n\gamma - [n\gamma]$ with the **Gauss bracket or floor function** $[a]$ denoting the greatest integer not exceeding a. Then f is **injective** and the **countable** set $f[\mathbb{N}]$ is **dense** in $[0; 1]$.

Proof: f is **injective** since $n\gamma - [n\gamma] = m\gamma - [m\gamma] \Leftrightarrow \gamma = \frac{[n\gamma]-[m\gamma]}{m-n} \in \mathbb{Q}$. Since the $f(n)$ are different from each other 9.3 yields that the sequence has at least one accumulation point in the **compact** (cf. 9.7) interval $[0; 1]$ and hence for every $\epsilon > 0$ there are natural numbers w.l.o.g. $n > m$ with $\delta := |n\gamma - [n\gamma] - (m\gamma - [m\gamma])| < \epsilon$. With $k = n - m$ and $z = [n\gamma] - [m\gamma]$ we get $|k\gamma - z| < \epsilon \Rightarrow z = [k\gamma]$ if $k\gamma > z$ resp. $z = [k\gamma] + 1$ if $k\gamma < z$. In the first case it follows that $0 < \delta = f(k) = k\gamma - [k\gamma] < \epsilon$ and for $\nu \cdot \delta < 1$ we get $[\nu k\gamma] = \nu[k\gamma]$ so that the **subsequence** $(f(\nu k))_{\nu\in\mathbb{N}^*}$ with $f(\nu k) = \nu(k\gamma - [k\gamma]) = \nu \cdot \delta$ is **increasing** in $[0; 1]$ with **increments** $f((\nu+1)k) - f(\nu k) = \delta < \epsilon$. In the second case we have $1 - \epsilon < 1 - \delta := f(k) = k\gamma - [k\gamma] < 1$ and for $\nu \cdot \delta < 1$ it follows that $[\nu k\gamma] = \nu[k\gamma] + 1$ so that the **subsequence** $(f(\nu k))_{\nu\in\mathbb{N}^*}$ is **decreasing** in $[0; 1]$ starting with $f(k) = 1 - \delta$ and **decrements** $f(\nu k\gamma) - f((\nu+1)k\gamma) = \delta < \epsilon$. In both cases the subsequence will meet the open neighborhood $B_\delta(x)$ of every $x \in [0; 1]$.

9.14 Dini's theorem

If the continuous real-valued functions $(f_n)_{n\in\mathbb{N}} \subset \mathcal{C}(X;\mathbb{R})$ on a topological space X **converge pointwise** to a **continuous** $f \subset \mathcal{C}(X;\mathbb{R})$ they also converge **uniformly on compact sets** to f.

Proof: Due to the hypothesis the **increasing** sequence $(\inf_{k\geq n} f_k)_{n\in\mathbb{N}}$ with $\inf_{k\geq n} f_k < f_k < f$ converges pointwise to f. Hence for every $\epsilon > 0$ and every $x \in K$ there is $\delta_x > 0$ and a $n_x \in \mathbb{N}$ with $|f(y) - \inf_{k\geq n_x} f_k(y)| \leq |f(y) - f_{n_x}(y)| \leq |f(y) - f(x)| + |f(x) - f_{n_x}(x)| + |f_{n_x}(x) - f_{n_x}(y)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$ for $y \in B_{\delta_x}(x)$. The finite subcover $(B_{\delta_x}(x_i))_{i\in K}$ yields a $m = \max\{n_{x_i} : i \in K\}$ with $|f(y) - \inf_{k\geq m} f_k(y)| \leq \epsilon$ for every $y \in K$ and hence the uniform convergence of $(\inf_{k\geq n} f_k)$ to f on K. With an analogous argument we obtain the uniform convergence of the **decreasing** sequence $\left(\sup_{k\geq n} f_k\right)_{n\in\mathbb{N}}$ to f and hence the proposition.

9.15 Lebesgue's Lemma

For every open cover $\bigcup_{i\in I} U_i \subset K$ of a compact set in a metric space $(X;d)$ there exists a positive **Lebesgue number** $\lambda > 0$ such that for every set $A \subset X$ with $A \cap K \neq \emptyset$ and diameter $\delta(A) < \lambda$ there is an $i \in I$ with $A \subset U_i$.

Proof: The open balls $\left(B_{\epsilon_x/2}(x)\right)_{x\in K}$ with $\epsilon_x > 0$ and $B_{\epsilon_x}(x) \subset U_i$ include a finite subcover over $x \in J$ such that $\lambda := \min\{\epsilon_x/2 : x \in J\}$ clearly satisfies the assertion.

10 Locally compact spaces

10.1 Locally compact spaces

A **Hausdorff space** is **locally compact** iff every point has a **compact neighborhood**. Due to 9.10 the most important representant is $\mathbb{R}^n$.

10.2 The Alexandrov compactification

A **locally compact** space X can be extended to a **compact** space $\overline{X} = X \cup \{\infty\}$ by adding a single **point at infinity** ∞ as well as the complements of all compact sets in X united with $\{\infty\}$. Every compact space which is homeomorphic to X up to a single point x is homeomorphic to $\overline{X}$.

Proof: The complements of compact sets in X united with $\{\infty\}$ already form a topology since arbitrary intersections and finite unions of compact sets are compact. Due to 9.4 **the added sets are compatible** with the existing topology as they are open in the subspace X. $\overline{X}$ is **Hausdorff** since X is Hausdorff and every $x \in X$ can be separated from ∞ by a compact neighborhood. $\overline{X}$ is **quasi-compact** since every open cover must include the complement of a compact set X. Let $\overline{X}'$ be a further space with infinite point ∞' such that $X' := \overline{X}' \backslash \{\infty'\}$ is homeomorph to X. Then according to 9.4 the complements $\overline{X}' \backslash U' = X' \backslash U'$ of open neighborhoods U' of ∞' must be compact and consequently the homeomorphism $f : X \to X'$ can be extended to $\overline{f} : \overline{X} \to \overline{X}'$ by means of $\overline{f}|_X := f$ resp. $\overline{f}(\infty) := \infty'$. It remains to show that $\overline{f}$ is continuous and open in ∞, i.e. $\forall U' \in \mathcal{U}(\infty') : \overline{f}^{-1}[U'] \in \mathcal{U}(\infty)$ and $\forall U \in \mathcal{U}(\infty) : \overline{f}[U] \in \mathcal{U}(\infty')$. But this is evident from $\overline{f}$ being bijective together with 9.8 and 9.4 since the neighborhoods of ∞' must be the complements of compact sets in X'.

10.3 The continuity of extended addition and multiplication

For $\delta = n + 2\|a\|$ for all $n \geq 2$ we have $\left\{(u+v) : u \in B_{1/\delta}(a) \wedge v \in \overline{\mathbb{C}} \setminus \overline{B_\delta(0)}\right\} \subset \overline{\mathbb{C}} \setminus \overline{B_n(0)}$ whence the **extended addition** from 4.2.3 $+ : \overline{\mathbb{C}}^2 \to \overline{\mathbb{C}}$ for $a \in \mathbb{C}$ is **continuous** at the added points $(a;\infty)$ resp. $(\infty;a)$ as well as $(\infty;\infty)$ and therefore on the entire extended plane $\overline{\mathbb{C}}^2$. Since we have $\left\{(u \cdot v) : u \in B_{1/\delta}(a) \wedge v \in \overline{\mathbb{C}} \setminus \overline{B_\delta(0)}\right\} \subset \overline{\mathbb{C}} \setminus \overline{B_n(0)}$ with $\delta = \frac{n+2}{\|a\|}$ for all $n \geq 3$ the **extended multiplication** $\cdot : \overline{\mathbb{C}}^2 \setminus \{(0;\infty);(\infty;0)\} \to \overline{\mathbb{C}}$ is **continuous**.

10.4 Meromorphic functions

A function $\bar{f} : \mathbb{C} \to \overline{\mathbb{C}}$ is continuous iff its restriction $f := \bar{f}|_{\{|\bar{f}|<\infty\}} : \mathbb{C} \to \mathbb{C}$ is continuous and for every $x \in \bar{f}^{-1}(\infty)$ and $n \in \mathbb{N}$ there is a $\delta > 0$ such that $\bar{f}\left[B_\delta(x)\right] \subset \overline{\mathbb{C}} \setminus \overline{B_n(0)}$, i.e., $\|u - x\| < \delta \Rightarrow \|f(u)\| > n$. Therefore all **meromorphic functions** are continuous at their **poles** with reference to the Alexandrov compactification.

10.5 Complete regularity

A **locally compact** space X is **completely regular** owing to 7.3 since it is the subspace of the completely regular Alexandrov compactification $\overline{X}$.

10.6 Compact neighborhoods

Due to 7.7, 9.4 and subsec:Alexandrov-compactification the **compact neighborhoods** of every point in a **locally compact** space form a **neighborhood basis.** Consequently in a locally compact space **open** resp. **closed** subsets as well as their **finite intersections** are **locally compact** with reference to their trace topology.

10.7 Regularity

For every **compact** set K and an **open** neighborhood $V \supset K$ in a **locally compact** space X there is a **continuous** $g : X \to [0;1]$ with $g^{-1}(\{1\}) = K$ and $g^{-1}(\{0\}) \supset X \setminus V$. In particular two **disjoint** and **compact** sets can be separated by **disjoint** and **open** neighborhoods. Moreover due to 7.7 and 9.4 the **compact** neighborhoods form a **neighborhood basis**.

Proof: Due to 10.6 every $x \in K$ possesses open neighborhoods with compact closures $x \in U_x \subset \overline{U_x}$ with $U_x \subset V$ such that the union $U := \bigcup_{x \in J} U_x$ with finite $J \subset K$ of the subcover $(U_x)_{x \in J}$ of K is open with $K \subset U \subset V$ and compact closure $\overline{U} = \bigcup_{x \in J} \overline{U_x}$. Due to 9.5 we can apply **Urysohn's lemma** 8.1 to find a $g' : \overline{U} \to [0;1]$ with $g'^{-1}(\{1\}) = K$ and $g'^{-1}(\{0\}) = \overline{U} \setminus U$ which can be extended by $g(x) := 0$ for every $x \in X \setminus U \supset X \setminus V$ to obtain the desired function.

10.8 σ-compact spaces

A locally compact space is **σ-compact** or **countable at infinity** iff it is a countable union of compact sets resp. iff the point at infinity has a countable neighborhood basis. A locally compact space X is **σ-compact** if it is **second countable** since every compact neigbourhood $K(x)$ includes an open basis set $U_n(x)$ as well as its compact closure $\overline{U}_n(x)$ and the resulting cover $\left(\overline{U}_n(x)\right)_{x \in X} \supset X$ consists of a countable number of compact sets.

10.9 Normal character of σ-compact spaces

Every **locally** and **σ-compact** space is **normal**.

Proof: Due to 9.4 the intersections $A \cap K_i$ resp. $B \cap K_i$ of disjoint closed sets $A, B \subset X$ with the sets of a w.l.o.g. **increasing compact** cover $(K_i)_{i\in\mathbb{N}}$ of X are closed and with 9.5 there are w.l.o.g. **increasing** sequences $(V_i)_{i\in\mathbb{N}}$ resp. $(W_i)_{i\in\mathbb{N}}$ of **open** sets with $V_i \supset A \cap K_i$ resp. $W_i \supset B \cap K_i$ as well as $V_i \cap W_i \cap K_i = \emptyset$. But then $V = \bigcup_{i\in\mathbb{N}} V_i \supset A$ resp. $W = \bigcup_{i\in\mathbb{N}} W_i \supset B$ are open disjoint neighborhoods of A bzw. B in X: Assuming $x \in V_i \cap W_j \neq \emptyset$ with w.l.o.g $i \leq j$ due to the ascending character of all set sequences concerned follows $x \in V_k \cap W_k \neq \emptyset \, \forall k \geq j$ but also $x \in K_k$ for $k \geq l$ and some $l \in \mathbb{N}$ contrary to the hypothesis $V_i \cap W_i \cap K_i = \emptyset$.

10.10 Countably compact spaces

A **Hausdorff space** is **countably compact**, i.e., every **countable** open cover has a finite subcover, iff every sequence has an **accumulation point**.

Proof:

$\Rightarrow$: Assume the sequence $(x_i)_{i\in\mathbb{N}}$ does not have an accumulation point. Then every point has a neighborhood meeting only finitely many members. In a Hausdorff space this neighborhood can be reduced to an open neighborhood meeting no members at all and consequently the complement of the sequence is an open set which taken together with the neighborhoods mentioned above forms a countable open cover of X. Since every set of the cover contains at most a single member a finite subcover is obviously impossible.

$\Leftarrow$: Let $(O_i)_{i\in\mathbb{N}}$ an open cover of the Hausdorff space X and $x_n \in X \setminus \bigcup_{0\leq i\leq n} O_i$. Then the accumulation point y of the sequence $(x_n)_{n\in\mathbb{N}}$ lies in an open set O_i which consequently contains infinitely many x_n contrary to the construction of the sequence..

10.11 Lindelöf spaces

A topological space is a **Lindelöf space** iff every open cover has a **countable** subcover. Consequently every **second countable** space is **Lindelöf**. A topological space is **compact** iff it is **countably compact** and **Lindelöf.**

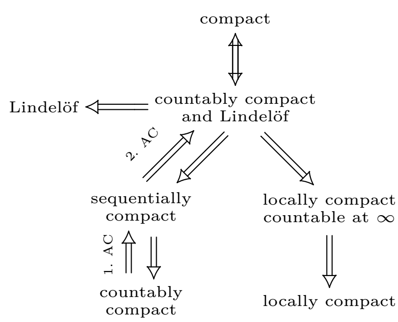
10.12 Sequentially compact spaces

A **Hausdorff space** is **sequentially compact** iff every sequence has a **convergent subsequence.** In a **first countable** space every accumulation point of a sequence allows the selection of a convergent subsequence. Such a space is **sequentially compact** iff it is **countably compact.**

10.13 Compactness on metric spaces

In a metric space the properties of being compact, countably compact and sequentially compact are equivalent.

Proof: Due to 10.10, 10.11 and 10.12 it remains to show that **sequentially compact metric spaces are second countable**. For an arbitrary $x_{0,n} \in X$ choose $x_{1,n} \in X \setminus B_{1/n}(x_{o,n})$ and for $x_{i,n} \in X$ subsequently choose $x_{i+1,n} \in X \setminus \bigcup_{0\leq j\leq i} B_{1/n}(x_{j,n})$. Because X is sequentially compact the sequence must end after finitely many $x_{i,n}$ and the corresponding $B_{1/n}(x_{j,n})$ obviously cover X. The set of all $x_{i,n}$ then is a countable dense subset of X and the $B_{1/m}(x_{i,n})$ for $m \in \mathbb{N}^*$ form a countable base of X.



10.14 Alexandrov's theorem

Every **path connected locally compact metric** space is **separable** and hence **second countable**.

Proof: In the case of $r_x = \sup\left\{\epsilon : \overline{B}_\epsilon(x) \text{ is compact}\right\} = \infty$ the space X is σ**-compact** disks. The assertion the follows from the facts that metric spaces are **first countable** and that compact metric spaces are **separable**. Due to 9.4 we have $r_x \geq r_y - d(x;y)$ such that $r_x < \infty$ for one $x \in X$ entails $r_y < \infty$ for every other $y \in X$ as well. Since X is **path connected** every $y \in X$ can be connectd to x by a path which is covered by finitely many $B_{r_z/4}(z)$ such that the sequence defined inductively by $A_1 = \overline{B}_{r_x/2}(x)$ and $A_{n+1} = \bigcup_{y \in A_n} \overline{B}_{r_y/2}(y)$ covers $X = \bigcup_{n \geq 1} A_n$. The set A_1 is compact and assuming a compact A_n for every sequence $(x_k)_{k \geq 1} \subset A_{n+1}$ exists a sequence $(y_k)_{k \geq 1} \subset A_n$ with $x_k \in \overline{B}_{r_{yk}/2}(y_k)$ and a subsequence $\left(y_{k(i)}\right)_{i \geq 1}$ converging to an $y = \lim_{i \to \infty} y_{k(i)} \in A_n$. Due to $d(y;x_k) \leq d(y;y_k) + d(y_k;x_k) \leq d(y;y_k) + \frac{1}{2} r_{y_k} \leq \frac{3}{2} d(y;y_k) + \frac{1}{2} r_y$ and $\lim_{i \to \infty} d(y;y_k) = 0$ there is a $K \in \mathbb{N}$ such that $x_k \in \overline{B}_{r_y}(y)$ for all $k \geq K$. According to 10.12 a subsequence $\left(x_{k(j)}\right)_{j \geq 1}$ converges to an $x = \lim_{j \to \infty} x_{k(j)}$. Again the preceding inequality and $\lim_{i \to \infty} d(y;y_k) = 0$ imply $x \in \overline{B}_{r_y/2}(y) \subset A_{n+1}$ whence A_{n+1} is **compact** by 10.13. Hence we have again proved that X is σ**-compact** and the assertion follows by the same argument as above.

10.15 Products of quotient maps with the identity

Every **product** $f \times \mathrm{id}_K : X \times K \to Y \times K$ of a **quotient map** $f : X \to Y$ and the **identity** on a **locally compact** space K is a **quotient map**.

Proof: For every $\left(f^{-1}(y_0);k_0\right) \in (f \times \mathrm{id}_K)^{-1}[U]$ exists an open neighborhood $\left(f^{-1}(y_0);k_0\right) \subset W_0 \times J_0 \subset (f \times \mathrm{id}_K)^{-1}[U]$. According to 10.6 there is a precompact neighborhood $x \in J \subset \bar{J} \subset J_0$ and due to $f^{-1}[f[W_0]] \times \bar{J} \subset (f \times \mathrm{id}_K)^{-1}[U]$ for every $x \in f^{-1}[f[W_0]]$ and every $k \in \bar{J}$ exists a neighborhood $(x;k) \in V_k \times J_k \subset (f \times \mathrm{id}_K)^{-1}[U]$. Finitely many of the J_k cover $\bar{J}$ and if V_x is the open intersection of the corresponding V_k we obtain a common neighborhood $(x;k) \in V_x \times J \subset (f \times \mathrm{id}_K)^{-1}[U]$ and $W_1 \times \bar{J} \subset (f \times \mathrm{id}_K)^{-1}[U]$ with the open set $W_1 = \bigcup_{x \in f^{-1}[f[W_0]]} V_x$ including $f^{-1}[f[W_0]]$. Repeating this construction yields a sequence $(W_i)_{i \geq 0}$ of open sets $W_i \subset f^{-1}[f[W_i]] \subset W_{i+1}$ and $W_i \times \bar{J} \subset (f \times \mathrm{id}_K)^{-1}[U]$ for $i \geq 0$. Then the **open** set $W = \bigcup_{i \geq 0} W_i$ is obviously **saturated**, i.e. $W = f^{-1}[f[W]]$ whence its image $f[W]$ is **open** in the **final topology** of f on Y. Therefore $(y_0;k_0) \in f[W] \times J \subset U$ is an open neighborhood whence U is open in the **final topology** on Y and consequently f is an **open** map.

11 Metrization

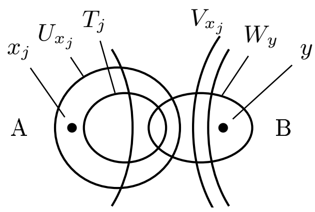
11.1 Paracompact spaces

A **Hausdorff-space** X is **paracompact** iff for every **open cover** $\mathcal{U} = (U_i)_{i \in I}$ of X there is a **locally finite refinement** $\mathcal{V} = (V_j)_{j \in J}$ such that every V_j is contained in an U_i.

11.2 A paracompact space is normal.

Proof: The assertion follows with regard to 7.4.2 from applying the following property twice: Two disjoint closed sets A and B in a paracompact space X have disjoint open neighborhoods iff every $x \in A$ can be separated from B by disjoint open neighborhoods U_x of x and V_x of B. In order to find these neighborhoods we take a look at the locally finite refinement $(T_i)_{i \in I}$ subordinate to the open cover of X provided by the set $X \setminus A$ together with the system of the supposed neighborhoods U_x for $x \in A$ with $U_x \cap V_x = \emptyset$:

For $A \cap T_i \neq \emptyset$ there is an $x_i \in A$ with $T_i \subset U_{xi}$ and $T := \bigcup_{A \cap T_i \neq \emptyset} T_i$ is an open cover of A. For every $y \in B$ there is an open neighborhood W_y such that $J_y := \{i \in I : A \cap T_i \neq \emptyset \neq W_y \cap T_i\}$ is finite. For each of these finitely many $j \in J_y$ there are open neighborhoods V_{xj} of B with $T_j \cap V_{xj} = \emptyset$ such that none of the open neighborhood $W'_y := W_y \cap \bigcap_{j \in Jy} V_{xj}$ of B meets any T_i. Then $W := \bigcup_{y \in B} W'_y$ is the desired open neighborhood of B with $T \cap W = \emptyset$.



11.3 Partitions of unity in paracompact spaces

A topological space X is **paracompact** iff it is **Hausdorff** and every **open cover** has a subordinate **partition of unity.**

Proof:

$\Rightarrow$: On account of 11.1 and 8.5 we already have a partition of unity $(g_j)_{j \in J}$ subordinate to the locally finite refinement $\mathcal{V} = (V_j)_{j \in J}$ of the given cover $\mathcal{U} = (U_i)_{i \in I}$. In order to extend this partition to $\mathcal{U}$ we choose for every $j \in J$ a $\phi(j) \in I$ with $V_j \subset U_i$ and combine the g_j assigned to $V_j \subset U_i$ into a sum $f_i := \sum_{\phi(j)=i} g_j$ if $\phi^{-1}(i) \neq \emptyset$ and $f_i = 0$ if $\phi^{-1}(i) \neq \emptyset$. The f_i are continuous, their supports are included in the $\bigcup_{\phi(j)=i} V_j \subset U_i$ and we have $\sum_{i \in I} f_i = \sum_{i \in I} \sum_{\phi(j)=i} g_j = \sum_{j \in J} g_j = 1$.

$\Leftarrow$: According to 8.5 the open sets $V_j = \{g_j > 0\}$ for a partition of unity $(g_j)_{j \in J}$ subordinate to the given open cover $\mathcal{U} = (U_i)_{i \in I}$ are a locally finite refinement of $\mathcal{U}$.

11.4 Closures of a locally finite systems

For a **locally finite** system $\mathcal{V}$ on a topological space X the family of the **closures** $\overline{\mathcal{V}}$ is again locally finite and $\overline{\bigcup \mathcal{V}} = \bigcup \overline{\mathcal{V}}$.

Proof: For $x \in \overline{\bigcup \mathcal{V}}$ and the neighborhood $U \in \mathcal{U}(x)$ meeting only finitely many $V_1, ..., V_n \in \mathcal{V}$ we have $x \in \bigcup_{1 \le m \le n} \overline{V}_m \subset \bigcup \overline{\mathcal{V}}$ since otherweise $U \setminus \bigcup_{1 \le m \le n} \overline{V}_m$ were an open neighborhood of x not meeting any $V \in \mathcal{V}$. Thus we have shown $\overline{\bigcup \mathcal{V}} \subset \bigcup \overline{\mathcal{V}}$ and since the converse is always true the equation follows.

11.5 Characterization of paracompact spaces

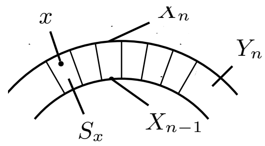
A topological space is **paracompact** iff it is **regular** and for every **open** cover $\mathcal{U} = (U_i)_{i \in I}$ of X exists a **σ-locally finite open refinemet** $\mathcal{S} = \bigcup_{n \in \mathbb{N}} \mathcal{S}_n$. Note that the subfamilies $\mathcal{S}_n$ are open and and locally finite but not necessarily a cover.

Proof: Note that in the following $\bigcup_{n \in \mathbb{N}} \mathcal{S}_n$ is the family of all sets **included** in any $\mathcal{S}_n$ whereas $\bigcup \mathcal{S}_n$ is the union of the sets **contained** in the single family $\mathcal{S}_n$. On account of 11.2 we only have to show $\Leftarrow$. This argument is split into three steps by demonstrating the equivalence of the following four statements for a **regular** space X with an **open cover** $\mathcal{U} = (U_i)_{i \in I}$:

1. There is a **σ-locally finite open refinement cover** $\mathcal{S} = \bigcup_{n \in \mathbb{N}} \mathcal{S}_n$ of $\mathcal{U}$.
2. There is a **locally finite refinement cover** $\mathcal{V}$ of $\mathcal{U}$.
3. There is a **closed locally finite refinement cover** $\mathcal{A}$ of $\mathcal{U}$.
4. There is an **open locally finite refinement cover** $\mathcal{O}$ of $\mathcal{U}$.

1. $\Rightarrow$ 2.: $\mathcal{V} := (Y_n \cap S)_{n \in \mathbb{N}, S \in \mathcal{S}_n}$ with $X_n := \bigcup_{0 \le m \le n} \bigcup \mathcal{S}_m$, $Y_0 := X_0$ and $Y_n := X_n \setminus X_{n-1}$ is a cover since for any $x \in X$ there is a n with $x \in Y_n \subset \bigcup \mathcal{S}_n$ because $(X_n)_{n \in \mathbb{N}}$ covers X and furthermore there is a $S_x \in \mathcal{S}_n$ with $x \in S_x$ because $\mathcal{S}_n$ covers $\bigcup \mathcal{S}_n$. Since $\mathcal{S}_m$ is locally finite for $0 \le m \le n$ and X_n is open there is a neighborhood $W_m \subset X_n$ of x meeting only finitely many sets from $\mathcal{S}_m$. Then $W := \bigcap_{0 \le m \le n} W_m$ is a neighborhood of x which intersects only a finite number of sets from $\mathcal{V}$ because for $k > n$ we have $V \cap Y_k = \emptyset$. Hence $\mathcal{V}$ is **locally finite**.

2. $\Rightarrow$ 3.: Since X is **regular** for every $x \in X$ and $U \in \mathcal{U}$ with $x \in U$ there is an open neighborhood $W_x \in \mathcal{U}(x)$ with $x \in W_x \subset \overline{W}_x \subset U$. According to the hypothesis there is a locally finite refinemet cover $\mathcal{V}$ of the open cover $\mathcal{W} := (W_x)_{x\in X}$. The family of the closures $\mathcal{A} := \overline{\mathcal{V}} = \left\{\overline{V} : V \in \mathcal{V}\right\}$ of $\mathcal{V}$ is again a refinement cover of $\overline{\mathcal{W}}$ and hence of $\mathcal{U}$. It is locally finite since for $x \in X$ and a neighborhood $U \in \mathcal{U}(x)$ intersecting only finitely many $V_1, ..., V_n \in \mathcal{V}$ the point x cannot be an accumulation point for any of the remaining $V \in \mathcal{V}$, i.e., $x \notin \bigcup\left(\overline{\mathcal{V}} \setminus \left\{\overline{V}_1, ..., \overline{V}_n\right\}\right) = \overline{\bigcup \mathcal{V} \setminus \{V_1, ..., V_n\}}$ due to 11.4 and $U \setminus \overline{\bigcup \mathcal{V} \setminus \{V_1, ..., V_n\}}$ is an open neighborhood intersecting only the closures $\overline{V}_1, ..., \overline{V}_n \in \overline{\mathcal{V}}$.



3. $\Rightarrow$ 4.: Let $\mathcal{V}$ be a **locally finite refinement cover** of $\mathcal{U}$,$\mathcal{W} := (W_x)_{x\in X}$ the cover formed by the **open neighborhoods** W_x of $x \in X$ each intersecting each finitely many $V \in \mathcal{V}$ and finally let $\mathcal{A}$ be a **closed locally finite refinement cover** of $\mathcal{W}$. For $V \in \mathcal{V}$ let $V' := X \setminus \bigcup\{A \in \mathcal{A} : A \cap V = \emptyset\}$. Since the A are closed and the system $\mathcal{A}$ is locally finite the V' must be **open** due to 11.4. On account of $V \subset V'$ the family $\mathcal{V}' := (V')_{V\in\mathcal{V}}$ is an **open cover** of X. In order to show that $\mathcal{V}'$ is **locally finite** let T_x be a neighborhood of $x \in X$ inersecting only finitely many $A_1, ..., A_n \in \mathcal{A}$. Since A is a cover it follows that $T_x \subset A_1 \cup ... \cup A_n$. In the case of T_x intersecting a V' we have $A_k \cap V' \neq \emptyset$ for some k with $1 \le k \le n$ and from the definition of the V' we infer $A_k \cap V \neq \emptyset$. Since $\mathcal{A}$ is a refinement of $\mathcal{W}$ every A_k meets only finitely many $V \in \mathcal{V}$ and consequently T_x meets only finitely many V'. We finally fit $\mathcal{V}'$ into $\mathcal{U}$ by choosing for every $V \in \mathcal{V}$ a $U_V \in \mathcal{U}$ with $V \subset U_V$ and defining $\mathcal{O} := (U_V \cap V')_{V\in\mathcal{V}}$. Due to $V \subset U_V \cap V'$ this is the desired open refinement cover of $\mathcal{U}$.

11.6 Countability of paracompact spaces

1. A **first countable and regular** space is **paracompact**.
2. The **countable union** of **closed** sets in a **paracompact** space is again **paracompact**.

11.7 Open sets in regular spaces with a σ-locally finite basis

In a **regular** space with **σ-locally finite basis** every **open** set is a F_σ-set.

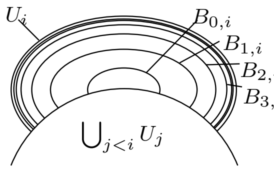
Proof: Since X is regular and due to 7.7 for every x there is an open $O\subset X$ and a neighborhood V_x with $x \in V_x \subset \overline{V}_x \subset O$ as well as elements S_{n_x,i_x} of the basis $\mathcal{S} = \bigcup_{n\in\mathbb{N}} \mathcal{S}_n$ with locally finite $\mathcal{S}_n$ satisfying $x \in S_{n_x,i_x} \subset V_x$ and therefore $\overline{S}_{n_x,i_x} \subset \overline{V}_x \subset O$. Since the $\mathcal{S}_k$ are locally finite and due to 11.4 the $\overline{S}_k = \bigcup\left\{\overline{S}_{n_x,i_x} : x \in O, n_x = k\right\}$ are closed and we have $\bigcup_{k\in\mathbb{N}} \overline{S}_k = O$.

11.8 The distance between sets in metric spaces

In a metric space $(X; d)$ we define the distance $d(x; A) := \inf\{d(x; y) : y \in A\}$ between a point $x \in X$ and a set $A \subset X$ and correspondingly $d(A; B) := \inf\{d(x; B) : x \in A\}$. The triangle inequality takes the form $d(x; A) \le d(x; y) + d(y; A)$ since for **all** $z \in A$ we have $d(x; A) \le d(x; z) \le d(x; y) + d(y; z)$ and consequently $d(x; A) \le \inf\{d(x; y) + d(y; z) : z \in A\} = d(x; y) + d(y; A)$.

11.9 Stone's Theorem

In a **metric** space $(X; d)$ every open cover $\mathcal{U} = (U_i)_{i\in I}$ has a **locally finite open refinement cover** $\mathcal{O}$.



Proof: We fill the U_i starting with the core $A_{0,i} := \{x \in U_i : 1 \le d(x; X \setminus U_i)\}$ and proceeding towards the rim with increasing layers $A_{n,i} := \{x \in U_i : 2^{-n} \le d(x; X \setminus U_i) \le 2^{-n+1}\}$ with $n \in \mathbb{N}^*$. Since the U_i are open we have $\bigcup_{n\in\mathbb{N}} A_{n,i} = U_i$. From

$2^{-n} \leq d(x;X \setminus U_i)$ for all $x \in A_{n,i}$ and $d(y;X \setminus U_i) \leq 2^{-n-k+1}$ for all $y \in A_{n+k,i}$ together with 11.8 we infer $d(x;y) \geq d(x;X \setminus U_i) - d(y;X \setminus U_i) \geq 2^{-n}\left(1-2^{1-k}\right) \geq 2^{-n-1}$ and hence $d(A_{n,i};A_{n+k,i}) \geq 2^{-n-1}$ for $k \geq 2$.

In a second stage we remove the overlapping parts of the U_j by means of taking $B_{n,i} := A_{n,i} \setminus \bigcup_{j<i} U_j$ to achieve the locally finite property. Since the $B_{n,j}$ are not yet open we add narrow perimeters by means of $C_{n,i} := \{x \in U_i : d(x;B_{n,i}) < 2^{-n-3}\}$ so that only adjacent $C_{n,i}$ intersect each other, i.e. $C_{n,i} \cap C_{m,i} = \emptyset$ if $|n-m| > 1$ because $d(C_{n,i};C_{n+k,i}) \geq d(B_{n,i};B_{n+k,i}) - 2 \cdot 2^{-n-3} \geq d(A_{n,i};A_{n+k,i}) - 2^{-n-2} \geq 2^{-n-1} - 2^{-n-2} = 2^{-n-2}$. In order to demonstrate the covering property of the $B_{n,i}$ and especially the $C_{n,i}$ we can assume a well-ordered index set I (cf. [19, p. 14.2]). For a $x \in X$ let $i \in I$ be the smallest index with $x \in U_i$. On account of $\bigcup_{n\in\mathbb{N}} A_{n,i} = U_i$ there is a $n \in \mathbb{N}$ with $x \in A_{n,i}$ but due to the choice of i we have $x \notin U_j$ for all $j < i$ and hence $x \in B_{n,i} \subset C_{n,i}$.

The $C_{n,i}$ are locally finite since for the above mentioned $x \in B_{n,i} \subset C_{n,i}$ due to the construction of the $B_{n,i}$ we have $x \notin B_{m,j}$ for all $j < i$ resp. $m \in \mathbb{N}$ and according to the choice of $i \in I$ and because of $x \in U_i$ this extends to all $j > i$. Especially we have $x \notin C_{m,j}$ for $j \neq i$ and $m \in \mathbb{N}$. Due to $d(C_{m,i};C_{m+2,i}) \geq 2^{-m-2}$ for $m \in \mathbb{N}$ the neighborhood $B_{2^{-n-3}}(x)$ apart from $C_{n,i}$ meets at most the two neighbours $C_{n-1,i}$ and $C_{n+1,i}$. Hence the system $\mathcal{O} := (C_{n,i})_{n\in\mathbb{N},i\in I}$ is the desired locally finite refinement cover of $\mathcal{U}$.

11.10 The metrization theorem of Bing, Nagata and Smirnow

A topological space X is **metrizable** iff it is **regular** and has a **σ-locally finite basis**.

Proof:

$\Rightarrow$: On account of 11.9 for every $n \in \mathbb{N}$ and the cover $\mathcal{U}_n = \left(B_{2^{-n}}(x)\right)_{x\in X}$ there is a locally finite refinement cover $\mathcal{O}_n$. The desired σ-locally finite base then is given by $\mathcal{O} := (\mathcal{O}_n)_{n\in\mathbb{N}}$ since for every open O and $x \in O$ there is a $n \in \mathbb{N}$ and an open ball $B_{2^{-n}}(x) \subset O$ as well as a $k \in \mathbb{N}$ with another open ball $B_{2^{-n-2-k}}(x)$ intersecting only finitely many sets of $\mathcal{O}_{n+2}$ all of them included in a $B_{2^{-n-2}}(y)$ and therefore in $B_{2^{-n}}(x) \subset O$.

$\Leftarrow$: Due to 11.5 X is **paracompact**: For an open cover $\mathcal{U} = (U_i)_{i\in I}$ the basis $\mathcal{S} = (\mathcal{S}_n)_{n\in\mathbb{N}}$ with locally finite $\mathcal{S}_n = (S_{n;i})_{i\in I_n}$ yields locally finite refinement systems $\mathcal{V}_n := \{V \in S_n : \exists i \in I : V \subset U_i\}$ whose union $\mathcal{V} := \bigcup_{n\in\mathbb{N}} \mathcal{V}_n$ covers X since $\mathcal{S}$ is a basis. Due to 11.7 the $S_{n,i}$are F_σ, the $X \setminus S_{n,i}$ are G_δ and so with 11.2 and 8.2 a continuous function $\phi_{n;i} : X \to [0;1]$ exists with $S_{n,i} = \{\phi_{n;i} > 0\}$. Since $\mathcal{S}_n$ is locally finite the function $\psi_{n;i} := \frac{2^{-n}\phi_{n;i}}{1+\sum_{i\in I_n}\phi_{n;i}}$ is well defined and continuous with $0 \leq \psi_{n;i} \leq 2^{-n}$, $S_{n;i} = \{\psi_{n;i} > 0\}$ as well as $0 \leq \sum_{i\in I_n} \psi_{n;i} \leq 2^{-n}$. Thus $d(x;y) := \sum_{n\in\mathbb{N}}\sum_{i\in I_n} |\psi_{n;i}(x) - \psi_{n;i}(y)|$ is the desired metric: For $x \neq y$ and due to T_1 there is a $S_{n;i} \in U(x)$ with $y \notin S_{n;i}$ and hence $\psi_{n;i}(x) - \psi_{n;i}(y) \neq 0$ resp. $d(x;y) \neq 0$. For $x = y$ we have of course $d(x;y) = 0$. Symmetry and triangle inequality directly follow from the definition. This metric is not uniquely determined but we have to show that it leads back to the given topology (cf. 1.8). For any $x \in X$ and a $S_{n;i} \in S$ with $x \in S_{n;i}$ we have $\delta := \psi_{n;i}(x) > 0$. Hence for all $y \in B_\delta(x)$ the estimate $|\psi_{n;i}(x) - \psi_{n;i}(y)| < \delta$ holds and therefore $\psi_{n;i}(y) > 0$ and consequently $y \in S_{n;i}$ resp. $B_\delta(x) \subset S_{n;i}$. Thus d induces a topology which is **stronger** than the given topology. Conversely for any $x \in X$ the limit $d(x;\,) : X \to \mathbb{R}^+$ of a uniformly convergent sequence of continuous functions (cf. 18.3) is again continuous with reference to the given topology, i.e. for $\delta > 0$ there is a basis set $S_{m;j}$ with $d(x;y) < \delta$ for all $y \in S_{m;j}$, especially $x \in S_{m;j}$ and hence $S_{m;j}\subset B_\delta(x)$ which shows that the given topology is **stronger** than the topology induced by d.

Remark: The construction of the metric is based on **Urysohn's lemma** 8.1 and its extension in 8.2. Urysohn's approach is based on the following improvement of theorem 7.9 on the **embedding** of a **completely regular** space into a product $\prod_{\varphi\in\Phi} I_\varphi$ **of real and therefore metrizable intervals:**

11.11 Embedding of normal spaces in product spaces

A **normal** space X with basis $\mathcal{B} = (V_j)_{j \in J}$ can be embedded into a **product space** $[0;1]^J$.

Proof: For every basis set $V_j \in \mathcal{B} = (V_j)_{j \in J}$ there is an $U_j \in \mathcal{B}$ with $\overline{U}_j \subset V_j \subset X$ and a continuous $f_j : X \to [0;1]$ with $f_j\left[\overline{U}\right] = \{0\}$ resp. $f_j[X \setminus V] = \{1\}$. The mapping $e : X \to [0;1]^J$ with $e(x) := (f_j(x))_{j \in J}$ is **continuous** due to 4.2 and **injective** owing to T_1. It is also **open** since for open $O \subset X$ and $x \in O$ there is are $U_j, V_j \in \mathcal{B}$ with $x \in U_j \subset \overline{U}_j \subset V_j \subset O$ as well as corresponding f_j as described above so that the cylinder set $\{f_j < 1\} = \pi_j^{-1}[0;1[\cap e[X]$ is open in $e[X]$ and contained in $e[O]$.

11.12 Metrizable product spaces

The **product** $\prod_{j \in J} Y_j$ of metric spaces Y_j is **metrizable** iff the index set J is **countable**.

Proof: Due to 4.2 the locally finite property of the basis sets $\mathcal{S}_{n;j} = (S_{n;i;j})_{i \in I_{n;j}}$ on the X_j is transferred to the subbasis sets $\mathcal{B}_{n;j} := \left(\pi_j^{-1}(S_{n;i;j})\right)_{i \in I_{n;j}}$ on the product $\prod_{j \in J} Y_j$. On account of [19, p. 17.6] the family $(\mathcal{B}_{n;j})_{n \in \mathbb{N}; j \in J}$ is countable iff J is countable. Due to 7.10 the regular character is transferred to the product in any case and hence the assertion follows from 11.11.

11.13 Urysohn's metrization theorems:

1. A topological space is **metrizable** and **separab**le iff it is **regular** and **second countable**.
2. A **compact** space is **metrizable** iff it is **second countable.**
3. A **locally compact** space is **metrizable** and **σ-compact** iff it is **second countable.**

Proof:

1. $\Rightarrow$ follows from 2.7 and 7.2 whereas $\Leftarrow$ follows from 11.11 resp. 11.12.
2. $\Rightarrow$ follows from 10.2 whereas $\Leftarrow$ follows from 9.5 resp. 11.10.
3. $\Rightarrow$: The countable basis is provided by the open balls $B_{1/n}(x_{n,m})$ which for every $n \in \mathbb{N}$ and $1 \leq m \leq m_n$ form a finite refinement cover of the open cover $\left(B_{1/n}(x)\right)_{x \in K_n}$ of the compact sets K_n covering themselves X. Indeed for any open $O \subset X$ and any $x \in O$ there is a $n \in \mathbb{N}$ with $B_{1/n}(x) \subset O$ and a $k \geq 2n$ with a $1 \leq m \leq m_k$ such that $x \in B_{1/k}(x_{k,m}) \subset B_{1/n}(x) \subset O$. $\Leftarrow$ follows from 10.9 resp. since the sets $K_n := X \setminus \bigcup_{k>n} U_k$ provided by the basis $(U_n)_{n \in \mathbb{N}}$ of X are compact and cover X.

12 Uniform spaces

12.1 Uniform structures

For arbitrary sets $A, B \subset X^2$ we define $A^{-1} = \{(x;y) \in X : (y;x) \in A\}$ and $AB = \{(x;z) \in X : \exists y \in X : (x;y) \in A \wedge (y;z) \in B\}$ with $A^2 := AA$. A is **symmetric** iff $A^{-1} = A$. We have $(AB)^{-1} = B^{-1}A^{-1}$ and $(AB)C = A(BC)$; from $A \subset B$ follows $A^{-1} \subset B^{-1}$ and $AC \subset BC$ for arbitrary C. The symmetry of A entails the symmetry of all A^n for $n \in \mathbb{N}^*$. A **neighborhood filter** $\mathcal{U}$ on a set X is a **filter** on the **product** X^2 whose **neighborhoods** or **entourages** $U \in \mathcal{U}$ contain the **diagonal** Δ and with every entourage U its **mirror image**U^{-1}as well as another entourage V with $V^2 \subset U$. The pair $(X;\mathcal{U})$ then is a **uniform space**. Two points x and y in X are **adjacent of order** $U \in \mathcal{U}$ iff $(x;y) \in U$; analogously a set $V \subset X$ is **small of order** $U \in \mathcal{U}$ iff $V^2 \subset U$. On account of $\Delta \subset U \in \mathcal{U}$ we have $U \subset U^n \in \mathcal{U}$ for $n \in \mathbb{N}^*$.

12.2 Neighbourhood basis

A subfamily $\mathcal{B}$ of a neighborhood filter $\mathcal{U}$ is a **neighborhood basis** iff every neighborhood from $\mathcal{U}$ contains a member of $\mathcal{B}$. With $\mathcal{B}$ the systems $\mathcal{B}' := \{B \cap B^{-1} : B \in \mathcal{B}\}$ and $\mathcal{B}_n := \{B^n : B \in \mathcal{B}\}$ are also neighborhood bases for the neighborhood filter $\mathcal{U}$.

12.3 Uniformization

For a neighborhood filter $\mathcal{U}$ on a set X the family $U(x) = \{y \in X : (x;y) \in U\}$ for $U \in \mathcal{U}$ and $x \in X$ is a neighborhood system defining a topology $\mathcal{O}$ on X according to 2.4. Correspondingly for a **neighborhood basis** $\mathcal{B}$ the sets $B(x)$ with $B \in \mathcal{B}$ define a **neighborhood basis** of x. The topology $\mathcal{O}$ **induced** by $\mathcal{B}(x)$ resp. $\mathcal{U}(x)$ is the **topology of the uniform space**. A topology $\mathcal{O}$ is **uniformizable** iff there is a neighborhood filter $\mathcal{U}$ which induces $\mathcal{O}$.

12.4 The discrete uniform structure

The **indiscrete topology** is induced by the set X^2 itself. The **discrete topology** on the one hand is generated by the set of all subsets of X^2 including the **diagonal** Δ. One the other hand it is generated by the **neighborhood filter of the finite partitions**: Indeed for every finite partition $P = \{A_1, ..., A_n\}$ of X the neighborhoods $U_P = \{(x;y) : \exists A_i \in P : x, y \in A_i\}$ satisfy the conditions 12.1 with $U_P \supset U_{P'}$ iff every set from P is the union of sets from P' and $U_P \cap U_{P'}$ is the neighborhood generated by all intersections of sets from P with sets from P'. Since every partition is a disjoint cover we also have $U_P^2 = U_P$.

12.5 Open interiors and closures of a neighborhood filter

The **open interiors** resp. the **closures** of a neighborhood filter with reference to the product X^2 are again a neighborhood filter in X.

Proof: According to 12.1 for any neighborhood U there is a symmetric neighborhood V with $V^3 \subset U$. For $(x;y) \in V$ the set $V(x) \times V(y)$ is open in X^2 and included in V^3. Hence V^3 is a neighborhood for each of its points, i.e. it is an open set in X^2. Consequently $V^3 \subset \overset{o}{U}$ and $\overset{o}{U}$ is also a neighborhood in X. For $(x;y) \in \overline{V}$ we have $V(x) \times V(y) \cap V \neq \emptyset$ and there is a pair $(x';y') \in V$ with $(x;x') \in V$ resp. $(y;y') \in V$ and on account of the symmetry of V follows $(x;y) \in V^3$. Especially we have $\overline{V} \subset V^3 \subset U$ which proves the statement with respect to the closures.

12.6 Separation axioms

The concepts introduced in section 7 tacitly refer to the topology induced by the uniform structure. An uniform space is **separated** iff it satisfies T_1 or equivalently T_2 (cf. 12.1).

1. An uniform space is **separated** iff the **diagonal** Δ is the intersection of all its neighborhoods.
2. Every uniform space is a T_3-space.

Proof:

1. $\Rightarrow$: For $x \neq y$ there is a neighborhood U with $U(x) \cap U(y) = \emptyset \Rightarrow (x;y) \notin U$. Hence the intersection of all neighborhoods does not contain $(x;y)$ and since this is true for arbitrary $x \neq y$ the statement follows. $\Leftarrow$: For $x \neq y$ there is a neighborhood U with $(x;y) \notin U$. Due to 12.1 there is another neighborhood V with $V^2 \subset U$ and especially $V(x) \cap V(y) = \emptyset$.
2. The propsition follows from 7.7 since owing to 12.1 for every neighborhood U there is a symmetric neighborhood V with $V^2 \subset U$ and $\overline{V(x)} \subset V^2(x) \subset U(x)$. Note that in this instance we mean the closure $\overline{V(x)} \subset V^2(x)$ with respect to X whereas in 12.5 we refer to the closure $\overline{V} \subset V^3$ in X^2.

12.7 Compactness

For a neighborhood U and a subset A the set $V(A) := \bigcup_{x\in A} V(x)$ is the **uniform neighborhood** of A. In a uniform space the two following statements hold:

1. Every neighborhood of a **compact** subset contains an uniform neighborhood.
2. Two disjoint sets K and A can be **separated** by uniform neighborhoods if K is **compact** and A is **closed**.

Proof:

1. For the neighborhood U of the compact subset $K \subset X$ and every $x \in K$ there is a neighborhood U_x with $U_x(x) \subset U$ and a second neighborhood V_x with $V_x^2 \subset U_x$. The $V_x(x)$ cover K and let V be the finite intersection of those V_x whose corresponding neighborhoods $V_x(x)$ cover K. For every$y \in V(K)$ there is a $z \in K$ with $(y;z) \in V$. This z is located in one of the $V_x(x)$, i.e. $(z;x) \in V_x$ such that $(y;x) \in \mathrm{VV_x} \subset V_x^2 \subset U_x$. Hence $y \in U_x(x) \subset U$ and consequently $V(K) \subset U$.
2. On account of 12.6.1 the set K can be separated by a uniform neighborhood $U(K)$ from A. Then the uniform neighborhoods $V(K)$ and $V(A)$ with $V^2 \subset U$ are disjoint.

12.8 Uniformly continuous functions

A function $f : X \to Y$ between uniform spaces $(X,\mathcal{U}_X)$ and $(Y,\mathcal{U}_Y)$ is **uniformly continuous** iff $\left(f^2\right)^{-1}[U] \in \mathcal{U}_X \forall U \in \mathcal{U}_Y$. For every neighborhood $U \in \mathcal{U}_Y$ there is a neighborhood $V \in \mathcal{U}_X$ with $\left(f^2\right)[V] \subset U$. Obviously it is sufficient to show this property for sets of a **neighborhood basis**. The **composition** $g \circ f : X \to Z$ of two uniformly continuous $f : X \to Y$ and $g : Y \to Z$ is again uniformly continuous. Obviously every uniformly continuous function is **continuous** with reference to the induced topologies.

12.9 Heine's theorem

A **continuous** function $f : X \to Y$ on a **compact** uniform space $(X,\mathcal{U}_X)$ into a uniform space $(Y,\mathcal{U}_Y)$ is **uniformly continuous**.

Proof: For any $U \in \mathcal{U}_Y$ we choose a symmetric $V \in \mathcal{U}_Y$ such that $V^2 \subset U$. Since f is continuous for every $x \in X$ there is a $V_x \in \mathcal{U}_X$ with $f[V_x(x)] \subset V(f(x))$ and subsequently another symmetrc $W_x \in \mathcal{U}_X$ with $W_x^2 \subset V_x$. The $W_x(x)$ possess a finite subcover and the intersection W of the corresponding W_x constitutes the desired neighborhood with $(f \times f)[W] \subset U$. For $(y;z) \in W$ there is a $W_x(x)$ of the finite subcover with $y \in W_x(x) \subset W_x^2(x) \subset V_x(x)$ and therefore $z \in WW_x(x) \subset W_x^2(x) \subset V_x(x)$. On account of $f[V_x(x)] \subset V(f(x))$ it follows that $f(y), f(z) \in V(f(x))$ and hence $(f(y); f(z)) \in V^2 \subset U$.

12.10 Initial neighborhood filter

The notations of stronger resp. weaker topologies are extended to **neighborhood filters:** $\mathcal{U}_1 \subset \mathcal{U}_2$ is denoted as $\mathcal{U}_2$ **stronger** than $\mathcal{U}_1$ resp. $\mathcal{U}_1$ **weaker** than $\mathcal{U}_2$. These relationships obviously transfer to the induced topologies. The identity $\mathrm{id} : (X;\mathcal{U}_1) \to (X;\mathcal{U}_2)$ is **uniformly continuous** iff $\mathcal{U}_1$ is **stronger** than $\mathcal{U}_2$. The **weakest filter** $\mathcal{U}$ on a set X such that the functions $f_i : (X,\mathcal{U}) \to (Y_i,\mathcal{U}_i)$ into the uniform spaces $(Y_i,\mathcal{U}_i)_{i\in I}$ are **uniformly contiuous** is the **initial neighborhood filter** with reference to the f_i. The **finite intersections** $\bigcap_{i\in E} \left(f_i^2\right)^{-1}[U_i]$ of inverse images of neighborhoods $U_i \in \mathcal{U}_i$ for finite $E \subset I$ form a **neighborhood basis** of $\mathcal{U}$ since $\left(f_i^2\right)^{-1}\left[U_i^{-1}\right] = \left(\left(f_i^2\right)^{-1}[U_i]\right)^{-1}$ and for $V_i^2 \subset U_i$ follows $\left(\bigcap_{i\in E} \left(f_i^2\right)^{-1}[V_i]\right)^2 \subset \bigcap_{i\in E} \left(f_i^2\right)^{-1}\left[V_i^2\right] \subset \bigcap_{i\in E} \left(f_i^2\right)^{-1}[U_i]$. The **initial neighborhood filter** $\mathcal{U}$ induces the initial topology with reference to the $f_i : (X,\mathcal{O}) \to (Y_i,\mathcal{O}_i)$

with the topologies O resp. $\mathcal{O}_i$ induced by $\mathcal{U}$ resp. $\mathcal{U}_i$. Indeed the sets $\left(\bigcap_{i\in E}\left(f_i^2\right)^{-1}[U_i]\right)(x) = \bigcap_{i\in E}\left(\left(f_i^2\right)^{-1}[U_i]\right)(x) = \bigcap_{i\in E} f_i^{-1}\left[U_i\left(f_i(x)\right)\right]$ form a **neighborhood basis** of the topology induced by $\mathcal{U}$. The topology induced by these neighborhoods is is the weakest topology so that all f_i are continuous.

12.11 Product of uniform spaces: Analogous to 4.2 the **product filter** on the **product** $\prod_{i\in I} X_i$ of the uniform spaces $(X_i, U_i)_{i\in I}$ is the **initial neighborhood filter** with reference to the **projections** $\pi_i : \prod_{i\in I} X_i \to X_i$. It is generated by the finite intersections $\bigcap_{i\in E}\left(\pi_i^2\right)^{-1}[U_i]$. The function $f : Y \to \prod_{i\in I} X_i$ is **uniformly contiuous** iff the inverse images $\left(f^2\right)^{-1}\left[\bigcap_{i\in E}\left(\pi_i^2\right)^{-1}[U_i]\right] = \bigcap_{i\in E}\left(\left(\pi_i \circ f\right)^2\right)^{-1}[U_i]$ of basis neighborhoods are again neighborhoods in $(Y,\mathcal{U})$. Hence f is uniformly continuous iff all **components** $\pi_i \circ f : (Y,\mathcal{U}) \to (X_i,\mathcal{U}_i)$ are uniformly continuous.

12.11 Uniform subspaces

Analogous to 4.3 the **trace filter** on the subset $A \subset X$ of the uniform space $(X;\mathcal{U})$ is the **initial neighborhood filter** with reference to the **injection** $j : A \to X$. It is constituted by the intersections $\left(j^2\right)^{-1}[U] = U \cap \left(A^2\right)$ of neighborhoods $U \in \mathcal{U}$ with A^2.

12.12 Dense subsets

For a **dense** subset A in X the closures with reference to X^2 of neighborhoods of the uniform subspace A form a basis for the neighborhoods in X.

Proof: For an **open** neighborhood U of X we have $U \subset \overline{U \cap A^2}$ since for $(x;y) \in U$ every neighborhood $V(x) \times W(y)$ w.l.o.g. contained in U intersects A^2 and hence $(x;y)$ is an accumulation point of $U \cap A^2$. Especially $\overline{U \cap A^2}$ is itself a neighborhood and contained in $\overline{U}$. The proposition then follows from 12.5.

13 Uniformization

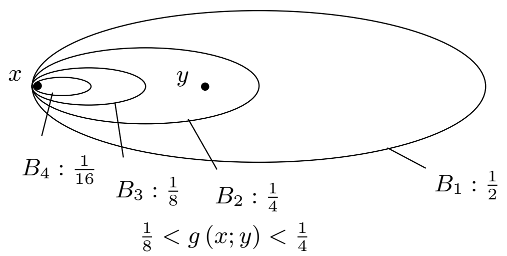
13.1 Uniformization of metric spaces

In a **metric space** $(X;d)$ the sets $U_{1/n} := \left\{(x;y) \in X^2 : d(x;y) < \frac{1}{n}\right\} = \left\{d < \frac{1}{n}\right\}$ for $n \in \mathbb{N}^*$ form a **countable neighborhood** basis of the corresponding neighborhood filter. This property is already sufficient in the general case for the construction of a **pseudometric** (cf. 1.2):

13.2 Uniformization of first countable spaces

A uniform space $(X;\mathcal{U})$ is induced by a **pseudometric** iff it has a **countable neighborhood basis**.

Proof: Owing to 12.2 we can assume that all elements B_n of the countable neighborhood basis $(B_n)_{n\in\mathbb{N}^*}$ of the neighborhood filter $\mathcal{U}$ are symmetric and decreasing with $B_{n+1}^3 \subset B_n$. For $x, y \in X$ let $g(x;y) := \begin{cases} 1 \text{ for } (x;y) \notin B_1 \\ \min\left\{2^{-k} : (x;y) \in B_k : k \in \mathbb{N}^*\right\} \end{cases}$ with M_{xy} be the set of all finite sequences $(x_i)_{i\in K}$ with index set $K = \{0;1;...;n\}$ for an $n \in \mathbb{N}$, starting point $x_0 = x$ and endpoint $x_n = y$.



The desired pseudometric is then given by $d(x;y) := \inf\left\{\sum_{0\leq i\leq n-1} g(x_i;x_{i+1}) : (x_i)_{i\in K} \in M_{xy}\right\}$. The properties 1.2.1 resp. 1.2.2 are trivial and the triangle inequality results from the fact that the paths from x to z via y are a subset of all paths from x to z. In order to show that d actually generates the $B_n = \{g \leq 2^{-n}\}$ we prove the estimate $\frac{1}{2}g(x;y) \leq d(x;y) \leq g(x;y)$ for $(x;y) \in B_1$. The right hand part directly follows from the definition.

The left hand part is equivalent to $\frac{1}{2}g(x;y) \leq \sum_{0\leq i\leq n-1} g(x_i;x_{i+1})$ for $(x_i)_{i\in K} \in M_{xy}$ and will be demonstrated by **induction** over n. The base case $n = 1$ is trivial. Now let $x_0 = x; ...; x_{n+1} = y$ and $a = \sum_{0\leq i\leq n-1} g(x_i;x_{i+1}) \neq 0$. In the case of $a \geq \frac{1}{2}$ the induction step immediately follows from $g(x;y) \leq 1$. So let $a < \frac{1}{2}$ and m the greatest index such that $\sum_{0\leq i\leq m-1} g(x_i;x_{i+1}) \leq \frac{a}{2}$. Then we have $\sum_{0\leq i\leq m} g(x_i;x_{i+1}) > \frac{a}{2}$ and hence the remaining sum is $\sum_{m+1\leq i\leq n} g(x_i;x_{i+1}) \leq \frac{a}{2}$. Applying the induction hypothesis to the partial sums on the left and the right hand sides of m we obtain $\frac{1}{2}g(x;x_m) \leq \frac{a}{2}$ and $\frac{1}{2}g(x_{m+1};y) \leq \frac{a}{2}$. As before he definition of a yields $\frac{1}{2}g(x_m;x_{m+1}) \leq \frac{a}{2}$. Let k be the smallest integer with $2^{-k} \leq a$, then $g(x;x_m), g(x_m;x_{m+1})$ and $g(x_{m+1};y)$ are less than or equal to 2^{-k}, i.e. $(x;x_m), (x_m;x_{m+1})$ and $(x_{m+1};y)$ are in B_k and hence in $(x;y) \in B_k^3 \subset B_{k-1}$. Especially we get $g(x;y) \leq 2^{-(k-1)}$ resp. $\frac{1}{2}g(x;y) \leq 2^{-k} \leq a$. In the case of $\sum_{0\leq i\leq n-1} g(x_i;x_{i+1}) = 0$ all $(x_i;x_{i+1})$ are in B_{k+n} and therefore $(x;y)$ in B_k for all $k \in \mathbb{N}^*$, i.e. $g(x;y) = 0$.

The above proved estimate entails $B_k \subset d^{-1}\left[\left[0;2^{-k}\right[\right] \subset B_{k-1}$, i.e. the given neighborhood basis $(B_n)_{n\in\mathbb{N}^*}$ is generated by d. Conversely the pseudometric d generates a countable neighborhood basis $(B_n)_{n\in\mathbb{N}^*}$ with $B_n := d^{-1}\left[\left[0;2^{-k}\right[\right]$ for the neighborhood filter induced by d.

13.3 The metrization theorem for uniform spaces

A uniform space is **metrizable** iff it is **separated** and **first countable**.

13.4 Uniformization by a system of pseudometrics

Every uniform space can be induced by a **system of pseudometrics**.

Proof: For every neighborhood V of the given neighborhood filter $\mathcal{U}$ there is a sequence of symmetric neighborhoods B_n such that $B_1 \subset V$ and $B_{n+1}^3 \subset B_n$ for all $n \in \mathbb{N}^*$. Each of these sequences is the basis for a neighborhood filter $\mathcal{U}_V$ which is induced by a pseudometric d_V as shown in 13.2. On account of $\mathcal{U}_V \subset \mathcal{U} \forall V \in \mathcal{U}$ and $\bigcup_{V\in\mathcal{U}} \mathcal{U}_V = \mathcal{U}$ the filter $\mathcal{U}$ is the weakest filter being stronger then any of the $\mathcal{U}_V$. Then $\mathcal{B} := \left\{\bigcap_{V\in\mathcal{E}\subset\mathcal{U}} d_V^{-1}\left[[0;a[\right] : |\mathcal{E}| \in \mathbb{N}; a \in]0;1]\right\} = \left\{d^{-1}\left[[0;a[\right] : a \in]0;1]\right\}$ for $d(x;y) := \min\{d_V(x;y) : V \in \mathcal{E} \subset \mathcal{U}; |\mathcal{E}| \in \mathbb{N}\}$is a neighborhood basis for $\mathcal{U}$. The d_V are pseudometrics on the topological space induced by $\mathcal{U}$ but they generate only a part of the neighborhoods in $\mathcal{U}_V$ whereas the function $d : X^2 \to [0;1]$ generates the system $\mathcal{U}$ by means of its inverse images but in general it is not a pseudometric any more.

13.5 Uniformization of T_{3a}-spaces

A topological space X is **uniformizable** iff it satisfies T_{3a}.

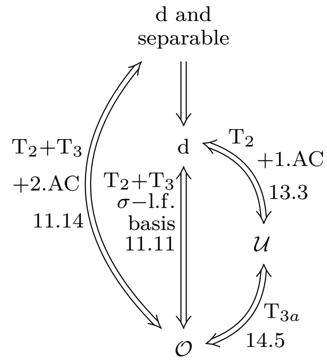
Proof:

$\Rightarrow$: On account of 13.4 for a closed $A \subset X$, $x_0 \in X \setminus A$ and $V \in \mathcal{U}$ with $V(x_0) \subset X \setminus A$ there is a pseudometric d_V and an $a \in]0;1]$ such that $d_V^{-1}[[0;a]] \subset V$. The function $f : X \to [0;1]$ with $f(x) := \sup\left\{0; 1 - \frac{1}{a}d_V(x;x_0)\right\}$ is continuous since $d_V^{-1}[[0;a[] \in U_V \subset U$ and furthermore we have $f[A] = \{0\}$ as well as $f(x_0) = 1$.

$\Leftarrow$: Let $(X;\mathcal{O})$ be a T_{3a}-space and I the set of continuous functions $f : X \to [0;1]$. On account of 7.8.2 the topology $\mathcal{O}$ is the **initial topology** with reference to the $f \in I$. The initial neighborhood filter $\mathcal{U}$ with reference to the $f \in I$ induces a topology $\mathcal{O}'$ coinciding with $\mathcal{O}$ on account of 12.2.

13.6 Metrization and Uniformization

As shown in 1.4 and 12.4 the relation metric $\to$ neighborhood filter $\to$ topology is not injective. Especially a topology $\mathcal{O}$ can be induced by different neighborhood filters $\mathcal{U}$ and $\mathcal{U}'$ not each of them being necessarily metrizable. E.g. the metric d with $d(x;x) = 0$ resp. $d(x;y) = 1$ generates the **discrete** neighborhood filter $\mathcal{U}$ and subsequently the **discrete topology** $\mathcal{O}$. The neighborhood filter $\mathcal{U}'$ of the **finite partitions** described in 12.4 also induces the discrete topology. But for an infinite set X the uniform structure $\mathcal{U}'$ is not metrizable since in that case there would exist a countable neighborhood basis $(P_n)_{n\in\mathbb{N}}$ owing to 13.3 , i.e. **every** set in a finite partition P of X would be the union of sets from a **single** partition $P_n \subset P$. But by means of unions a single partition P_n can only generate a finite number of weaker partitions and hence the set of partitions of the infinite set X would be countable contrary to e.g. [19, p. 17.9].



14 Completion

14.1 Cauchy filter

A subset A of an uniform space $(X;\mathcal{U})$ resp. metric space $(X;d)$ is **small of order** $V \in \mathcal{U}$ resp. $\epsilon > 0$ iff $A^2 \subset V$ resp. $A^2 \subset d^{-1}\left[[0;\epsilon[\right]$. A filter $\mathcal{F}$ on X is a **Cauchy filter** iff for every neighborhood V there is a $F \in \mathcal{F}$ small of order V.

1. Every convergent filter $\mathcal{F}$ is a Cauchy filter since for every $V \in \mathcal{U}$ the neighborhood filter $\mathcal{U}(x) \subset \mathcal{F}$ contains an element $U(x) \in \mathcal{U}(x) \subset \mathcal{F}$ with $U^2 \subset V$, i.e. $U(x)$ is small of order V.
2. Every Cauchy filter **converges to its accumulation points** since for every accumulation point x and $U(x) \in \mathcal{U}(x)$ there is a $F \in \mathcal{F}$ small of order $V \in \mathcal{U}$ with $V^2 \subset U$ and $F \cap V(x) \neq \emptyset$, hence $F \subset U(x)$ and therefore $U(x) \in \mathcal{F}$.
3. The **image** $f(\mathcal{F})$ of a Cauchy-filter $\mathcal{F}$ under the uniformly continuous function $f : (X;\mathcal{U}_X) \to (Y;\mathcal{U}_Y)$ is again a Cauchy filter since for every $V \in \mathcal{U}_Y$ we have $\left(f^2\right)^{-1}[V] \in \mathcal{U}_X$ such that there is a $F \in \mathcal{F}$ with $F^2 \subset \left(f^2\right)^{-1}[V]$ and therefore $f^2\left[F^2\right] = \left(f[F]\right)^2 \subset V$.
4. A filter $\mathcal{F}$ on X **is a Cauchy filter** with reference to the **initial neighborhood filter** $\mathcal{U}$ for the functions $f_i : X \to (Y_i\mathcal{U}_i)$ iff its **filter images** $f_i(\mathcal{F})$ are Cauchy filters. Indeed on the one hand for a given neighborhood basis set $B = \bigcap_{i\in E}\left(f_i^2\right)^{-1}[U_i]$ with $U_i \in \mathcal{U}_i$, finite $E \subset I$ (cf. 12.10) and if all filter images are Cauchy there are $F_i \in \mathcal{F}$ with $f_i(F_i) \in f_i(\mathcal{F})$ with $f_i^2\left(F_i^2\right) \subset U_i \forall i \in E$ and the filter set $F := \bigcap_{i\in E} F_i$ satisfies $f_i^2\left(F^2\right) \subset U_i \forall i \in E$. Hence $F^2 \subset B$ such that $\mathcal{F}$ must be a Cauchy filter. The converse follows from 12.8.

14.2 Complete spaces

An uniform space $(X;\mathcal{U})$ is **complete** iff every Cauchy filter $\mathcal{F}$ on X converges to a limit $x \in X$.

1. The **product** $X = \prod_{i\in I} X_i$ of **complete** spaces X_i is **complete** since the images $\pi_i(\mathcal{F})$ of a Cauchy filter $\mathcal{F}$ under the **projections** $\pi_i : X \to X_i$ are again Cauchy filters on the X_i and converge each to an $x_i \in X_i$ such that $\mathcal{F}$ converges to $(x_i)_{i\in I} \in X$.
2. Every closed subspace A of a **complete** space X is again **complete** since the image filter $i(\mathcal{F})$ of a Cauchy filter $\mathcal{F}$ under the **injection** $i : A \to X$ is again a Cauchy filter on X and converges to an $x \in X$ which because of $\mathcal{U}(x) \subset i(\mathcal{F})$ is an accumulation point of the set A as well as of the filter $\mathcal{F}$ on A so that under the hypothesis $\mathcal{F}$ converges to $x \in A$.

3. Conversely every **complete subspace** A of a **separable** space X is **closed** since for every accumulation point x of A the nonempty intersections $\mathcal{U}(x) \cap A$ of a neighborhood basis $\mathcal{U}(x)$ with A generate a Cauchy filter on A which on account of T_2 has a single accumulation point x and converges to x due to 14.1. Since A is complete x must lie in A.

14.3 Minimal Cauchy filter:

For every Cauchy filter $\mathcal{F}$ on a uniform space $(X;\mathcal{U})$ there is a unique minimal Cauchy filter $\mathcal{F}_0 \subset \mathcal{F}$.

Proof: $\mathcal{F}_0$ is generated by $\mathcal{B} := \{U(F) : U \in \mathcal{U}, F \in \mathcal{F}\}$. On account of $(U_1 \cap U_2)(F_1 \cap F_2) \subset U_1(F_1) \cap U_2(F_2)$ the family $\mathcal{B}$ is a filter basis. $\mathcal{F}_0$ is Cauchy-filter since for $U \in \mathcal{U}$ there is a $V \in \mathcal{U}$ with $V^3 \subset U$ and due to the hypothesis a $F \in \mathcal{F}$ with $F \times F \subset V$ such that $V(F) \times V(F) \subset V^3 \subset U$. Obviously we have $\mathcal{F}_0 \subset \mathcal{F}$ and for every other Cauchy filter $\mathcal{F}_0' \subset \mathcal{F}$ we have $\mathcal{F}_0 \subset \mathcal{F}_0'$ since for every $U(F) \in \mathcal{B}$ there is a $F' \in F_0'$ with $F'^2 \subset U$ and $F' \cap F \neq \emptyset$ such that $F' \subset U(F)$ and hence $\mathcal{F}_0 \subset \mathcal{F}_0'$.

14.4 Properties of minimal Cauchy filters

1. Every **neighborhood filter** $\mathcal{U}(x)$ is a minimal Cauchy filter.
2. A minimal Cauchy filter containing the filter set F also contains its **interior** $\mathring{F}$.
3. The **basis** $\mathcal{B}$ of a Cauchy filter $\mathcal{F}$ is a **basis** for the corresponding **minimal** Cauchy filter $\mathcal{F}_0$.

14.5 Characterization of complete spaces by Cauchy filters

A uniform space X is **complete** iff the **trace** $\mathcal{F} \cap A$ of every Cauchy filter $\mathcal{F}$ on a **dense subset** $A \subset X$ converges to an $x \in X$.

Proof: Due to 14.4.2 the minimal Cauchy filter $\mathcal{F}_0$ of $\mathcal{F}$ with every set $F \in \mathcal{F}_0$ also contains its nonempty open interior $\mathring{F} \in \mathcal{F}_0$ intersecting the dense subset A. Consequently $\mathcal{F}_0 \cap A$ is again a Cauchy filter converging to a $x \in X$ due to the hypothesis, i.e. $\mathcal{U}(x) \cap A \subset \mathcal{F}_0 \cap A$. Hence x is also accumulation point of $\mathcal{F}_0$ and owing to 14.1.2 $\mathcal{F}_0$ converges to x and so does $\mathcal{F}$.

14.6 Extension of uniformly continuous functions

A **uniformly continuous** function $f : A \to Y$ on a **dense subset** A of the uniform space $(X;\mathcal{U}_X)$ into the complete and separated space $(Y;\mathcal{U}_Y)$ can be extended to a unique and uniformly continuous $\overline{f} : X \to Y$.

Proof: Every neighborhood of an arbitrary $x \in X$ intersects the set A and hence the trace $\mathcal{U}(x) \cap A$ is a Cauchy filter on A. According to the hypothesis the uniformly continuous image $f(\mathcal{U}(x) \cap A) = (f \circ i)(U(x))$ is again a Cauchy filter converging on Y to a uniquely determined $\overline{f}(x) := \lim (f \circ i)(U(x))$ with the canonical injection $i : A \to X$. $\overline{f}$ is uniformly continuous since for $U \in \mathcal{U}_Y$ there is a $V \in \mathcal{U}_Y$ wit $V^3 \subset U$ and on account of f being uniformly continuous a $W \in \mathcal{U}_X$ with $f^2\left[W \cap (A^2)\right] \subset V$. For $(x;y) \in M$ with $M^3 \subset W$ there are neighborhoods M_x resp. M_y w.l.o.g. small of order M and points $x_A \in M_x(x) \cap A$ resp. $y_A \in M_y(y) \cap A$ with $f(x_A) \in f[W_x(x) \cap A] \subset V\left(\overline{f}(x)\right)$ resp. $f(y_A) \in f[W_y(y) \cap A] \subset V\left(\overline{f}(y)\right)$. So we have $(x_A;y_A) \in M^3 \subset W$, hence $(f(x_A);f(y_A)) \in V$ and finally $\left(\overline{f}(x);\overline{f}(x)\right) \in V^3 \subset U$. According to 6.6.3 the extension $\overline{f}$ coincides on A with f. The uniqueness follows from 7.13.

14.7 Completion of separated spaces

A **separated** space $(X;\mathcal{U})$ can be **embedded** into a **complete separated** space $\left(\widetilde{X};\widetilde{\mathcal{U}}\right)$. The space $\left(\widetilde{X};\widetilde{\mathcal{U}}\right)$ is unique up to **homeomorphism** and every **uniformly continuous** $f : X \to \widetilde{Y}$ into another complete separated space $\widetilde{Y}$ can be **extended** to a uniformly continuous $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ such that $f = \widetilde{f} \circ i$ with the **embedding** $i : X \to \widetilde{X}$. The image $e\,[X]$ is **dense** in $\widetilde{X}$, i.e. the completion is its **closure**: $\overline{e\,[X]} = \widetilde{X}$. In particular every **closed** subset of a **completely regular** space is **complete**.

$$\begin{tikzcd} \widetilde{X} \arrow[r, "\widetilde{f}"] & \widetilde{Y} \\ X \arrow[u, "i"] \arrow[ur, "f"'] & \end{tikzcd}$$

Proof: Let $\widetilde{X}$ be the set of all **minimal Cauchy filters** on X with the uniform structure $\widetilde{\mathcal{U}}$ generated by the neighborhoods $\widetilde{V}$ of the pairs $(\mathcal{F};\mathcal{G})$ having a common set M small of order $V \in \mathcal{U}$ with **symmetric** V. On account of 6.1 the filters $\mathcal{F}$ and $\mathcal{G}$ then coincide on all sets including M such that possible limit points must be adjacent of order V. The **symmetry** of the V transfers to the $\widetilde{V}$ and they all include the **diagonal** Δ since every Cauchy filter contains a set small of order V. From $W^2 \subset V$ follows $\widetilde{W}^2 \subset \widetilde{V}$ since for $(\mathcal{F};\mathcal{G}),(\mathcal{G};\mathcal{H}) \in \widetilde{W}$ there is a $M \in \mathcal{F} \cap \mathcal{G}$ and $N \in \mathcal{G} \cap \mathcal{H}$ with $M^2, N^2 \subset V$ such that $M \cup N \in \mathcal{F} \cap \mathcal{H}$ and $(M \cup N)^2 \subset V$. In an analogous way we see that from $U \subset V \cap W$ follows $\widetilde{U} \subset \widetilde{V} \cap \widetilde{W}$.

$\widetilde{X}$ is **separated** since assuming $(\mathcal{F};\mathcal{G}) \in \widetilde{V}$ for all $V \in \mathcal{U}$ the sets $M \cup N$ with $M \in \mathcal{F}$ and $N \in \mathcal{G}$ induce a Cauchy filter included in $\mathcal{F}$ as well as in $\mathcal{G}$ and since $\mathcal{F}$ and $\mathcal{G}$ are **minimal** it follows that $\mathcal{F} = \mathcal{G}$.

The mapping $i : X \to \widetilde{X}$ with $i(x) := \mathcal{U}(x) \in \widetilde{X}$ is **well defined** on account of 14.1.1 and also **injectiv**e for $U(x) = U(y) \Rightarrow x = y$ due to Y being **separated**. i is **uniformly continuous** since for every $\widetilde{V} \in \widetilde{\mathcal{U}}$ there is a symmetric $W \in \mathcal{U}$ with $W^3 \subset V$ and for $(x;y) \in W$ the set $W(x) \cap W(y) \in \mathcal{U}(x) \cap \mathcal{U}(y)$ is small of order W^3, i.e. $(i(x);i(y)) \in \widetilde{V}$. i is **open** since for every open $O \subset X$ and $\mathcal{U}(x) \subset i\,[O]$ we have $x \in O$ and hence $U(x) \subset O$ for a $U \in \mathcal{U}$ such that $\widetilde{U}(\mathcal{U}(x)) \subset i\,[O]$, i.e. $i\,[O]$ open in $i\,[X]$. **The set $i\,[X]$ of all neighborhood filters is dense in the set $\widetilde{X}$ of all minimal Cauchy filter**s since for $\mathcal{F} \in \widetilde{X}$ and every neighborhood $\widetilde{V}(\mathcal{F})$ there is a w.l.o.g. **open** $U \in \mathcal{F}$ (cf. 14.4.2) small of order V, i.e. $\mathcal{U}(x) \in \widetilde{V}(\mathcal{F})$ for all $x \in U \in \mathcal{F}$. But that means $i\,[U] \subset \widetilde{V}(\mathcal{F})$ and since for every $\widetilde{V}(\mathcal{F}) \in \widetilde{\mathcal{U}}(\mathcal{F})$ there exists a $U \in \mathcal{F}$ small of order V we conclude that the image $i(\mathcal{F})$ of the minimal Cauchy filter $\mathcal{F}$ converges to $\mathcal{F} \in \widetilde{X}$.

$\widetilde{X}$ is **complete**: Due to $(i^2)^{-1}\left[\widetilde{V}\right] \subset V \forall V \in \mathcal{U}$ for every Cauchy filter $\widetilde{\mathcal{G}}$ on $\widetilde{X}$ the inverse image $i^{-1}\left(\widetilde{\mathcal{G}}\right)$ forms a basis for a Cauchy filter $\mathcal{F}'$on X including a minimal Cauchy filter $\mathcal{F}$. The uniformly continuous image $i(\mathcal{F})$ is again a Cauchy filter converging to the element $\mathcal{F} \in \widetilde{X}$ as shown above. On account of 14.4.3 we have $i^{-1}\left(\widetilde{\mathcal{G}}\right) \subset \mathcal{F} \Rightarrow \widetilde{\mathcal{G}} = i\left(i^{-1}\left(\widetilde{\mathcal{G}}\right)\right) \subset i(\mathcal{F})$ and due to $\mathcal{U}(\mathcal{F}) \subset i(\mathcal{F})$ the element $\mathcal{F}$ is an **accumulation point** of $\widetilde{\mathcal{G}}$; hence the Cauchy filter $\widetilde{\mathcal{G}}$ also converges to $\mathcal{F}$ and is in fact **identical** to $i(\mathcal{F})$.

For a uniformly continuous $f : X \to \widetilde{Y}$ into a second complete and separated space $\widetilde{Y}$ we may define $\widetilde{f}_0 : i(X) \to \widetilde{Y}$ by means of $\widetilde{f}_0(x) := \lim f(\mathcal{U}(x))$ on account of the completeness of $\widetilde{Y}$ and the uniform continuity of f. We have $f = \widetilde{f}_0 \circ i$ and $\widetilde{f}_o$ is uniformly continuous since for every neighborhood $\widetilde{U}$ in $\widetilde{Y}$ there is a symmetric neighborhood V in X with $(f^2)^{-1}\left[\widetilde{U}\right] \subset V$ such that for $(i(x);i(y)) \in \widetilde{V} \Rightarrow (x;y) \in V \Rightarrow (f(x);f(y)) = \left(\widetilde{f}_0(i(x));\widetilde{f}_0(i(y))\right) \in \widetilde{U}$. The mapping $\widetilde{f}_0$ is uniquely determined by $f = \widetilde{f}_0 \circ i$ and due to 14.6 it can be extended in an unique way to a uniformly continuous $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ with $f = \widetilde{f} \circ i$.

Finally let $\left(\widetilde{X}';i'\right)$ be a second pair satisfying the hypothesis. By applying the above proved extension to $i' := f : X \to \widetilde{Y} := \widetilde{X}'$ we get $\widetilde{i}' : \widetilde{X} \to \widetilde{X}'$with $i' = \widetilde{i}' \circ i$ on the one hand and on the other hand by applying it a second time to $i := f : X \to \widetilde{Y} := \widetilde{X}$ we obtain $\widetilde{i} : \widetilde{X}' \to \widetilde{X}$ with $i = \widetilde{i} \circ i'$. We can substitute to $\widetilde{i}' \circ \widetilde{i} = id_{\widetilde{X}'}$ as well as to $\widetilde{i} \circ \widetilde{i}' = id_{\widetilde{X}}$, i.e. $\widetilde{X}'$ is **homeomorphic** to $\widetilde{X}$.

14.8 Completion of metric spaces

Every **metric space** $(X;d)$ is homeomorphic to a **dense subset** of a **complete metric space** $\left(\widetilde{X};\widetilde{d}\right)$.

Proof: Let $\left(\widetilde{X};\widetilde{\mathcal{U}}\right)$ be the complete closure of the uniform space $(X;\mathcal{U})$ induced by the metric d according to 14.7. The image $i^2[X] := i[X] \times i[X]$ is dense in $\widetilde{X}^2$ and the **function** $d \circ \left(i^{-1}\right)^2 : i^2[X] \to \mathbb{R}$ is **uniformly continuous** since for $\epsilon > 0$ and a neighborhood base set $U_\epsilon = \{d < \epsilon\} \in \mathcal{U}$ (in this instance d is regarded as a **metric** on X!) resp. the corresponding neighborhood basis set $\widetilde{U}_\epsilon = \left\{(\mathcal{F};\mathcal{G}) \in \widetilde{X}^2 : \exists M \in \mathcal{F} \cap \mathcal{G} \wedge M^2 \subset U_\epsilon\right\} \in \widetilde{\mathcal{U}}$ we have $\left(d \circ \left(i^{-1}\right)^2\right)\left[\widetilde{U}_\epsilon\right] = \tilde{d}\left[\widetilde{U}_\epsilon \cap i^2[X]\right] < \epsilon$. According to 14.6 the **function** $d \circ \left(i^{-1}\right)^2$ can be extended to a uniformly continuous **function** $\tilde{d} : \widetilde{X}^2 \to \mathbb{R}$ with $\tilde{d}\left[\widetilde{U}_\epsilon\right] = \tilde{d}\left[\overline{\widetilde{U}_\epsilon \cap i^2[X]}\right] \subset \overline{\tilde{d}\left[\widetilde{U}_\epsilon \cap i^2[X]\right]} \leq \epsilon$ due to 3.5.1. We show that $\tilde{d}$ is **metric** on $\widetilde{X}$: As to the **positive definiteness** from the assumption $\tilde{d}(\mathcal{F};\mathcal{G}) < \epsilon \forall \epsilon > 0$ follows that $\mathcal{F} \cap \mathcal{G}$ is a basis for a Cauchy filter including both $\mathcal{F}$ and $\mathcal{G}$ which then must be equal due to 14.3. Concerning the **triangle inequality** let $\epsilon > 0$ and $U_{\epsilon/3}$ resp. $\widetilde{U}_{\epsilon/3}$ as above defined. For $\mathcal{F}, \mathcal{G}, \mathcal{H} \in \widetilde{X}$ there are $x, y, z \in X$ with $(\mathcal{U}(x);\mathcal{F})$, $(\mathcal{U}(y);\mathcal{G})$, $(\mathcal{U}(z);\mathcal{H}) \in \widetilde{U}_{\epsilon/3}$ since $i^2[X]$ is dense in $\widetilde{X}^2$ and $\tilde{d}\left[\widetilde{U}_{\epsilon/3}\right] \leq \frac{\epsilon}{3}$. On account of $\tilde{d}(\mathcal{U}(x);\mathcal{U}(y)) = d \circ \left(i^{-1}\right)^2(\mathcal{U}(x);\mathcal{U}(y)) = d(x;y)\ \forall x, y \in X$ follows $\tilde{d}(\mathcal{F};\mathcal{G}) + \tilde{d}(\mathcal{G};\mathcal{H}) - \tilde{d}(\mathcal{F};\mathcal{H}) > \tilde{d}(\mathcal{U}(x);\mathcal{U}(y)) + \tilde{d}(\mathcal{U}(y);\mathcal{U}(z)) - \tilde{d}(\mathcal{U}(x);\mathcal{U}(z)) + 2\epsilon = d(x;y) + d(y;z) - d(x;z) + 2\epsilon > 2\epsilon$ and this being true for every $\epsilon > 0$ proves the triangle inequality. The **symmetry** is obvious. Finally we must show that the neighborhood filter $\widetilde{\mathcal{U}}^{\tilde{d}}$ induced by the the metric $\tilde{d}$ is identical to $\widetilde{\mathcal{U}}$. In both cases we will use 12.5: The definition of $\tilde{d}$ already yielded $\tilde{d}\left[\widetilde{U}_\epsilon\right] \leq \epsilon \Leftrightarrow \widetilde{U}_\epsilon \subset \left\{\tilde{d} \leq \epsilon\right\} \Leftrightarrow \widetilde{\mathcal{U}}^{\tilde{d}} \subset \widetilde{\mathcal{U}}$. On the other hand with $\{d < \epsilon\} \cap i^2[X] < \epsilon = \left\{d \circ \left(i^{-1}\right)^2 < \epsilon\right\}$ dense in $\left\{\tilde{d} < \epsilon\right\}$ we obtain $\left\{\tilde{d} < \epsilon\right\} \subset \overline{\{d < \epsilon\} \cap i^2[X]}$ $= \overline{\{(U(x);U(y)) \in i^2(X) : (x;y) \in U_\epsilon\}} = \overline{\widetilde{U}_\epsilon \cap i^2[X]} = \overline{\widetilde{U}_\epsilon} \in \widetilde{\mathcal{U}}$, hence $\widetilde{\mathcal{U}} \subset \widetilde{\mathcal{U}}^{\tilde{d}}$.

14.9 Complete metric spaces

In a metric space $(X;d)$ the **diameter** of a set $A \subset X$ is defined as $\delta(A) := \sup\{d(x;y); x, y \in A\}$. Then the following statements are equivalent:

1. X is **complete**.
2. The **intersection** of a **decreasing** sequence $(A_n)_{n\geq 1}$ of nonempty **closed** sets $A_{n+1} \subset A_n \subset X$ for all $n \geq 1$ with $\inf\limits_{n\geq 1} \delta(A_n) = 0$ contains exactly **one point**: $\bigcap_{n\geq 1} A_n = \{x\} \subset X$.
3. Every **Cauchy sequence** converges.

Proof:

1. $\Rightarrow$ 2.: On account of $\inf \delta(A_n) = 0$ the A_n form the basis for a Cauchy filter converging to a single point owing to the hypothesis and 7.5.2.

2. $\Rightarrow$ 3.: The closures of the tails $A_n := \overline{\bigcup_{m\geq n}\{x_n\}}$ satisfy the conditions of 2. and the intersection $\bigcap_{n\geq 1} A_n = \{x\}$ contains the limit point $x = \lim (x_n)_{n\geq 1}$.

3. $\Rightarrow$ 1.: A Cauchy filter $\mathcal{F}$ includes a nonempty set $\emptyset \neq F_n \in \mathcal{F}$ with $\delta(F_n) < \frac{1}{n}$ for every $n \in \mathbb{N}$. Every sequence $(x_n)_{n\geq 1}$ with $x_n \in F_n$ is a Cauchy sequence and converges to a point $x = \lim (x_n)_{n\geq 1} \in \bigcap_{n\geq 1} F_n$ which is an accumulation point and hence resp. due to 14.1.2 and 7.5.2 the single limit point of the whole filter $\mathcal{F}$.

14.10 The dilation principle

A **continuous** function $f : X \to (Y;d_2)$ on a **complete** metric space $(X;d_1)$ into an arbitrary metric space $(Y;d_2)$ being **dilated** with $d_2(f(x);f(y)) \geq d_1(x;y)$ for all $x, y \in X$ is **closed**.

Proof: For a **closed** set $A \subset X$ and every $y \in \overline{f[A]}$ there is a sequence $(x_n)_{n\in\mathbb{N}} \subset A$ with $\lim_{n\to\infty} f(x_n) = y$. Owing to the hypothesis $(x_n)_{n\in\mathbb{N}}$ is a **Cauchy sequence** on the **complete** set A (cf.14.2.2) and hence converges to an $x \in A$. Since f is **continuous** we infer $y = f(x) \in f[A]$.

14.11 The contraction principle

For every **contracting** function $f : X \to X$ on a **complete** metric space $(X; d)$ with $d(f(x); f(y)) \le K \cdot d(x; y)$ for some $0 < K < 1$ and all $x, y \in X$ exists a uniquely determined **fixed point** $z = f(z) = \lim_{n\to\infty} f^n(x) \in X$ for every $x \in X$.

Proof: According to the hypothesis for $x \in X$ we have $d\left(f^{n+1}(x); f^n(x)\right) \le K^n \cdot d(f(x); x)$ whence $(f^n(x))_{n\ge 1}$ is a **Cauchy sequence** converging to a $z \in X$. Then for every $\epsilon > 0$ there is an $n \ge 1$ such that $d\left(f^{n+1}(x); f(z)\right) < d(f^n(x); z) \le \frac{\epsilon}{3}$, $d\left(f^{n+1}(x); f^n(x)\right) \le \frac{\epsilon}{3}$ and consequently $d(f(z); z) \le d\left(f(z); f^{n+1}(x)\right) + d\left(f^{n+1}(x); f^n(x)\right) + d(f^n(x); z) < \epsilon$. Hence $f(z) = z$ and for every other $u = f(u)$we have $d(u; z) = d(f(u); f(z)) < d(u; z) = 0$ whence $z = u$.

14.12 Isometric embeddings

A **continuous** map $f : X \to Y$ from a **complete metric** space $(X; d_X)$ into a **metric** space $(Y; d_Y)$ is an **isometric homeomorphism** iff its **restriction** $f|_{X_0}$ is an **isometry** from a **dense** subset $X_0 \subset X$ into a **dense** subset $f[X_0] \subset Y$.

Proof: Due to the **isometry** of f on the **dense** subset X_0 and the **continuity** of $(f; f) : X^2 \to Y^2$ resp. $d_X : X^2 \to \mathbb{R}^+$ and $d_Y : Y^2 \to \mathbb{R}^+$ for every $x; y \in X$ with approximating sequences $(x_n)_{n\ge 1} \subset X_0$ and $(y_n)_{n\ge 1} \subset X_0$ such that $\lim_{n\to\infty} x_n = x$ resp. $\lim_{n\to\infty} y_n = y$ we have $d_Y(f(x); f(y)) = d_Y\left(\lim_{n\to\infty} f(x_n); \lim_{n\to\infty} f(y_n)\right) = d_Y\left(\lim_{n\to\infty}(f(x_n); f(y_n))\right) = \lim_{n\to\infty} d_Y(f(x_n); f(y_n)) = \lim_{n\to\infty} d_X(x_n; y_n) = d_X\left(\lim_{n\to\infty} x_n; \lim_{n\to\infty} y_n\right) = d_X(x; y)$. Hence f is an **isometry** on X. Since $f[X_0]$ is **dense** in Y for every $y \in Y$ there is a sequence $(x_n)_{n\ge 1} \subset X_0$ with $\lim_{n\to\infty} f(x_n) = y$. Since $(f(x_n))_{n\ge 1}$ is **Cauchy** in Y and f is an **isometry** $(x_n)_{n\ge 1}$ is Cauchy in X whence exists a limit $\lim_{n\to\infty} x_n = x \in X$ since X is **complete**. From the **continuity** of f follows $\lim_{n\to\infty} f(x_n) = f(x) = y$ which shows that f is **surjective**.

14.13 The supremum property

Every **bounded** set $(x_i)_{i\in I}$ in a **linearly ordered complete metric space** X has a **supremum** $\sup_{i\in I} x_i$.

Proof: Due to the hypothesis and the **well-ordering** of the **natural numbers** for every $n \in \mathbb{N}$ there is a **Minimum** p_n of the set of all natural numbers $p \in \mathbb{N}$ such that $\frac{p}{2^n}$ is an upper bound of $(x_i)_{i\in I}$. The upper bounds $\left(\frac{p_n}{2^n}\right)_{n\in\mathbb{N}}$ form a **decreasing Cauchy sequence** with **limit** $x \in X$ which clearly is the minimum of all upper bounds, i.e. $x = \sup_{i\in I} x_i$.

15 Polish spaces

15.1 Definitions

A topological space X is **polish** iff it is **completely metrizable** and **second countable**.

1. On account of 14.2 these properties extend to **closed subsets** and **countable products**.

2. **Open subsets** $O \subset X$ are also polish: If we pair all points $x \in O$ with the inverse $\frac{1}{d(x;X\setminus O)}$ of their distance to the boundary we obtain the set $f^{-1}\{1\} = \left\{(t;x) : d(x;X\setminus O) = \frac{1}{t}\right\} \subset \mathbb{R}\times X$ which as the reverse image of the closed set $\{1\}$ under the continuous mapping $f : \mathbb{R}\times X \to \mathbb{R}$ with $f(t;x) = t \cdot d(x;X\setminus O)$ is closed and hence polish. Due to the uniqueness off the distance $d(x;X\setminus O)$ of a given point x the **projection** $\pi_2 : \mathbb{R}\times X \to X$ is **injective** on the subset $f^{-1}\{1\} \subset \mathbb{R}\times X$ and hence a **homeomorphism** due to 4.2. Thus the image $O = \pi_2\left[f^{-1}\{1\}\right] \subset X$ is **polish**. Note that π_2 is **continuous** and **open** on the set $\mathbb{R}\times X$ but obviously **not closed** since it is not injective for arbitrary pairs (t,x).
3. The most important representants of polish spaces besides $\mathbb{C}^n$ are the family $\mathbb{C}^{\mathbb{N}}$ of all **complex valued sequences** and the **continuous complex valued functions** $C(I,\mathbb{C}) \subset \mathbb{C}^I$ (cf. remark 18.12) on a **compact** interval $I \subset \mathbb{R}$ resp. the subspace of **sequences** $(X_n(\omega))_{n\in\mathbb{N}}$ of **random variables** $X_n : \Omega \to \mathbb{C}$ resp. **families** $(X_t(\omega))_{t\in I}$ of **continuous** random variables occurring as realisations of **stochastic processes** for a given event $\omega \in \Omega$. By means of the **Skorokhod metric** ([1, ch. 3, p. 121]) the **càdlàg** (vgl. 3.3) **real valued** functions $D(I,\mathbb{R})$ are provided with a topology of a **polish space** coinciding with the **trace** of the **product topology** on $C(I,\mathbb{C}) \subset D(I,\mathbb{C}) \subset \mathbb{C}^I$. Due to [20, th. 6.7 and 6.8] the spaces L^p **of real valued p-integrable** functions are **polish** too.

15.2 Locally compact polish spaces

Every **locally compact** and **second countable** space X is **polish**.

Proof: Owing to 10.8 the space X is **σ-compact** whence the **Alexandrov compactification** 10.2 $\overline{X}$ is **second countable** and due to **Urysohns metrization theorem** 11.14.2 **metrizable**. According to 10.13, 10.10 and 14.1.2 the **compact metric** space $\overline{X}$ is **complete** and consequently **polish**. Since $X \subset \overline{X}$ is an **open subspace** the assertion follows from the preceding section 15.1.2.

Note: The three **equivalent metrics** d,d' and d'' defined by $d(x;y) := |x-y|$, $d' = \frac{d}{1+d}$ and $d''(x;y) = \left|\frac{x}{1+|x|} - \frac{y}{1+|y|}\right|$ for $x;y \in \mathbb{R}$ illustrate the fact mentioned in [12, p. 21 and ex. 12 on p. 39] that different metrics **may induce the same topology with ot without completeness** depending on their **translation invariance**: d and d' are translation invariant and produce the same complete topology on $\mathbb{R}$; d' by $d'(x;\infty) = 1$ for every $x \in \mathbb{R}$ also covers the **Alexandrov compactification** $\overline{\mathbb{R}} = \mathbb{R}\cup\{\infty\}$. The metric d'' generates the same topology on $\overline{\mathbb{R}} = \mathbb{R}\cup\{\infty\}$ with $d''(x;\infty) = 1$ for every $x \in \mathbb{R}$ but as the d''-Cauchy sequences $(n)_{n\in\mathbb{N}} \to \infty \in \overline{\mathbb{R}}$ and $(-n)_{n\in\mathbb{N}} \to -\infty \notin \overline{\mathbb{R}}$ show neither space is complete any more. For $(x;y)$ close to the origin d' and d'' behave similarly to d, i.e. they generates the same neighborhood basis; with increasing distance from the origin d' and d'' tend to 1 so that $\infty \in B''_\epsilon(x)$ for $|x| > \frac{1}{\epsilon} - 1$. The distances $d'(0;n)$ and $d''(0;n)$ both tend to 1 as $n \to \infty$ but due to the translation invariance $d'(n;n+1) = \frac{1}{2}$ stays constant while $d''(n;n+1) \to 0$ whence $(n)_{n\in\mathbb{N}}$ is d''-Cauchy but not d'-Cauchy.

15.3 Mazurkiewicz' theorem

A **subspace** of a **polish** space is **polish** iff it is G_δ.

Proof:

$\Rightarrow$: For a polish subspace $A \subset X$ and $n \in \mathbb{N}^*$ let $A_n = \left\{x \in \overline{A} : \exists U \in \mathcal{U}(x) : \delta_A(U\cap A) < \frac{1}{n}\right\}$. Obviously we have $A \subset \bigcap_{n\in\mathbb{N}^*} A_n \subset \overline{A}$ and for an $x \in \bigcap_{n\in\mathbb{N}^*} A_n$ the family $\mathcal{U}(x)\cap A$ generates a Cauchy filter on A converging to a unique $x' \in A$ with $x' = x$ and consequently $A = \bigcap_{n\in\mathbb{N}^*} A_n$. Every A_n is open in $\overline{A}$ and $U_n = \bigcup\left\{U : \exists x \in \overline{A} : U \in \mathcal{U}(x) : \delta_A(U\cap A) < \frac{1}{n}\right\}$ is open in X with $U_n \cap \overline{A} = A_n$. The neighborhoods $V_n = \left\{x \in X : d\left(x;\overline{A}\right) < \frac{1}{n}\right\}$ of $\overline{A}$ are open in X with $\overline{A} = \bigcap_{n\in\mathbb{N}^*} V_n$. Hence $A = \bigcap_{n\in\mathbb{N}^*}(U_n \cap V_n)$ provides the desired representation as a G_δ-set.

$\Leftarrow$: Let $A = \bigcap_{n\in\mathbb{N}} O_n$ with open $O_n \subset X$ which are already **polish** according to 15.1. The function $f : \bigcap_{n\in\mathbb{N}} O_n \to \Delta \subset \prod_{n\in\mathbb{N}} O_n$ with $f(x) = (x)_{n\in\mathbb{N}}$ is **bijective** and with its components$p_n \circ f = \mathrm{id}$ being **continuous** as well as **open** these additional properties transfer to f. Thus $f : A \to \Delta$ is a **homeomorphism** on the diagonal Δ which is closed due to 7.5.4 and therefore a polish subset of the polish space $\prod_{n\in\mathbb{N}} O_n$ according to 15.1. Hence A is polish..

15.4 Homeomorphism to a G_δ-set in the Hilbert cube

A topological space X is **polish** iff it is **homeomorphic** to a G_δ-set in the **Hilbert cube** $\mathbb{H} := [0,1]^{\mathbb{N}}$.

Proof:

$\Rightarrow$: Follows directly from 7.2, **Urysohn's metrization theorem** 11.11 and **Mazurkiewicz** 15.3. The following alternative proof provides the **construction** of a concrete **embedding**: Assume (X, d) polish with a dense subset $(z_n)_{n\in\mathbb{N}} \subset X$ and $(\mathbb{H}, d')$ with the product metric $d'(x; y) := \sum_{n\in\mathbb{N}} \frac{|x_n - y_n|}{2^{n+1}}$ for $x = (x_n)_{n\in\mathbb{N}}, y = (y_n)_{n\in\mathbb{N}} \in \mathbb{H}$ according to 1.8. Due to 1.6 we can assume $d < 1$ and define $f : X \to [0,1]^{\mathbb{N}}$ by means of $f(x) = (d(x; z_n))_{n\in\mathbb{N}}$. On account of $d(f(x); f(y)) = \sum_{n\in\mathbb{N}} \frac{|d(x;z_n) - d(y;z_n)|}{2^{n+1}}$ $\leq \sum_{n\in\mathbb{N}} \frac{|d(x;y)|}{2^{n+1}} = \frac{1}{2} d(x; y)$ the function f **is continuous.** f is **open** and especially **injective** since for every $x \in X$ there is an $m \in \mathbb{N}$ with $d(x; z_m) < \epsilon$ and for $d(f(x); f(y)) < \frac{\epsilon}{2^{m+1}}$ we have $|d(x; z_m) - d(y; z_m)| < \epsilon$ hence $d(y; z_m) < 2\epsilon$ and therefore $d(x; y) \leq d(x; z_m) + d(y; z_m) \leq 3\epsilon$.

$\Leftarrow$: Consider again **Mazurkiewicz** 15.3 resp.15.1.

15.5 The Baire space $\mathcal{N}$

The **Baire space** $\mathcal{N} := \mathbb{N}^{\mathbb{N}}$ is the set of all sequences of natural numbers furnished with the product of the **discrete topologies** induced by the **basis** $\mathcal{B} = \left\{U_s : s \in \mathbb{N}^{<\mathbb{N}}\right\}$ with $\mathbb{N}^{<\mathbb{N}} = \bigcup_{n\in\mathbb{N}} \mathbb{N}^n$ and $U_s := \{x \in \mathcal{N} : s \subset x\}$. Since a countable union of countable sets is again countable (cf. [19, th 17.6]) this construction already yields the **2nd axiom of countability**. $\mathcal{N}$ is **regular** (cf. 7.10) and therefore **metrizable** as well as **separable** (cf. 11.14.1). A **(Ultra)metric** is given by means of $d(x, y) = \frac{1}{2^{n(x,y)}}$ with $n(x, y) = \min\{n \in \mathbb{N} : x_n \neq y_n\}$ whereas $D = \{x \in \mathcal{N} : \exists m \in \mathbb{N} : x_n = 0 \forall n \geq m\}$ provides a **countable** and **dense** subset. $\mathcal{N}$ is **complete** since a given Cauchy family $(x_n)_{n\in\mathbb{N}} \subset \mathcal{N}$ for every $m \in \mathbb{N}$ must possess a finite subsequence $s_m \in \mathbb{N}^m$ with $n_m \in \mathbb{N}$ such that $x_n \in U_{s_m} \forall n \geq n_m$ and hence converges to the uniquely determined limit $x := \bigcup \{s_m : m \in \mathbb{N}\}$. Thus $\mathcal{N}$ is **polish** and according to 11.12 resp. 14.2.1 resp. the above mentioned argument concerning the second axiom of countability these properties transfer to **countable products** $\mathcal{N}^{\mathbb{N}}$. Every $x \in \mathcal{N}$ has a **countable neighborhood basis** constituted by the sets $U_n(x) := \{y \in \mathcal{N} : y|_n \subset x\}$ with $x|_n := \{x_0; ...; x_{n-1}\}$ for $n \in \mathbb{N}$. Their **complements** $\mathcal{N} \setminus U_n(x) = \{y \in \mathcal{N} : \exists k \in \mathbb{N} : y_k \neq x_k\}$ include the neighborhood $U_{\{y_k\}} \in \mathcal{U}(y)$ of every $y \in \mathcal{N} \setminus U_n(x)$ and hence are **open** too, i.e. $\mathcal{N}$ is **totally disconnected** (cf. 5.1). A mapping $f : \mathcal{N} \to \mathcal{N}$ is **continuous** iff for every $n \in \mathbb{N}$ there is an $m_n \in \mathbb{N}$ such that the images $f(y)$ of all sequences $y \in \mathcal{N}$ coinciding on the **first** m coordinates with x on the **first** n coordinates coincide with $f(x)$. This pleasantly computational space is extremly useful for the study of the function spaces mentioned in 15.1 since in 15.8 we will show that every polish space is the **continuous** image of a **closed subset** of $\mathcal{N}$. To this end we need the following definitions and two lemmas being of interest in their own right:

15.6 Trees and paths

A **tree** is a family $T \subset \mathbb{N}^{<\mathbb{N}}$ with $t \subset s \Rightarrow t \in T$ for all $s \in T$ and a **path through T** is a sequence $x \in \mathcal{N}$ with $x|_n \in T \forall n \in \mathbb{N}$. Then $[T] \subset \mathcal{N}$ is the set of all paths trough T and for $s = \{x_0; ...; x_n\} \in \mathbb{N}^{<\mathbb{N}}$ resp. $k \in \mathbb{N}$ we abbreviate $s\hat{}k := \{x_0; ...; x_n; k\}$. A tree is **pruned** iff for every $s \in T$ there is a $k \in \mathbb{N}$ with $s\hat{}k \in T$ resp. iff for every $s \in T$ there is an $x \in [T]$ with $s \subset x$. Hence in a figurative sense a pruned tree doesn't have dead ends. For a tree $T \subset \mathbb{N}^{<\mathbb{N}}$ the subtree

$T' := \{s \in T : \exists x \in [T] : s \subset x\}$ is a **pruned tree** with $T' \subset T$ and $[T'] = [T]$. For a closed set $F \subset \mathcal{N}$ every $x \in X \setminus F$ possesses a finite subsequence $s \subset x$ with $U_s \subset X \setminus F$ resp. for $x \in F$ and every $s \subset x$ there is a $y \in F$ with $s \subset y$, i.e. $T_F := \left\{s \in \mathbb{N}^{<\mathbb{N}} : \exists x \in F : s \subset x\right\}$ is a pruned tree with $F = [T_F]$. Thus we have the following lemma:

15.7 Characterization of open and closed sets

1. $U \subset \mathcal{N}$ is **open** iff there is a family $S \subset \mathbb{N}^{<\mathbb{N}}$ of **finite** sequences with $U = \bigcup_{s \in S} U_s$.
2. $F \subset \mathcal{N}$ is **closed** iff there is a **pruned tree** $T \subset \mathbb{N}^{<\mathbb{N}}$ with $F = [T]$.

15.8 Coverings

In a polish space $(X; d)$ for every $\epsilon > 0$ and every

1. **open** set $O \subset X$ there is a cover $\mathcal{O} := (O_n)_{n \in \mathbb{N}}$ of open sets with **diameter** $\delta\left(\overline{O_n}\right) < \epsilon$ and $O = \bigcup_{n \in \mathbb{N}} O_n = \bigcup_{n \in \mathbb{N}} \overline{O}_n$.
2. F_σ-set $A \subset X$ there is a cover$\mathcal{A} := (A_n)_{n \in \mathbb{N}}$ of **disjoint** F_σ-sets with **diameter** $\delta\left(\overline{A_n}\right) < \epsilon$ and $A = \bigcup_{n \in \mathbb{N}} A_n = \bigcup_{n \in \mathbb{N}} \overline{A}_n$.

Particularly in a polish space every **open** set is F_σ and every **closed** set is G_δ**.**

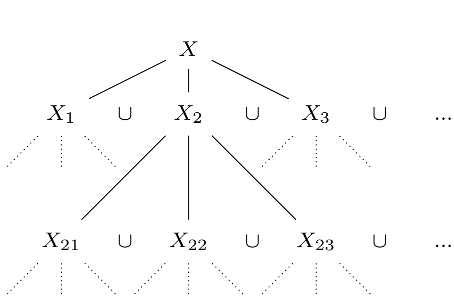
Proof:

1. Choose $\mathcal{O} = \left\{B_{\frac{1}{n}}(y) : y \in Y \cap O \wedge \overline{B}_{\frac{1}{n}}(y) \subset O \wedge \frac{1}{n} < \frac{\epsilon}{2}\right\}$ with a **countable dense subset** $Y \subset X$. Then for each $x \in O$ there is an $n \in \mathbb{N}$ with $\frac{1}{n} < \epsilon$ and $B_{\frac{1}{n}}(x) \subset O$ as well as a $y \in Y$ with $x \in B_{\frac{1}{3n}}(y) \subset \overline{B}_{\frac{1}{3n}}(y) \subset B_{\frac{1}{n}}(x) \subset O$, i.e. $B_{\frac{1}{3n}}(y) \in \mathcal{O}$. Concerning the diameter we note that in metric spaces $\delta\left(\overline{A}\right) = \delta(A)$.
2. For $A = \bigcup_{n \in \mathbb{N}} C_n$ with closed C_n we obtain closed disjoint $B_n := C_n \setminus \bigcup_{0 \le k < n} C_k$ with $A = \bigcup_{n \in \mathbb{N}} B_n$. According to 1. for every $n \in \mathbb{N}$ we can find open sets $O_{n,m}$ with $\delta(O_{n,m}) < \epsilon$ and $X \setminus \bigcup_{0 \le k < n} C_k = \bigcup_{m \in \mathbb{N}} O_{n,m} = \bigcup_{m \in \mathbb{N}} \overline{O_{n,m}}$. Hence the sets $A_{n,m} := C_n \cap O_{n,m} \setminus \bigcup_{0 \le i < m} O_{n,i}$ are **disjoint** with $\delta(A_{n,m}) < \epsilon$ and $B_n = \bigcup_{m \in \mathbb{N}} A_{n,m} = \bigcup_{m \in \mathbb{N}} \overline{A}_{n,m}$. The $A_{n,m}$ are F_σ since they are formed by intersections of the closed sets $C_n \cap X \setminus \bigcup_{0 \le i < m} O_{n,i}$ with the open sets $O_{n,m}$ which are F_σ according to 1.

15.9 Characterization of polish spaces as closed subsets of the Baire space

Every **polish** space X is the image of a **closed** set $F \subset \mathcal{N}$ under a **uniformly continuous bijection** $\Phi : F \to X$.

Proof: According to 15.8 there is a family $\left\{X_\sigma \subset X : \sigma \in \mathbb{N}^{<\mathbb{N}}\right\}$ of F_σ-sets on X with the following properties:



1. $X_\emptyset = X$
2. $X_\sigma = \bigcup_{n \in \mathbb{N}} X_{\sigma\hat{}n} = \bigcup_{n \in \mathbb{N}} \overline{X_{\sigma\hat{}n}}$
3. $\delta\left(\overline{X_\sigma}\right) < \frac{1}{|\sigma|}$
4. $\sigma \subset \tau \Rightarrow \overline{X_\tau} \subset X_\sigma$
5. $n \neq m \Rightarrow X_{\sigma\hat{}n} \cap X_{\sigma\hat{}m} = \emptyset$

Due to 14.9.3 the set $F = \left\{ f \in \mathcal{N} : \exists x \in X : x \in \bigcap_{n\in\mathbb{N}} \overline{X_{f|_n}} \right\}$ is not empty and $\Phi : F \to X$ is well defined by means of $\Phi(f) \in \bigcap_{n\in\mathbb{N}} \overline{X_{f|_n}}$. On account of 5. the mapping Φ is **injective**. It is **surjective** since for every $x \in X$ and every $\sigma \in \mathbb{N}^{<\mathbb{N}}$ with $x \in X_\sigma$ due to 2. and 5. there is a unique $n \in \mathbb{N}$ with $x \in X_{\sigma\hat{}n}$ so that we find an increasing sequence $\emptyset = \sigma_0 \subset \sigma_1 \subset ...$ of finite sequences $\sigma_n \in \mathbb{N}^n$ and a corresponding decreasing sequence $X = X_\emptyset \supset X_{\sigma_1} \supset ...$ of F_σ-sets with $x \in \bigcap_{n\in\mathbb{N}} X_{\sigma_n}$. The mapping Φ is **uniformly continuous** since for $f, g \in F$ with $d(f; g) < \frac{1}{2^n}$ we have $f|_n = g|_n$ and hence $d(\Phi(f); \Phi(g)) < \frac{1}{n}$ on due to 3. and 4. According to 3.1 the set $F = \Phi^{-1}[X]$ is **open** and **closed**. For a better understanding of the mapping Φ it is useful to demonstrate the closed character of F **directly**: Let $(f_n)_{n\in\mathbb{N}} \subset F$ be a **Cauchy** sequence converging to a $f \in \mathcal{N}$, i.e., $\forall n \in \mathbb{N} \exists m \in \mathbb{N} : f_k|_n = f|_n \forall k \geq m$. According to 14.1.3 $\Phi(f_n)_{n\in\mathbb{N}} \subset X$ is again a Cauchy sequence converging to a $x \in X$, i.e. $\forall n \in \mathbb{N}\ \exists m \in \mathbb{N} : d(x; \Phi(f_k)) < \frac{1}{n}\ \forall k \geq m \Leftrightarrow \forall n \in \mathbb{N}\ \exists m \in \mathbb{N} : x \in X_{f_k|_n} = X_{f|_n}\ \forall k \geq m \Rightarrow x \in \bigcap_{n\in\mathbb{N}} X_{f|_n} \Rightarrow x = \Phi(f) \Rightarrow f \in F$ whence F is **closed** and also open due to 15.5**.**

16 Baire spaces

16.1 Baire Categories

A set $A \subset X$ is **meager** or of **first category** in the space X iff it is a countable union $A = \bigcup_{n\in\mathbb{N}} A_n$ of **nowhere dense** sets A_n resp. iff its **complement** $X \setminus A = \bigcap_{n\in\mathbb{N}} X \setminus A_n$ is the **countable intersection of dense sets** $X \setminus A_n$ and **not meager** resp. of **second category** otherwise.

1. If a subset $A \subset B \subset X$ is of **first category** in a **set** $B \subset X$ then it is also of **first category** in X since for a $\overline{A_n}$ not nowhere dense in X there is an x and a neighborhood U with $x \in U \subset \overline{A_n}$ such that x cannot be a boundary point and especially $x \in U \cap A_n \neq \emptyset$. Since $A_n \subset B$ it follows that $x \in U \cap B \subset \overline{A_n} \cap B$ hence $\overline{A_n}$ is not nowhere dense in B.

2. The **first category** of any set obviously extends to **subsets**, **finite intersections** und **countable unions**.

3. The **rational numbers** $\mathbb{Q}$ are of **first category** in $\mathbb{R}$ and likewise the **Cantor set** T in $[0; 1]$ (cf. 2.9). Both have the **Lebesgue measure** $\lambda(\mathbb{Q}) = \lambda(T) = 0$. On the other hand there are sets $U \subset \mathbb{R}$ of first category with Lebesgue measure $\lambda(U) = 1$. (cf. [19, p. 3.7])

4. For a **measure** μ being **bounded from above**, i.e. $\mu(A) < M \forall A \in \mathcal{A}$ the space L^q is of **first category** in L^p for $p < q$: According to [19, p. 6.6.1]) in this case we have $\|f\|_p \leq \|f\|_q \cdot M^{\frac{1}{p}-\frac{1}{q}}$. The closures $\overline{B_n^q}$ (with reference to $\|\|_p$) of the balls $B_n^q = \left\{ f \in L^q : \|f\|_q \leq n \right\} \subset L^q \subset L^p$ have no **interior points** with reference to $\|\|_p$ since for every $m \geq 2n$ there is an $h = n^\alpha|_{[0;m^{-p}n^{-\alpha p}]}$ with $\alpha = \frac{q+p}{q-p} \cdot \frac{\ln(m)}{\ln(n)}$ such that on the one hand $\|h\|_q = (n^{\alpha q} \cdot m^{-p} \cdot n^{-\alpha p})^{\frac{1}{q}} = \left(n^{\alpha(q-p)} m^{-p} \right)^{\frac{1}{q}}$ $= \left(m^{q+p} \cdot m^{-p} \right)^{\frac{1}{q}} = m > 2n$ and on the other hand $\|h\|_p = (n^{\alpha p} \cdot m^{-p} \cdot n^{-\alpha p})^{\frac{1}{p}} = \frac{1}{m}$. Hence for every $f \in \overline{B_n^q}$ in every neighborhood $B^p_{\frac{1}{2m}}(f)$ there is a $g \in B_n^q \cap B^p_{\frac{1}{2m}}(f)$ and a further $g + h \in B^p_{\frac{1}{2m}}(g) \subset B^p_{\frac{1}{m}}(f)$ with $g + h \notin B_n^q$ so that f cannot be an interior point of $\overline{B_n^q}$. Thus $\mathrm{L}^q = \bigcup_{n\geq 1} B_n^q$ is of first category in L^p.

16.2 Baire spaces

A topological space X is a **Baire space** iff it satisfies one of the following equivalent conditions:

1. Every set of **first category** is **nowhere dense**.

2. A **countable intersection** of **open** and **dense** sets ist again **dense**.

3. A **countable intersection** of dense G_δ-sets is again dense and G_δ.

4. Every nonempty **open** subset is of **second category**.
5. The **complement** of every set of **first category** is **dense**.

Proof:

1. $\Rightarrow$ 2. : Consider the complements.

2. $\Rightarrow$ 3. : If the intersection $A_n = \bigcap_{m\in\mathbb{N}} U_{m,n}$ is dense the $U_{m,n}$ must be dense too.

3. $\Rightarrow$ 4. : For a nonempty, open and meager $O = \bigcup_{n\in\mathbb{N}} A_n \neq \emptyset$ the open sets $X \setminus \overline{A}_n$ are dense and due to the hypothesis the intersection $\bigcap_{n\in\mathbb{N}} \left(X \setminus \overline{A}_n\right) = X \setminus \bigcup_{n\in\mathbb{N}} \overline{A}_n \subset X \setminus \bigcup_{n\in\mathbb{N}} A_n = X \setminus O$ is dense too. Since $X \setminus O$ is closed $X \setminus O = X$ contrary to the assumption.

4. $\Rightarrow$ 5. : Assuming the complement $X \setminus A$ of the meager set A is not dense the open set $\mathring{A} \subset A$ would be nonempty and due to 16.1.2 be of first category contrary to 4. again.

5. $\Rightarrow$ 1. : For a meager $A = \bigcup_{n\in\mathbb{N}} \overline{A}_n$ the set $X \setminus \overline{A} = \emptyset$ and hence A is nowhere dense.

16.3 Baire's category theorem

For a nonempty **Baire** space X the following statements hold:

1. Every **countable closed cover** $(A_n)_{n\in\mathbb{N}}$ contains at least one A_n with $\mathring{A}_n \neq \emptyset$.
2. Every nonempty **open** subset $O \subset X$ is **Baire**.
3. The **complement** of every set of **first category** is **Baire** and hence of **second category**.

Proof:

1. Follows directly from 16.2.1.
2. Assuming there is a nonempty open subset $\emptyset \neq U = \bigcup_{n\in\mathbb{N}} A_n \subset O$ of first category in O with nowhere dense A_n in O. Since U is nonempty and open in X and X is Baire one of the A_n must contain a neighborhood $V \in \mathcal{U}(x)$ in X of an interior point $x \in \mathring{\overline{A}}_n \subset A_n$ of the closure $\overline{A}_n$ in X. But then $V \cap O$ is a nonempty neighborhood in O and x would be an interior point of the closure $\overline{A}_n \cap O$ in O contrary to the assumption.
3. According to 16.2.5 the complement $X \setminus A$ of a set $A \subset X$ of first category in a Baire space X is dense in X. Due to 16.1.1 a subset $B \subset X \setminus A$ being of first category in $X \setminus A$ is also of first category in X. From 16.1.2 we infer that $A \cup B$ is of first category in X such that the complement $(X \setminus A) \setminus B = X \setminus (A \cup B)$ is dense in X and especially in $X \setminus A$.

16.4 Baire's theorem

A topological space X is **Baire** if it is

1. **completely metrizable** or
2. **locally compact**.

Proof:

1. For a countable intersection $O = \bigcap_{n\in\mathbb{N}} O_n \neq \emptyset$ of w.l.o.g. **decreasing open** dense sets $(O_n)_{n\in\mathbb{N}}$ and an arbitrary **open** $U \subset X$ there is a a **decreasing** sequence $(B_n)_{n\in\mathbb{N}}$ with $\delta(B_n) < \frac{1}{n+1}$ und $B_{n+1} \subset \overline{B}_n \subset O_n \cap U$. Due to 14.9.2 we have $\emptyset \neq \bigcap_{n\in\mathbb{N}} \overline{B}_n \subset U \cap O$ and the statement follows from 16.2.2.
2. On account of 10.6 and 9.4 the $\overline{B}_n$ from above may be chosen as **closed** and **compact** sets such that the proposition follows from 9.2.2.

16.5 Banach's category theorem

In any topological space the **union of open sets of first catogory** is again of **first category**.

Proof: On account of 16.2.4 the topological space is **not a Baire** space. Let G be the union of a family $\mathcal{G}$ of nonempty open sets of first category. According to **Zorn's lemma** (cf.[19, p. 14.2.4]) there is a maximal family $\mathcal{F} := (U_\alpha)_{\alpha\in A}$ of **disjoint, nonempty** and **open** sets U_α such that each of them is included in a set from $\mathcal{G}$ and that any set of any other such family is included in a set of $\mathcal{F}$. Then the closed set $\overline{G} \setminus \bigcup_{\alpha\in A} U_\alpha$ is nowhere dense since any interior point could be added to an U_α contrary to the maximality of $\mathcal{G}$. For $U_\alpha = \bigcup_{n\in\mathbb{N}} N_{\alpha,n}$ with nowhere dense $N_{\alpha,n}$ let $N_n := \bigcup_{\alpha\in A} N_{\alpha,n}$. Any open set U intersecting N_n particularly meets a $N_{\alpha,n}$ such that $\emptyset \neq (U \cap U_\alpha)\setminus N_{\alpha,n} \subset U \setminus N_n$. Hence N_n does not include any open set and consequently is nowhere dense and the set $G \subset \left(\overline{G} \setminus \bigcup_{\alpha\in A} U_\alpha\right) \cup \bigcup_{\alpha\in A} U_\alpha = \left(\overline{G} \setminus \bigcup_{\alpha\in A} U_\alpha\right) \cup \bigcup_{n\in\mathbb{N}} N_n$ is of first category.

16.6 Decomposition into sets of first and second category

Every topological space con be decomposed into an **open** set G of **first category** and a **closed Baire** set $X \setminus G$.

Proof: The system of all families $\mathcal{G}$ of nonempty open sets of first category is inductively ordered with regard to inclusion such that on account of **Zorn's lemma** (cf. [19, p. 14.2.4]) there is a maximal family with open union G of first category owing to the preceding theorem16.5. On account of the **maximal character** of G and 16.2.4 the **closed complement** $X \setminus G$ is **Baire**.

16.7 Examples

$\mathbb{Q}$ is dense and if first category in $\mathbb{R}$ and $\mathbb{R}$ is not dense but still of first category in $\mathbb{C}$. $\mathbb{Q}$ is not Baire since $(r)_{r\in\mathbb{Q}}$ is a countable closed cover without interior points; $\mathbb{R}^n$ is Baire owing to 16.4.2. The **category theorem** 16.3 and **Baire's theorem** 16.4 in suitable situations may reduce the proof of a proposition for a Baire space X to the case of a single element A_n of a closed cover $(A_n)_{n\in\mathbb{N}}$:

16.8 Continuous functions on Baire spaces

For a family $(f_i)_{i\in I}$ of **real valued, continuous** and **pointwise bounded** functions on a **Baire space** X for every **open** $U \subset X$ there is a nonempty and **open** set $V \subset U$ so that the f_i are **uniformly bounded** on V.

Proof: On account of 16.3.2 the set U is **Baire** and with $A_n := \{\sup_{i\in I} f_i \leq n\}$ we obtain a closed cover $(A_n)_{n\in\mathbb{N}}$ containing at least one A_n with nonempty interior V due to 16.3.1.

Note: The statement of 16.8 can be extended to the whole space X in the case of X being **compact** and **convex** (cf. [16, th. 4.3]). Further applications of the Baire theorem are the **Banach-Steinhaus theorem** [16, th. 4.1], the **open mapping theorem** [16, th. 4.4] and the **closed graph theorem** [16, th. 4.6] concerning **continuity** and **open character** of **linear** mappings between **Fréchet** spaces.

17 Compactification

17.1 Precompact spaces

A uniform space is **precompact** iff for every neighborhood V on X there is a finite cover of X with sets small of order V. A subset $A \subset X$ is precompact iff the uniform subspace A is precompact. **Metric** precompact spaces are denoted as **totally bounded**.

17.2 Separated precompact spaces

For a **separated** space X the following statements are equivalent:

1. X is **precompact**.
2. The **complete closure** $\widetilde{X}$ is **compact**.
3. Every **ultrafilter** on X is a **Cauchy filter**.

Proof:

1. $\Rightarrow$ 2. : For an ultrafilter $\widetilde{\mathcal{F}}$ and a w.l.o.g. (cf. 12.5) closed neighborhood $\widetilde{V}$ on the complete closure $\overline{i(X)} = \widetilde{X}$ (cf. 14.7) there is a finite cover $X = \bigcup_{1\leq k\leq n} A_k$ with $A_k^2 \subset \left(i^{-1}\right)^2\left[\widetilde{V}\right]$. Hence we have $i(X) = \bigcup_{1\leq k\leq n} i(A_k)$ with $i^2(A_k) \subset \widetilde{V}$ and consequently $\widetilde{X} = \bigcup\limits_{k=1}^{n} \overline{i(A_k)}$ with $\overline{i(A_k)} \times \overline{i(A_k)} \subset \widetilde{V}$ since $\widetilde{V}$ is closed. Because of $\bigcap_{1\leq k\leq n} X \setminus A_k = \emptyset$ the ultrafilter $\widetilde{\mathcal{F}}$ must contain one of the $\overline{i(A_k)}$ and since this is true for every $\widetilde{V}$ the filter $\widetilde{\mathcal{F}}$ is Cauchy converging on the complete space $\widetilde{X}$ which must be compact due to 9.2.4.

2. $\Rightarrow$ 3. : Due to 6.6 the continuous image $i(\mathcal{F})$ of an ultrafilter $\mathcal{F}$ on X is again an ultrafilter on $\widetilde{X}$ converging on account of 9.2.4. Hence for every $V \subset X^2$ the image $i(\mathcal{F})$ contains a filter set $\widetilde{F} \in \widetilde{\mathcal{F}}$ with $\widetilde{F}^2 \subset i^2(V)$ and agin owing to 6.6 there is a $F \in \mathcal{F}$ with $i(F) \subset \widetilde{F}$. Hence $i^2(F) \subset i^2(V) \Rightarrow F^2 \subset V$ and consequently $\mathcal{F}$ is a Cauchy filter.

3. $\Rightarrow$ 1. : Assuming there is a neighborhood V and no finite covering X with sets A_i small of order V the sets $X \setminus \bigcup_{i\in L} A_i$ with finite L and A_i small of order V constitute a basis for an ultrafilter $\mathcal{F}$ containing every $X \setminus A_i$ and hence none of the A_i. Thus $\mathcal{F}$ does not contain any set small of order V and cannot be a Cauchy filter.

17.3 Complete precompact spaces

A **separated** space is **compact** iff it is **precompact** and **complete**.

Proof: $\Rightarrow$: A separated compact space X is obviously precompact (cf. 17.1), as a compact subspace of the separated complete closure $\widetilde{X}$ it is closed (cf. 14.7 and 9.4) and hence itself complete owing to 14.2.2. $\Leftarrow$: Follows directly from 17.2.2.

17.4 Neighbourhood filters of compact spaces

The uniquely determind **neighborhood filter** $\mathcal{U}$ of a **compact** space X consists of all **neighborhoods** of the **diagonal** Δ in X^2.

Proof: Due to 9.5 and 8.1.2 the space X is **completely regular**, **complete** (cf. 17.3) and **uniformizable** (cf. 13.5). For every neighborhood $V \in \mathcal{U}$ in the uniform space $(X,\mathcal{U})$ there is a neighborhood W with $W(W) \subset V$. Then we have $\Delta \subset \bigcup_{x\in X} (W \times W)(x;x) \subset V$ and $\bigcup_{x\in X} (W \times W)(x;x)$ is open in the product space $\left(X^2,\mathcal{U}^2\right)$, i.e. V is a neighborhood of Δ in X^2. Assuming there is a neighborhood V of Δ with $V \notin \mathcal{U}$ then the sets$\left\{U \cap \left(X^2 \setminus V\right) : U \in \mathcal{U}\right\}$ are the basis for a neighborhood filter $\mathcal{F}$ stronger than $\mathcal{U}$. The product space $\left(X^2,\mathcal{U}^2\right)$ is **compact** (Tychonov 9.9) such that $\mathcal{F}$ and particularly $\mathcal{U}$ must have an accumulation point $(x,y) \in \bigcap_{U\in\mathcal{U}} \overline{U}$ with $(x,y) \notin \Delta$ according to its construction (cf. 9.2.3). But according to 12.5 and 12.6.1 we have $\Delta = \bigcap_{U\in\mathcal{U}} \overline{U}$ since X is **separated**.

17.5 Locally compact spaces with a countable basis

Every **locally compact** space with a **countable basis** is **polish**.

Proof: On account of 10.8 the **Alexandrov compactification** $X \cup \{\infty\}$ of the locally compact space X is **second countable**, hence **metrizable** due to 11.14.3 and **complete** owing to 17.3. Thus $X \cup \{\infty\}$ is **polish** as well as the open subset $X \subset X \cup \{\infty\}$ due to 15.1.

17.6 Totally bounded spaces

For a metric space $(X; d)$ the following statements are equivalent:

1. X is **totally bounded**.
2. For every $\epsilon > 0$ there is a **finite cover** $(U_k)_{1\leq k\leq n}$ of X with $\delta(U_k) \leq \epsilon$.
3. Every sequence has a partial **Cauchy sequence**.

Proof:

1\. $\Rightarrow$ 2.: In a metric space a set A is small of order ϵ iff $\delta(A) \leq \epsilon$.

2\. $\Rightarrow$ 3.: For every $k \in \mathbb{N}$ there is at least one set U_k of every finite cover of sets small of order $\frac{1}{k}$ containing infinitely many members of the given sequence $(x_n)_{n\in\mathbb{N}}$. W.l.o.g. choose $n_0 := 0$, $n_1 := 1$ and $V_1 := U_1$ such that it contains infinitely many members x_n with $n \geq 1$. For already chosen $x_{n1}; ...; x_{nk}$ and $V_1; ...; V_k$ take U_{k+1} such that $V_{k+1} := U_{k+1} \cap V_k$ contains infinitely many members x_n with $n \geq n_k$ and $n_{k+1} := \min\{n \in \mathbb{N} : x_n \in V_{k+1} \setminus \{x_{n1}; ...; x_{nk}\}\}$. Thus we get a partial Cauchy sequence $(x_{nk})_{k\in\mathbb{N}}$ with $x_{nk} \in V_{k0}$ for $k > k_0$ and $V_{k+1} \subset V_k$ as well as $\delta(V_k) = \frac{1}{k}$.

3\. $\Rightarrow$ 1.: Assuming there is an $\epsilon > 0$ such that no finite union of sets small of order ϵ covers the set X. Choose an arbitrary $x_0 \in X$ and $U_0 := B_\epsilon(x_0)$. For already chosen $x_1, ..., x_n$ and $U_1, ..., U_n$ take $x_{n+1} \in \overline{U}_n \setminus U_n$ and $U_{n+1} := U_n \cup B_\epsilon(x_{n+1})$ such that the sequence $(x_n)_{n\in\mathbb{N}}$ satisfies $d(x_n; x_k) \geq \epsilon$ for all $k, n \in \mathbb{N}$ with $k < n$ and consquently does not contain a partial Cauchy sequence.

17.7 The Stone-Čech-compactification

Every **completely regular** space X has an **embedding** $e : X \to \beta X$ into a **compact** space βX which is uniquely determined up to homeomorphism and allows the unique **extension** of any continuous $f : X \to Y$ into a compact space Y to a continuous $\beta f : \beta X \to Y$ with $f = \beta f \circ e$. The image $e[X]$ is **dense** in $\widetilde{X}$, i.e. the compactification is its **closure**: $\overline{e[X]} = \widetilde{X}$. In particular every **closed** subset of a **completely regular** space is **compact**.

Proof: For every $\varphi : X \to \mathbb{R}$ from the set $C^*(X)$ of real-valued, continuous and bounded functions on X the image $\varphi[X]$ lies in a minimal closed interval $I_\varphi \subset \mathbb{R}$. According to 7.9 the evaluation function $e : X \to \prod_{\varphi\in C^*(X)} I_\varphi$ with $e(x) := (\varphi(x))_{\varphi\in C^*(X)}$ is an embedding and $\beta X := \overline{e[X]}$ is compact owing to 9.4 and **Tychonov's theorem** 9.9.

Let Y be compact with a continuous $f : X \to Y$ and an **embedding** $a : Y \to \prod_{\psi\in C^*(Y)} I_\psi$ with $a(y) := (\psi(y))_{\psi\in C^*(Y)}$ into $\beta Y = a[Y] = \overline{a[Y]}$. Now we simply **change coordinates** by means of $F : \prod_{\varphi\in C^*(X)} I_\varphi \to \prod_{\psi\in C^*(Y)} I_\psi$ with $F(e(x)) := a(f(x))$ resp. $(p_\psi \circ F)\left((t_\varphi)_{\varphi\in C^*(X)}\right) := t_{\psi\circ f}$. Thus the functions $\psi \in C^*(Y)$ are identified with $\varphi = \psi \circ f \in C^*(X)$ such that the φ-th coordinate t_φ of $\prod_{\varphi\in C^*(X)} I_\varphi$ can be assigned to the ψ-th coordinate t_ψ of $a[Y] \subset \prod_{\psi\in C^*(Y)} I_\psi$. The mapping F is **continuous** since $F^{-1}\left[p_\psi^{-1}[U_\psi]\right] = (p_\psi \circ F)^{-1}[U_\psi] = p_{\psi\circ f}^{-1}[U_{\psi\circ f}]$ is **open** in $\prod_{\varphi\in C^*(X)} I_\varphi$. Therefore and since $F(e(x)) = a(f(x))$ we obtain $F[\beta X] = F\left[\overline{e[X]}\right] \subset \overline{F[e[X]]} = \overline{a[f[X]]} \subset \overline{a[Y]} = \beta Y$.

$$\begin{tikzcd}
\prod_{\varphi\in C^*(X)} I_\varphi \arrow[r, "F"] & \prod_{\psi\in C^*(Y)} I_\psi \\
\cup & \cup \\
\beta X \arrow[r, "\beta f"] & \beta Y \\
X \arrow[u, "e"] \arrow[r, "f"] & Y \arrow[u, "a"']
\end{tikzcd}$$

The desired continuous extension is $\beta f := a^{-1} \circ F|_{\beta X}$ with $(\beta f \circ e)(x) = (a^{-1} \circ F|_{\beta X} \circ e)(x) = (a^{-1} \circ a \circ f)(x) = f(x)$. The restriction $F|_{e[X]} = a \circ f \circ e^{-1}$ resp. $\beta f|_X = f \circ e^{-1}$ is determined by $F|_{e[X]} \circ e = a \circ f$ resp. $f = \hat{f}|_X \circ e$ and on account of 7.13 the continuous extension βf is uniquely determined. The uniqueness of the compactification $\beta X := \overline{e[X]}$ follows as in the proof of 14.7 by invoking twice the uniqueness of the extension with regard to an alternative embedding $f := e' : X \to \beta X'$.

17.8 Application to the ordinal numbers

Due to the request for a continuous extension the Stone-Čech-compactification in general needs a lot more additional points than the **Alexandrov compactification** according to 10.2. The exact cardinality even of apparently simple examples like $\mathbb{N}$ is still the subject of current research. But there are also simple cases: On the set ω of the **ordinal numbers** (vgl. [19, p. 11.1]) with the **order topology** generated by the open intervals $]a;b[$with $a \subsetneq b \in \omega$ we have the smallest **infinite** ordinal $\aleph_0 = \mathbb{N}$ and smallest **non countable** ordinal $\aleph_1$. Every ordinal $x \in \omega$ represents a half open interval $[\emptyset, x[$ (cf. [19, p. 11.3.3]). It can be shown that the Stone-Čech-compactification of the subset of the countable ordinals $\aleph_1 = [\emptyset; \aleph_1[$ is achieved by simply adding $\aleph_1$ itself: $\beta\aleph_1 = \aleph_1 \cup \{\aleph_1\}$.

18 Compact convergence

18.1 Uniform convergence

For a family $F(X;Y)$ of functions $f : X \to Y$ on a set X into a uniform space $(Y;\mathcal{U})$ the finite intersections of the sets $W(U) := \{(f;g) \in F^2(X;Y) : (f(x);g(x)) \in U \,\forall x \in X\}$ for $U \in \mathcal{U}$ generate the neighborhood filter $\mathcal{W}(\mathcal{U})$ **of uniform convergence**. The resulting **topology $\mathcal{O}_X$ of uniform convergence on** X is induced by the finite intersections of $W(U)(f) := \{g \in F(X;Y) : g(x) \in U(f(x))\}$ and the topological space is denoted as $F_{\mathcal{U}}(X;Y) := (F(X;Y);\mathcal{O}_X)$.

18.2 Examples

On a **metric space** $(Y;d)$ the neighborhood filters $W\left(U_{1/n}\right) = \left\{D(f;g) < \frac{1}{n}\right\}$ with the **supremum metric** $D(f;g) := \sup\{d(f(x);g(x)) : x \in X\}$ constitute a **countable basis** for $\mathcal{W}(\mathcal{U})$. For **compact** X the **continuous image** $f[X]$ is **compact** (cf. 9.8) and hence **bounded** with reference to the metric d on Y. On account of 12.9 the continuous functions $f \in C_{\mathcal{U}}(X;Y)$ are **uniformly continuous** for **compact** X.

18.3 Uniform limits of continuous functions

$C_{\mathcal{U}}(X;Y)$ is **closed** in $F_{\mathcal{U}}(X;Y)$. Particularly every **limit function** of a **uniformly convergent filter** of **continuous functions** is again **continuous**.

Proof: For every filter $\mathcal{F}$ on X the family$A_{\mathcal{F}}$ of all $f \in F_{\mathcal{U}}(X;Y)$ with $f(\mathcal{F})$ being a Cauchy filter is closed in $F_{\mathcal{U}}(X;Y)$: For an accumulation element g on $A_{\mathcal{F}}$ and $V^3 \subset U \in \mathcal{U}$ there is an $f \in A_{\mathcal{F}}$ with $(f,g) \in V$ and a filter set $F \in \mathcal{F}$ with $f[F] \times f[F] \subset V$ such that $g[F] \times g[F] \subset V^3 \subset U$, i.e. $g(\mathcal{F})$ is a Cauchy filter too and hence $g \in A_{\mathcal{F}}$. For a Cauchy filter $f(\mathcal{U}(x))$ and every neighborhood V on Y there is a neighborhood $U(x) \in \mathcal{U}(x)$ with $f[U(x)] \times f[U(x)] \subset V$, i.e. $f[U(x)] \subset V(f(x))$ so that we have $\mathcal{U}(f(x)) \subset f[\mathcal{U}(x)]$ and due to 6.6.2 we infer $\bigcap_{x \in X} A_{\mathcal{U}(x)} \subset C_{\mathcal{U}}(X;Y)$. The converse inclusion is trivial and the closed character of the $A_{\mathcal{U}(x)}$ extends to their intersection $C_{\mathcal{U}}(X;Y)$.

18.4 Completeness of spaces of uniform convergence

Completeness transfers from Y to $F_{\mathcal{U}}(X;Y)$.

Proof: For a Cauchy filter $\mathcal{F}$ on $F_{\mathcal{U}}(X;Y)$ and every $x \in X$ the sets $F(x) := \{g(x) : g \in F\}$ for $F \in \mathcal{F}$ constitute a Cauchy filter $\mathcal{F}(x)$ on Y **uniformly** converging to a $f(x) := y \in Y$. The resulting function $f : X \to Y$ is the limit element of $\mathcal{F}$ since for every neighborhood $U \in \mathcal{U}$ there is an $F \in \mathcal{F}$ with $F(x) \subset U(f(x)) \,\forall x \in X$, i.e. $F \subset W(U)(f) \in \mathcal{F}$.

18.5 Uniform $\mathcal{S}$-convergence

For a subsystem $\mathcal{S} \subset P(X)$ let $\mathcal{W}(\mathcal{S};\mathcal{U})$ be the **initial neighborhood filter** on the set $F(X;Y)$ with reference to the **projections** $p_S : F(X;Y) \to F_{\mathcal{U}}(S;Y)$ with $p_S(f) := f|_S$ for $S \in \mathcal{S}$. According to 12.10 this **neighborhood filter** of **uniform $\mathcal{S}$-convergence** $\mathcal{W}(\mathcal{S};\mathcal{U})$ is generated by finite intersections of sets $W(S;U) := \left(p_S^2\right)^{-1}[W(U)] = \left\{(f;g) \in F^2(X;Y) : (f(x);g(x)) \in U \,\forall x \in S\right\}$ for $U \in \mathcal{U}$ and $S \in \mathcal{S}$ while the finite intersections of sets $p_S^{-1}[U(f)] = \{g \in F(X;Y) : g(x) \in U(f(x)) \;\forall x \in S\}$ induce the **topology $\mathcal{O}_{\mathcal{S}}$ of uniform $\mathcal{S}$-convergence**. Note that for **particular sets** $S_1 \subset S_2$ we have $W(S_1;U) \supset W(S_2;U)$ but for **set families** $\mathcal{S}_1 \subset \mathcal{S}_2$ the inclusion is converse: $\mathcal{W}(\mathcal{S}_1;\mathcal{U}) \subset \mathcal{W}(\mathcal{S}_2;\mathcal{U})$ and $\mathcal{O}_{\mathcal{S}_2}$ is **stronger** than $\mathcal{O}_{\mathcal{S}_1}$.

18.6 Examples

1. For the set $\mathcal{S} = \mathcal{E}$ of **finite** subsets on X we obtain the **neighborhood filter** $\mathcal{W}(\mathcal{E};\mathcal{U})$ resp. the **topology** $\mathcal{O}_{\mathcal{E}}$ of **pointwise convergence** coinciding with the product filter generated by the sets $W(\{x\};U) := (\pi_x \times \pi_x)^{-1}[U]$ on $\prod_{x \in X} Y = Y^X$. A filter converges on $F_{\mathcal{E}}(X;Y)$ iff every component $\pi_x(\mathcal{F}) = \mathcal{F}(x) := \{f(x) : f \in \mathcal{F}\}$ converges **pointwise**.
2. For a topology on X and the family $\mathcal{S} = \mathcal{K}$ of **compact** subsets we get the neighborhood filter $\mathcal{W}(\mathcal{K};\mathcal{U})$ resp. the **topology** $\mathcal{O}_{\mathcal{K}}$ of **compact convergence**. A filter converges with reference to the corresponding **topology of compact convergence** on $F_{\mathcal{K}}(X;Y)$ iff it **uniformly** converges on every compact subset of X. The subspace $C_{\mathcal{K}}(X;Y) \subset F_{\mathcal{K}}(X;Y)$ contains the functions that are uniformly continuous on compact sets (cf. 12.9)
3. For $\mathcal{S}=\{X\}$ we have the neighborhood filter $\mathcal{W}(\mathcal{U})$ resp. the **topology** $\mathcal{O}_{\mathrm{X}}$ of **uniform convergence** on X.
4. Due to $\mathcal{E} \subset \mathcal{K}$ and $W(X;U) \subset W(K;U) \subset W(\{x\};U) \forall U \in \mathcal{U}, x \in K \in \mathcal{K}$ the inclusion $\mathcal{W}(\mathcal{E};\mathcal{U}) \subset \mathcal{W}(\mathcal{K};\mathcal{U}) \subset \mathcal{W}(\mathcal{U})$holds. (cf. 18.5): Uniform convergence ist stronger than compact convergence which is stronger than poitwise convergence.
5. On account of 4. the projections $\pi_x : F(X;Y) \to Y$ with $\pi_x(f) := f(x)$ are not only continuous on $\mathcal{W}(\mathcal{E};\mathcal{U})$ but also on $\mathcal{W}(\mathcal{K};\mathcal{U})$ and $\mathcal{W}(\mathcal{U})$. Hence for any set $A \subset F(X;Y)$ we have $\overline{A}(x) = \pi_x\left[\overline{A}\right] \subset \overline{\pi_x[A]} = \overline{A(x)}$ owing to 3.5.

18.7 Properties of the function spaces $F_{\mathcal{S}}(X;Y)$

For a set X and a uniform space Y the following statements hold:.

1. For a **separated** Y and a **cover** $\mathcal{S}$ of X the function space $F_{\mathcal{S}}(X;Y)$ is **separated** too since for $f \neq g \in F_{\mathcal{S}}(X;Y)$ there is a $x \in S \subset \mathcal{S}$ with $f(x) \neq g(x)$ and hence $U, V \in \mathcal{U}_Y$ with $U(f(x)) \cap V(g(x)) = \emptyset$ so that $W(S;U)(f) \cap W(S;V)(g) = \emptyset$.
2. For a family $\mathcal{S}$ of subsets whose **interiors** already cover X the **continuous functions** $C(X;Y)$ are **closed** in $F_{\mathcal{S}}(X;Y)$ since owing to 18.3 the set $C_{\mathcal{U}}(S;Y)$ is closed in $F_{\mathcal{U}}(S;Y)$ and on account of the projections $p_S : F(X;Y) \to F_{\mathcal{U}}(S;Y)$ being continuous this is also true for $C(X;Y) = \bigcap_{S \in \mathcal{S}} p_S^{-1}[C_{\mathcal{U}}(S;Y)]$. Note that due to the hypothesis $f^{-1}[O]$ is open in X iff $\left(f^{-1}[O]\right) \cap \mathring{S}$ is open in X for all $S \in \mathcal{S}$, cf. 4.3.3.
3. The **completeness** of Y transfers to $F_{\mathcal{U}}(S;Y)$ due to 18.4 and hence to $F_{\mathcal{S}}(X;Y)$ on account of 6.7 since for a Cauchy filter $F \subset F_{\mathcal{S}}(X;Y)$ every projection $p_S(F) \subset F_{\mathcal{U}}(S;Y)$ is again a Cauchy filter converging to a $f_S \in F_{\mathcal{U}}(S;Y)$ such that F converges to $f := \bigcap_{S \in \mathcal{S}} p_S^{-1}(f_S) \in F_{\mathcal{S}}(X;Y)$.
4. All **separation axioms** with the exception of T_4 transfer to $F_{\mathcal{E}}(X;Y)$ due to 7.10.
5. For X **locally compact** as well as **σ-compact** and Y **metrizable** the function spaces $F_{\mathcal{K}}(X;Y)$ and $C_{\mathcal{K}}(X;Y)$ are **metrizable:** Each f has a **countable neighborhood basis** $p_{K_n}^{-1}\left[B_{1/m}(f)\right]$

with compact $K_n \subset X$ for $n \in \mathbb{N}$, $K_n \subset K_{n+1}$ and $\bigcup_{n\in\mathbb{N}} K_n = X$ resp. $B_{1/m} = \left\{d < \frac{1}{m}\right\}$. Furthermore it is **separated** due to 18.7.1 and hence **metrizable** on account of 13.3. A metric on $F_{\mathcal{K}}(X;Y)$ is provided by $D(f;g) := \max\limits_{n\in\mathbb{N}} \frac{d_n(f;g)}{n(1+d_n(f;g))}$ with $d_n(f;g) := \sup\{d(f(x);g(x)) : x \in K_n\}$: As in 1.6 the **triangle inequality** follows from the corresponding relation of the components: $D(f;g) + D(g;h) = \max\limits_{n\in\mathbb{N}} \frac{d_n(f;g)}{n(1+d_n(f;g))} + \max\limits_{n\in\mathbb{N}} \frac{d_n(g;h)}{n(1+d_n(g;h))} \geq \max\limits_{n\in\mathbb{N}} \left(\frac{d_n(f;g)}{n(1+d_n(f;g))} + \frac{d_n(g;h)}{n(1+d_n(g;h))}\right) \geq \max\limits_{n\in\mathbb{N}} \frac{d_n(f;h)}{n(1+d_n(f;h))} = D(f;h)$. On the one hand for every $\epsilon > 0$ and $n_\epsilon = \left[\epsilon^{-1}\right]$ the basis set $\{D < \epsilon\} = \bigcap\limits_{n=0}^{n_\epsilon} \left\{d_n < \frac{n\epsilon}{1-n\epsilon}\right\} = \bigcap\limits_{n=0}^{n_\epsilon} W\left(K_n; \left\{d < \frac{n\epsilon}{1-n\epsilon}\right\}\right) \in \mathcal{W}(\mathcal{K};\mathcal{U})$ and on the other hand for every neighborhood $W(K;U) \in \mathcal{W}(\mathcal{K};\mathcal{U})$ there is an $n \in \mathbb{N}$ with $K \subset K_n$ and an $\epsilon > 0$ with $\{d < \epsilon\} \subset U$ such that $\left\{D < \frac{\epsilon}{n(1+\epsilon)}\right\} \subset W(K;U)$.

18.8 The compact open topology

For a topological space X and a uniform space Y the sets $(K;O) := \{f : X \to Y : f[K] \subset O\}$ for **compact** $K \subset X$ and **open** $O \subset Y$ form a **subbasis** for the **compact open topology** on the set of continuous functions $\mathcal{C}(X;Y)$. Due to 18.6.2 the compact open topology is identical with the **topology** $\mathcal{O}_{\mathcal{K}}$ of **compact convergence** on the subspace $\mathcal{C}_{\mathcal{K}}(X;Y) \subset F_{\mathcal{K}}(X;Y)$.

Proof:

For an arbitrary $(K;O)$, $f \in (K;O)$ and every $y \in f[K]$ there is a neighborhood $V_y(y) \subset O$ and a finite $E \subset f[K]$ with $f[K] \subset \bigcup_{y\in E} U_y(y)$ and $U_y^2 \subset V_y$ since $f[K]$ is quasi-compact due to 9.8. For $U := \bigcap_{y\in E} U_y$ we have $U(f[K]) \subset O$ such that for $g \in W(K;U)(f)$ with $f[K] \times g[K] \subset U$ follows $g[K] \subset U(f[K]) \subset O$, i.e. $g \in (K;O)$. Hence we have shown that $W(K;U)(f) \subset (K;O)$.

Conversely for an arbitrary $W(K;U)(f)$ there is a closed and symmetric $V^3 \subset U$ as well as a finite $E \subset f[K]$ with $f[K] \subset \bigcup_{i\in E} V(f(x_i))$. On account of 3.1 and 9.4 the sets $K_i := K \cap f^{-1}(V(f(x_i)))$are again compact and cover K. For $g \in \bigcap_{i\in E} (K_i;O_i)$ and $x \in K$ there is an $i \in E$ with $x \in K_i$ such that $g(x) \in O_i := V^2 (\mathring{f}(x_i))$ and since $f(x) \in V(f(x_i))$ we infer $(f(x);g(x)) \in V^3$ hence $\bigcap_{i\in E} (K_i;O_i) \subset W(K;U)(f)$.

18.9 Uniform approximation of the absolute value function by real polynomials

1. The polynomials $p_n : \mathbb{R} \to \mathbb{R}$ defined by $p_0(t) := 0$ and $p_{n+1}(t) := p_n(t) + \frac{1}{2}\left(t - p_n^2(t)\right)$ uniformly converge on $[0;1]$ to $f(t) = \sqrt{t}$ since for $t \in [0;1]$ we have $\sqrt{t} - p_{n+1}(t) = \left(\sqrt{t} - p_n(t)\right) \cdot \left(1 - \frac{1}{2}\left(\sqrt{t} + p_n(t)\right)\right)$ and by means of **induction** over n it is easily shown that $0 \leq \sqrt{t} - p_n(t) \leq \frac{2\sqrt{t}}{2+n\sqrt{t}} \leq \frac{2}{n}$ whence follows uniform convergence.
2. The polynomials $q_n : \mathbb{R} \to \mathbb{R}$ defined by $q_n(t) := a \cdot p_n\left(\frac{t^2}{a^2}\right)$ uniformly converge on $[-a;a]$ to $h(t) = |t|$ since for $t \in [-a;a]$ we have $||t| - q_n(t)| = \left|a\sqrt{\frac{t^2}{a^2}} - q_n(t)\right| \leq \frac{2a\sqrt{\frac{t^2}{a^2}}}{2+n\sqrt{\frac{t^2}{a^2}}} \leq \frac{2a}{n}$.

18.10 Algebrae of continuous functions on compact spaces

The **closure** $\overline{A}$ of a subalgebra $A \subset \mathcal{C}(K;\mathbb{C})$ of **continuous complex-valued functions** on a **compact** space K with regard to **uniform convergence** has the following properties:

1. $\overline{A}$ is again a **subalgebra** since for $f, g \in \overline{A}$ in every ϵ-neighbourhood there are $f_\epsilon, g_\epsilon \in A$ such that $|\|\alpha \cdot f\| - \|\alpha \cdot f_\epsilon\|| \leq |\alpha| \cdot \|f - f_\epsilon\| \leq |\alpha|\,\epsilon \Rightarrow \alpha \cdot f \in \overline{A}$; $\leq \|f - f_\epsilon + g - g_\epsilon\| \leq 2\epsilon \Rightarrow f + g \in \overline{A}$ and finally $|\|f+g\| - \|f_\epsilon + g_\epsilon\|| \leq |\|f \cdot g\| - \|f_\epsilon \cdot g_\epsilon\|| \leq |\|g\| \cdot \|f - f_\epsilon\| + \|f_\epsilon\| \cdot \|g - g_\epsilon\|| \leq \|g\| \cdot \epsilon + (\|f\| + \epsilon) \cdot \epsilon \Rightarrow f \cdot g \in \overline{A}$ since f and g are bounded on K due to 9.8 and 9.10.

2. For every $f;g\in\overline{A}$ we have $|f|\in\overline{A}$, $\max\{f;g\}=\frac{1}{2}((f+g)+|f-g|)\in\overline{A}$ and $\min\{f;g\}=\frac{1}{2}((f+g)+|f-g|)\in\overline{A}$ since with $a:=\|f\|$ we can apply 18.9.2, i.e. for every $\epsilon>0$ there is a polynomial p_ϵ without constant term satisfying $\||f(x)|-p_\epsilon(f(x))\|<\epsilon$ and $p_\epsilon(f(x))\in\overline{A}$ on account of 1.

18.11 The Stone Weierstrass theorem on compact spaces

The subalgebra $A(D)$ generated by a subfamily $D\subset\mathcal{C}(K;\mathbb{C})$ of **continuous complex-valued functions** on a **compact** space K by means of **polynomials** without **constant term** is **dense** in $\mathcal{C}(K;\mathbb{C})$ with regard to **uniform convergence** if D satisfies the following conditions:

1. D separates every point in K from 0: $\forall y\in K\,\exists g_y\in D:\ g_y(y)\neq 0$.
2. D **separates points** in K: $\forall y,z\in K\,\exists g\in D:\ g(y)\neq g(z)$
3. With every $f\in D$ we also have the **complex conjugate** $\overline{f}\in D$.

Proof: Owing to **condition 3.** and on account of $\overline{g+h}=\overline{g}+\overline{h}$ resp. $\overline{g\cdot h}=\overline{g}\cdot\overline{h}$ the subalgebra $A(D)$ can be decomposed into a **real part** $\mathrm{Re}A(D)=\left\{\frac{1}{2}\left(h+\overline{h}\right):h\in A(D)\right\}$ and an **imaginary part** $\mathrm{Im}A(D)=\left\{\frac{1}{2}\left(h-\overline{h}\right):h\in A(D)\right\}$. For a given $f\in\mathcal{C}(K;\mathbb{C})$ we have $\mathrm{Re}f\in\mathcal{C}(K;\mathbb{R})$ and for every $y,z\in K$ due to **condition 1.** we obtain $g_y;g_z\in A(D)$ with $g_y(y)=g_z(z)=1$ and hence a $g_{yz}=g_y+g_z-g_y\cdot g_z\in A(D)$ with $g_{yz}(y)=g_{yz}(z)=1$. Thus the function $h_{yz}=\frac{\mathrm{Re}f(y)(g-g(z)\cdot g_{yz})-\mathrm{Re}f(z)(g-g(y)\cdot g_{yz})}{g(y)-g(z)}\in A(D)$ satisfies $h_{yz}(y)=\mathrm{Re}f(y)$ and $h_{yz}(z)=\mathrm{Re}f(z)$ for $g\in A(D)$ being either the real part or the imaginary part of the separating function for y und z depending on which part actually separates the two points according to **condition 2.** For every $z\in K$ the neighborhoods $U_{z\epsilon}(y)=\{h_{yz}-\mathrm{Re}f<\epsilon\}$ cover the compact set K for finitely many $y\in L\subset K$ and $h_{z\epsilon}=\max\{h_{yz}:y\in L\}\in A(D)$ with $h_{z\epsilon}(x)-\mathrm{Re}f(x)<\epsilon\,\forall x\in K$ due to 18.10.2. The neighborhoods $U_\epsilon(z)=\{h_{z\epsilon}-\mathrm{Re}f>-\epsilon\}$ again cover K for finitely many $z\in M\subset X$ and as above we have $\mathrm{Re}h_\epsilon:=\max\{h_{z\epsilon}:z\in M\}\in A(D)$ with $|\mathrm{Re}h_\epsilon-\mathrm{Re}f|<\epsilon$. In the same way we find $\mathrm{Im}h_\epsilon\in A(D)$ with $|\mathrm{Im}h_\epsilon-\mathrm{Im}f|<\epsilon$ and hence for $h_\epsilon=\mathrm{Re}h_\epsilon+i\mathrm{Im}h_\epsilon\in A(D)$ we have $\|h_\epsilon-f\|<\sqrt{2}\epsilon$.

18.12 Properties of $\mathcal{C}(K;\mathbb{C})$

The algebra $A(D)$ induced by the three functions $D=\{x\mapsto 1;x\mapsto x;x\mapsto\overline{x}\}$ has the same cardinality as the set of all finite subsets of $\mathbb{N}$ and hence is **countable** due to [19, p. 17.6]. According to the preceding theorem it is also **dense** in $\mathcal{C}(K;\mathbb{C})$. Due to 18.7.5 the topological vector space $\mathcal{C}(K;\mathbb{C})$ is **metrizable** whence on account of 11.14.1 it is **second countable**. Finally because of 18.7.3 it is **complete** and hence a **polish space**.

18.13 The Stone-Weierstrass theorem for locally compact spaces

Let X be a σ**-compact** space X and $\mathcal{C}_0(X;\mathbb{C})$ the **algebra** of all **continuous** functions **vanishing at infinity**, i.e. $\lim\limits_{|\boldsymbol{x}|\to\infty}|f(\boldsymbol{x})|=0\,\forall f\in\mathcal{C}_0(X;\mathbb{C})$. Then the subalgebra $A(D)$

1. generated by $D\subset\mathcal{C}(X;\mathbb{C})$ is **dense** in $\mathcal{C}(X;\mathbb{C})$ with regard to **compact convergence.**
2. generated by $D\subset\mathcal{C}_0(X;\mathbb{C})$ on is **dense** in $\mathcal{C}_0(X;\mathbb{C})$ with regard to **uniform convergence.**

if D satisfies the conditions 1. - 3. from 18.11.

Proof:

1. This is just a paraphrasing of 18.11.
2. By the **Alexandrov compactification** 10.2 and the **contintuous extension** $f(\infty)=0$ for every $f\in\mathcal{C}_0(X;\mathbb{C})$ by he conditions for 18.11 are obviously satisfied.

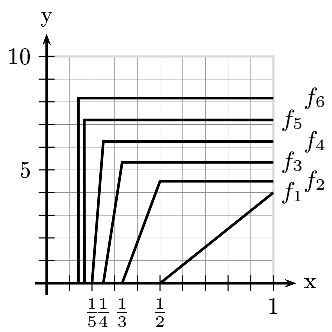
19 Equicontinuity

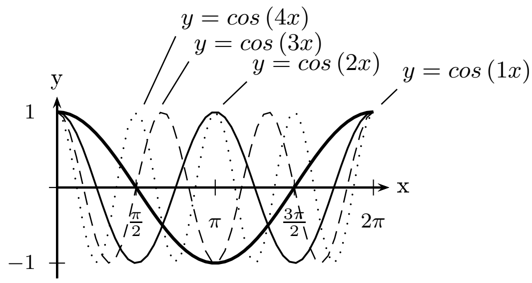
19.1 Equicontinuity

For a topological space X and a uniform space Y the family H$\subset$ $F\left(X;Y\right)$ is **equicontinuous** in $x\in X$ iff for evey neighborhood U in Y there is a neighborhood $V(x)$ of x such that $f\left[V(x)\right]\subset U\left(f(x)\right)\forall f\in H$. The family H is simply equicontinuous iff H is equicontinuous in every $x\in X$. For an uniform space X and V independent of x we have a case of **uniform equicontinuity**

19.2 Examples

1. For two metric spaces $(X;d)$ resp. $(Y;d')$ and $k,\alpha\in\mathbb{R}_+^*$ the **Lipschitz continuous** functions $H_{k;\alpha}:=\left\{f:X\to Y:\,d'\left(f(x);f(y)\right)\leq k\cdot d\left(x;y\right)^{\alpha}\;\forall x,y\in X\right\}$ are **equicontinuous** since $f\left[V_\delta(x)\right]\subset U_\epsilon\left(f(x)\right)\forall f\in H\wedge\epsilon>0$ with $\delta=\left(\frac{\epsilon}{k}\right)^{\frac{1}{\alpha}}$.
2. For $a<b$ and $k\in\mathbb{R}_+^*$ the family $H_k:=\left\{f:[a;b]\to\mathbb{R}:\,|f'(x)|\leq k\,\forall x\in[a;b]\right\}$ is **equicontinuous** since $f\left[V_\delta(x)\right]\subset U_\epsilon\left(f(x)\right)\forall f\in H\wedge\epsilon>0$ with $\delta=\frac{\epsilon}{k}$.
3. The family $(f_n)_{n\in\mathbb{N}}:\;[0;1]\to\mathbb{R}$ with $f_n(x)=n$ is **uniformly equicontinuous** but **not uniformly bounded** although every single f_n is **bounded**.
4. The family $(f_n)_{n\in\mathbb{N}}:\,[0;1]\to\mathbb{R}$ with $f_n(x)=\begin{cases}0 & \text{für } x<\frac{1}{n}\\ n^3\left(x-\frac{1}{n}\right) & \text{für } \frac{1}{n}<x<\frac{1}{n-1}\\ \frac{n^2}{n-1} & \text{für } \frac{1}{n-1}<x\end{cases}$ is **equicontinuous** at every x (the choice of V is independent of f) and each f_n is **uniformly continuous** (the choice of V is independent of x) as well as bounded, but the family is **neither uniformly equicontinuous** (there is no common V for all x and f) nor bounded with the exception of $x=0$. (counterexample to [10, Aufgabe 14.11]!)
5. The family $(f_n)_{n\in\mathbb{N}}:\;[0;1]\to[-1;1]$ with $f_n(x)=\cos(nx)$ is **not equicontinuous** but **uniformly bounded** and all f_n are **uniformly continuous**.

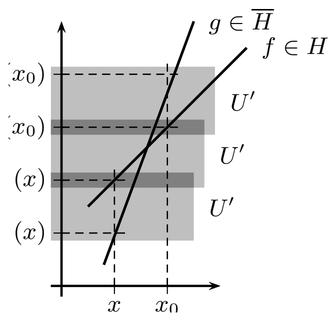




19.3 Characterization of equicontinuous families

For a topological space X and a uniform space Y the family $H\subset F(X;Y)$ is **equicontinuous** in $x_0\in X$ iff the **closure** $\overline{H}$ in $F_{\mathcal{E}}(X;Y)$ is **equicontinuous** in x_0. In less abstract words: **The pointwise limit of a family of equicontinuous functions is again equicontinuous**.

Proof: We only have to show $\Rightarrow$: For a neighborhood U in Y there is a neighborhood $U'^3\subset U$ and a neighborhood $V(x_0)$ of x_0 with $f\left[V\left(x_0\right)\right]\subset U'\left(f(x_0)\right)\forall f\in H$. For $g\in\overline{H}$ and every $x\in V\left(x_0\right)$ there is a $f\in H\cap W\left(\{x_0;x\};U'\right)(g)=H\cap W\left(\{x_0\};U'\right)(g)\cap W\left(\{x\};U'\right)(g)$ $\neq\emptyset$ with $\left(f\left(x_0\right);g\left(x_0\right)\right)\in U'$ und $\left(f\left(x\right);g\left(x\right)\right)\in U'$. From the above stated $\left(f\left(x_0\right);f\left(x\right)\right)\in U'$ we infer $\left(g\left(x_0\right);g\left(x\right)\right)\in U'^3$, i.e. $g\left[V\left(x_0\right)\right]\subset U'^3\left(g(x_0)\right)$.



19.4 Pointwise and compact convergence

On every **equicontinuous** family $H \subset C(X;Y)$ between a **topological** space X and a **uniform** space Y the neighborhood filters of **compact** convergence and of **pointwise** convergence are identical.

Proof: On account of 18.6.4 we only hace to show that $C_{\mathcal{K}}(X;Y) \subset C_{\mathcal{E}}(X;Y)$: For every neighborhood $W(K;U)$ let $U'^3 \subset U$ be a neighborhood in Y and $(V(x_i))_{1\leq i\leq n}$ finitely many neighborhoods in X with $f[V(x_i)] \subset U'(f(x_i))\,\forall f \in H$ as well as $K \subset \bigcup_{1\leq i\leq n} V_i$. For an arbitrary $f \in W(E;U')$ with $E := \{x_1;\ldots;x_n\}$ and $x,y \in K$ there are $x_i, x_j \in E$ with $x \in V(x_i)$ resp. $y \in V(x_j)$ such that $(f(x_i);f(x)) \in U'$ resp. $(f(x_j);f(y)) \in U'$. On account of $f \in W(E;U')$ we have $(f(x_i);f(x_j)) \in U'$ and hence $(f(x);f(y)) \in U'^3 \subset U$ consequently $W(E;V) \subset W(K;U)$ resp. $\mathcal{W}(\mathcal{E};\mathcal{U}) \supset \mathcal{W}(\mathcal{K};\mathcal{U})$.

19.5 Closure of equicontinuous families

For a **topological** space X and a **uniform** space Y the closures $\overline{H}$ of every **equicontinuous** family $H \subset C(X;Y)$ with reference to **compact** resp. **pointwise** convergence coincide.

Proof: On account of 18.6.4 for the closures of arbitrary families $H \subset F(X;Y)$we have $\overline{H}_{\mathcal{E}} \supset \overline{H}_{\mathcal{K}}$ with reference to $\mathcal{W}(\mathcal{E};\mathcal{U}) \subset \mathcal{W}(\mathcal{K};\mathcal{U})$. Due to 19.3 the closure $\overline{H}_{\mathcal{E}}$ in $F_{\mathcal{E}}(X;Y)$ is equicontinuous and in particular$\overline{H}_{\mathcal{K}} \subset \overline{H}_{\mathcal{E}} \subset C(X;Y)$ such that we can apply subsec:Pointwise-and-compact convergence to $\overline{H}_{\mathcal{E}}$ and hence obtain the proposition.

19.6 The Arzela-Ascoli theorem

For a **locally compact** space X and a **separated** space Y the **closure** $\overline{H}$ of a family $H \subset C(X;Y)$ is **compact** in $C_{\mathcal{K}}(X;Y)$ iff H is **equicontinuous** and $\overline{H(x)}$ is **compact** in Y for every $x \in X$.

Proof:

$\Rightarrow$: On account of 9.4 $\overline{H(x)}$ is **compact** in Y. In order to show the **equicontinuity** let U be a neighborhood in Y, U' **symmetric** with $U'^3 \subset U$ and $x_0 \in X$. Due to the hypothesis for a **compact** neighborhood K of x_0 there are finitely many $(f_i)_{1\leq i\leq n}$ with $H \subset \overline{H} \subset \bigcup_{1\leq i\leq n} W(K;U')(f_i)$. Since the f_i are **continuous** there are neighborhoods $V_i \subset K$ of x_0 with $f_i[V_i(x_0)] \subset U'(f_i(x_0))$. For $x \in V(x_0) \subset K$ with $V := \bigcap_{1\leq i\leq n} V_i$ and $f \in H$ there is a j with $(f_j(x_0);f_j(x)) \in U'$, $(f(x);f_j(x)) \in U'$, $(f(x_0);f_j(x_0)) \in U'$ hence $(f(x);f(x_0)) \in U'^3$, i.e. $f[V(x_0)] \subset U'^3(f(x_0)) \subset U(f(x_0))$.

$\Leftarrow$: Due to 19.4 for **equicontinuous** $H \subset C(X;Y)$ the neighborhood filters of **compact** and **pointwise** convergence coincide such that H can be regarded as a subset of the **product space** $Y^X = \prod_{x\in X} Y$ with $H \subset \overline{H} \subset \prod_{x\in X} \overline{H}(x) \subset \prod_{x\in X} \overline{H(x)}$ on account of 18.6.5. Since the components $\overline{H(x)}$ are **compact** this transfers to the **product** $\prod_{x\in X} \overline{H(x)}$ (c.f. 9.9) and hence to the **closed** subset $\overline{H}$ (c.f. 9.4).

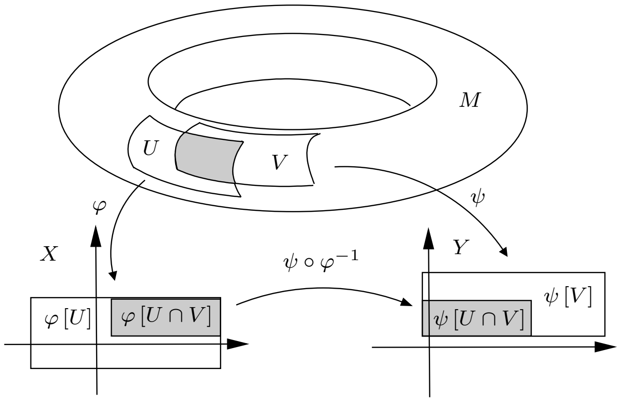
19.7 Examples

1. Due to 18.7.5 for **metrizable** Y and **locally compact** X being also **σ-compact** the family $C_{\mathcal{K}}(X;Y)$ is **metrizable** and owing to 10.13 the properties of being **compact**, **countably compact** and **sequentially compact** are equivalent on Y as well as on$C_{\mathcal{K}}(X;Y)$. Thus we obtain the classical formulation of the **Arzela-Ascoli theorem**: A sequence $(f_n)_{n\in\mathbb{N}} \subset C(X;Y)$ of continuous functions has a **subsequence uniformly converging on compact sets to a continuous** $f \in C(X;Y)$ iff it is **equicontinuous** and the point sequences $(f_n(x))_{n\in\mathbb{N}}$ on Y have a **convergent subsequence** for every $x \in X$. This variant is used in the proof of the following theorem [15, th. 8.3] of **Peano** to show the existence of solutions for a wide class of **differential equations**.

2. For any open $G \subset \mathbb{C}$ the family $\mathcal{F} \subset C(G;\mathbb{C})$ is **normal** iff its closure $\overline{\mathcal{F}}$ is **compact** resp. **sequentially compact** (see above) in $C_{\mathcal{K}}(G;\mathbb{C})$. Due to 19.6 this is equivalent to $\mathcal{F}$ being **equicontinuous** und $\overline{\mathcal{F}(x)}$ being **sequentially compact** for every $x \in G$. The Arzela-Ascoli theorem together with **Cauchy's integral formula** [15, th. 7.5] serve to prove **Montel's theorem** [15, th. 7.15] which states that the normal character of a family $\mathcal{F} \subset H(G;\mathbb{C})$ of **holomorphic** functions is equivalent to it being locally bounded, i.e. that for every $x \in G$ there is a neighborhood $U(x)$ and an $M > 0$ such that $|f(x)| < M$ for every $x \in U(x)$ and $f \in \mathcal{F}$.

20 Manifolds

20.1 Atlases and charts



An **atlas** on a set M is a family $\mathfrak{A}$ of **charts** $(U;\varphi;X)$ each consisting of a **subset** $U \subset M$ with $M \subset \bigcup\{U : (U;\varphi;X) \in \mathfrak{A}\}$ and an **injective coordinate function** $\varphi : U \to X$ into a **Banach space** X such that for every pair of charts $(U;\varphi;X)$ and $(V;\psi;Y) \in \mathfrak{A}$ the **coordinate sets** $\varphi[U \cap V] \subset X$ resp. $\psi[U \cap V] \subset Y$ are **open** and the **change of coordinates** or **transition map** $\psi \circ \varphi^{-1} : \varphi[U \cap V] \to \psi[U \cap V]$ is a **homeomorphism**. In particular $\varphi[U]$ is **open** in X and U is open in M for every $(U;\varphi;X)$. Every **chart** $(U;\varphi;X)$ consisting of a subset $U \subset M$ and a **bijection** $\varphi : U \to \varphi[U]$ onto an open $\varphi[U] \subset X$ in some **Banach space** Y is **admissible** to the **atlas** $\mathfrak{A}$ iff is satisfies the above stated conditions for every other chart $(V;\psi;Y) \in \mathfrak{A}$. Two atlases $\mathfrak{A}$ and $\mathfrak{B}$ are **compatible** to each other if each chart of $\mathfrak{A}$ is admissible to every chart of $\mathfrak{B}$. The compatibility obviously defines an **equivalence relation** and each equivalence class of atlases obviously is **inductively ordered** by **inclusion** such that **Zorn's lemma** [19, th. 41.2.4] implies the existence of a **maximal atlas** $\mathfrak{C}(M)$ on M. In [15, section 6] we examine **differentiable manifolds** of class C^r with r times **continuously differentiable** transition functions. The vector space of all admissible charts $(U;\varphi;X) \in \mathfrak{C}(M)$ at a point $m \in M$ is denoted as $\mathcal{C}_m M$ and the corresponding family of neighborhoods U is $\mathcal{U}_m M$.

According to [16, th. 1.2] and due to $\boldsymbol{a} + X \cong X$ the topology on the Banach space X is uniquely determined by the **local basis** $\mathcal{B}(\mathbf{0})$ of the **origin**: $\boldsymbol{a} + U = t_{\boldsymbol{a}}[U] \in \mathcal{U}(\boldsymbol{a}) \Leftrightarrow U \in \mathcal{U}(\mathbf{0})$. Hence the homeomorphic coordinate changes $\psi \circ \varphi^{-1} : X \to Y$ show that any two Banach spaces X and Y of two charts $(U;\varphi;X)$ and $(V;\psi;Y)$ with nonempty intersection $U \cap V \neq \emptyset$ are **homeomorphic** to each other. Furthermore the set of points $m \in M$ for which exist a chart $(V;\psi;Y)$ such that Y is homeomorphic to a given Banach space X is both **open** and **closed**. Consequently modulo linear transformations there is a **common Banach space** X on each **connected component** of M and the maximal class of all compatible charts on such a connected component is a **topological X-manifold** on M. Up to further notice we will examine connected manifolds with coordinates on a common Banach space X.

The manifold is furnished with the **final topology** $\mathcal{O}_{\mathfrak{C}(M)}$ with regard to the **parametrizations** $\varphi^{-1} : \varphi[U] \to M$ for $(U;\varphi;X) \in \mathfrak{C}(M)$ according to 4.5. Hence a subset $O \subset M$ is **open** iff its coordinate set $\varphi[O \cap U]$ in every chart $(U;\varphi)$ is **open** in $\varphi[U]$, hence **open** in X, and conversely the **parametrizations** $\varphi_i^{-1}[O_i] = \varphi_i^{-1}[O_i \cap \varphi_i[U_i]]$ of open coordinate sets $O_i \subset X_i$ constitute a **basis** of the open sets in M. The final topology is already determined by any subcollection $\mathfrak{C}_I(M) = (U_i;\varphi_i;X_i)_{i \in I} \subset \mathfrak{C}(M)$ of charts **covering** $M = \bigcup_{i \in I} U_i$ since for any $O \in \mathcal{O}_{\mathfrak{C}_I(M)}$ and any other **admissible** chart $(V;\psi;Y)$ the relations $\psi[O \cap V] = \bigcup_{i \in I} \psi(O \cap U_i \cap V)$ and $\psi[O \cap U_i \cap V]$ $= \left(\psi \circ \varphi_i^{-1} \circ \varphi_i\right)[O \cap U_i \cap V] = \left(\psi \circ \varphi_i^{-1}\right)(\varphi_i[O \cap U_i \cap V])$ in the case of $O = M$ imply that $V \in$

$\mathcal{O}_{\mathfrak{C}_I(M)}$ whereas in the case of any $O \in \mathcal{O}_{\mathfrak{C}_I(M)}$ we see that $\psi[O \cup V]$ is open in Y whence follows $O \in \mathcal{O}_{\mathfrak{C}_I(M)\cup\{V;\psi;Y\}}$. Since the definition of the final topology implies $\mathcal{O}_{\mathfrak{C}_I(M)\cup\{V;\psi;Y\}} \subset \mathcal{O}_{\mathfrak{C}_I(M)}$ the equality of the two topologies follows.

By the definition of the open sets in M every **coordinate function** $\varphi : U \to X$ is an **open** map and also **continuous** since for every open $O \subset X$ and every chart $(V;\psi)$ the image $\psi\left[\varphi^{-1}[O] \cap V\right] = \left(\psi \circ \varphi^{-1}\right)[O] \cap \psi[V]$ is open in X whence $\varphi^{-1}[O]$ is open in M. Thus every coordinate function $\varphi : U \to \varphi[U]$ is a **homeomorphism** with regard to the **trace topologies** $\mathcal{O}_{\mathfrak{C}(M)} \cap U$ on the manifold M and $O_{\|\|} \cap \varphi[U]$ on the Banach space $(X; \|\ \|)$ and since U resp. $\varphi[U]$ are itself open the homeomorphy extends to the topologies on M resp. X itself in accordance with the definitions.

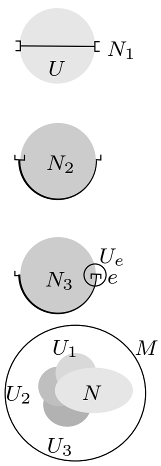
In the case of a given **topology** $\mathcal{O}$ on M the restriction to **continuous parametrizations** implies $\mathcal{O} \subset \mathcal{O}_{\mathfrak{C}(M)}$ and conversely if only **continuous coordinate functions** $\varphi : M \to X$ are admitted we have $\mathcal{O}_{\mathfrak{C}(M)} \subset \mathcal{O}$ since every open $O \in \mathcal{O}_{\mathfrak{C}(M)}$ as the union $O = \bigcup\left\{\varphi^{-1}[\varphi[O \cap U]] : (U;\varphi) \in \mathfrak{C}(M)\right\}$ of open preimages of open sets under continuous maps is also open in $\mathcal{O}$.

Since every point $m \in M$ has a neighborhood which is homeomorphic to an open neighborhood of $\varphi(m)$ in a Banach space X the manifold M is **Hausdorff** and according to 7.7 even **regular**. Note that in this text **second countability** and **connectedness** are **not included** a priori in the definition

Examples:

1. Every **open** subset $O \subset M$ is a manifold with the atlas $\{(O \cap U; \varphi|_{O\cap U}) : (U;\varphi) \in \mathfrak{C}(M)\}$.
2. Every **open** subset $O \subset X$ is a manifold with the atlas $(O; \mathrm{id}|_O)$.
3. The **torus group** $\mathbb{R}^n/\mathbb{Z}^n$ is a **compact** manifold with the atlas of the two charts $(]\boldsymbol{a}; \boldsymbol{a}+\mathbf{1}[\,; \pi_{\boldsymbol{a}})$ and $(]\boldsymbol{b}; \boldsymbol{b}+\mathbf{1}[\,; \pi_{\boldsymbol{b}})$ for $\boldsymbol{a} - \boldsymbol{b} \notin \mathbb{Z}^n$ and the bijections $\pi_{\boldsymbol{a}} :]\boldsymbol{a}; \boldsymbol{a}+\mathbf{1}[\to \mathbb{R}^n/\mathbb{Z}^n$ defined by $\pi_{\boldsymbol{a}}(\boldsymbol{x}) = \boldsymbol{x} + \mathbb{Z}^n$ and likewise for $\pi_{\boldsymbol{b}}$.

20.2 Submanifolds



A subset $N \subset M$ of a topological space M is **locally closed** iff every $n \in N$ has an **open neighborhood** $n \in U_n \subset M$ such that $U_n \cap N$ is **closed in** U_n. Hence N_1 and N_2 are locally closed but N_3 is not since every neighborhood of the closed endpoint e of its boundary $\delta N_3 = \overline{N}_3 \setminus N_3$ includes an open part of the upper half so that $U_e \cap N$ can never be closed in U_e. In that case $U_n \setminus N$ is **open** in U_n and therefore open in M. Consequently $\bigcup_{n\in N} U_n \setminus N \subset M \setminus N$ is open in M so that every locally closed N set is the **intersection** of an **open** set $\bigcup_{n\in N} U_n$ and a **closed** set $N \subset M \setminus \left(\bigcup_{n\in N} U_n \setminus N\right) = \bigcap_{n\in N} M \setminus (U_n \setminus N)$ in M. Since M is both closed and open we conclude that every open set and also every closed set are locally closed. The converse is not true since the open interval N_1 is locally closed but neither closed nor open in $\mathbb{R}^2$.

A Banach space Y **splits** iff there are two **closed** subspaces Y_1 and Y_2 such that $Y = Y_1 \oplus Y_2$. As a consequence of the **closed graph theorem** [16] the identity $Y_1 \times Y_2 \to Y_1 \oplus Y_2$ then is a **homeomorphism** such that the direct sum is furnished with the **product topology** and in particular every **open set** $O \subset Y_1 \oplus Y_2$ has the form $O = O_1 \times O_2$ with O_i open in Y_i. An **injective continuous linear** map $\boldsymbol{f} : X \to Y$ between topological vector spaces X and Y **splits** iff there exist topological vector spaces Y_1 and Y_2 with an **isomorphic homeomorphism** $\boldsymbol{\alpha} : Y \to Y_1 \oplus Y_2$ such that $\boldsymbol{\alpha} \circ \boldsymbol{f} : X \to Y_1$ is a **continuous isomorphism**. In the **finite-dimensional case** with $X = \mathbb{R}^m$ and $Y = \mathbb{R}^n$ with $n \leq m$ the passive basis vectors in a vector subspace $Y_1 \times \{\mathbf{0}\} \subset Y$ with $Y_1 = \operatorname{span}\left\{\boldsymbol{e}^1; ...; \boldsymbol{e}^m\right\}$ are sometimes explicitely written down in the form $\mathbf{0} = \sum_{i=n+1}^{m} 0 \cdot \boldsymbol{e}^i$ such that the **projection** has the form $\pi_1 : Y_1 \times Y_2 \to Y_1 \times \{\mathbf{0}\}$. In the finite dimensional case **every** injective continuous linear map splits since $Y_1 = \boldsymbol{f}[X]$ is a vector subspace and the **Steinitz basis exchange lemma** [17] implies the existence of the **complementary** space Y_1 such that the **product** $Y = Y_1 \times Y_2 = Y_1 \oplus Y_2$ is a **direct sum** with the representing matrix $\mathcal{M}^{\mathcal{A}}_{\mathcal{B}}(\boldsymbol{f}) = \begin{pmatrix} A \\ 0 \end{pmatrix} \in M(m \times n)$ and $A \in M(n \times n)$.

A subset M_1 of an X-manifold M is an X_1**-submanifold** of M iff $X = X_1 \oplus X_2$ **splits** and for every $m \in M_1$ exists a **chart** $(U;\varphi) \in \mathcal{C}_m M$ such that $\varphi[U] = U_1 \times U_2$ and $\varphi[U \cap M_1] = U_1$ for some **open** $U_i \subset X_i$. In particular for every **change of coordinates** $\psi \circ \varphi^{-1} : \varphi[U \cap V \cap M_1] \to \psi[U \cap V \cap M_1]$ between charts $(U;\varphi)$ and $(V;\psi)$ the intersection $U \cap V \cap M_1 \subset M$ is **homeomorphic** to its **open coordinate sets** $\varphi[U \cap V \cap M_1] \subset U_1 \subset X_1$ resp. $\psi[U \cap V \cap M_1] \subset X$. According to [17] the homeomorphy of the neighborhoods extends to the entire vector spaces X resp. X_1 whence the coordinates $\varphi[U \cap M_1]$ of M_1 on **every** chart $(U;\varphi) \in \mathcal{C}_m M$ lie in X_1. Since $\varphi[U] \setminus \varphi[U \cap M_1] = (U_1 \times U_2) \setminus U_1 = U_1 \times (U_2 \setminus \{\mathbf{0}\})$ is **open** in $\varphi[U]$ the set $\varphi[U \cap M_1]$ is **closed** in $\varphi[U]$ for every **open** $U \subset M$. Hence every intersection $U \cap M_1$ is **closed** in U and consequently the submanifold M_1 is **locally closed** in M. It is a **closed submanifold** if it is closed with regard to the topology of M.

Every X_1**-submanifold** M_1 of the X-manifold M with charts $(U;\varphi)$ for $\varphi : U \to \varphi[U] \subset X$ is itself an X_1**-manifold** with the charts $(U \cap M_1; \varphi \circ \iota_1)$ for the injection $\iota_1 : M_1 \to M$ with $\iota_1^{-1}[U] = \pi_1[U] = U \cap M_1$ for every subset $U \subset M$ since the **coordinate functions** $\varphi \circ \iota_1$ are still **injective** with the **parametrization** $\pi_1 \circ \varphi^{-1} : U_1 \to M_1$. For another chart $(V \cap M_1; \psi \circ \iota_1)$ the **change of coordinates** $(\psi \circ \iota_1) \circ (\varphi \circ \iota_1)^{-1} = \psi \circ \varphi^{-1} : \varphi[U \cap V \cap M_1] \subset U_1 \subset X_1 \to \psi[U \cap V \cap M_1] \subset V_1 \subset X_1$ is a **homeomorphism** on X_1. According to 4.3 The resulting topology $\mathcal{O}_{M_1 \subset M}$ of the X_1-submanifold M_1 is generated by the sets $U \cap M_1$ with **open** $\varphi[U] \subset X$ and **open** $\varphi[U \cap M_1] = U_1 = U_1 \times \{\mathbf{0}\} \subset X_1$ which are **closed** in $X = X_1 \times X_2$. It is also the **final topology** induced by the parametrizations $\varphi^{-1} : U_1 \to M_1$ for $(U;\varphi) \in \mathcal{C}_m(M)$ and $m \in M_1$. According to the **universal property** 4.1 and due to the continuous character of $\iota_1 \circ \varphi^{-1} : U_1 \to M$ these functions are also **continuous** with regard to the **trace topology** $\mathcal{O}_M \cap M_1$ generated by **all** sets $U \cap M_1$ with open $\varphi[U] \subset X$ which is also the **initial topology** induced by the injection $\iota_1 : M_1 \to M$. Hence the **submanifold topology** $\mathcal{O}_{M_1 \subset M} \subset \mathcal{O}_M \cap M_1$ is **weaker** than the **subspace topology** $\mathcal{O}_M \cap M_1$. It is also included in the topology $\mathcal{O}_{M_1}$ of the X_1**-manifold** which is the initial topology of **all** coordinate functions $\varphi_1 : U_1 \to X_1$ with $(U_1;\varphi_1) \in \mathcal{C}_m(M_1)$. Indeed since every $(U \cap M_1; \varphi \circ \iota_1) \in \mathcal{C}_m(M_1)$ we have $\mathcal{O}_{M_1 \subset M} \subset \mathcal{O}_{M_1}$ but the converse is not true since e.g. every open ball $B_\epsilon(\mathbf{0}) \subset \mathbb{R}^2$ contains three open pairwise disjoint subsets $U_i \subset L$ of the **lemniscate** L in 20.11 such that there is no open $U \subset M = \mathbb{R}^2$ with $\varphi^{-1}[U] = U_i \cap L$ whence $U_i \in \mathcal{O}_C \setminus (\mathbb{R}^2 \cap L)$ and therefore in general we have $\mathcal{O}_{M_1 \subset M} \subsetneq \mathcal{O}_{M_1}$.

$$\begin{tikzcd}
(M_1;\mathcal{O}_{M_1}) \arrow[r, phantom, "\supset"] \arrow[ddr, "\varphi_1"'] & (M_1;\mathcal{O}_{M \subset M_1}) \arrow[r, phantom, "\subset"] \arrow[dd, "\varphi \circ \iota_{M_1}"'] & (M_1;\mathcal{O}_M \cap M_1) \arrow[d, "\iota_{M_1}"] \\
 & & (M;\mathcal{O}_M) \arrow[d, "\varphi"] \\
 & \left(X_1;\mathcal{O}_{\|\,\|} \cap X_1\right) \arrow[r, "\iota_{X_1}"] & \left(X;\mathcal{O}_{\|\,\|}\right)
\end{tikzcd}$$

Due to 20.1 the two topologies coincide if we restrict the charts to **continuous coordinate functions** with respect to the **subspace topology** $\mathcal{O}_M \cap M_1$ which then becomes the **original topology** such that $\mathcal{O}_M \cap M_1 = \mathcal{O}_{M_1}$.

Note that according to 4.1 the initial topology $\mathcal{O}_M \cap M_1$ is **uniquely determined** by the **universal property**: Any map $f : N \to M_1$ between an Y**-manifold** N and a X_1**-submanifold** $M_1 \subset M$ of an X-manifold M on a proper subspace $X_1 \subsetneq X$ is **continuous** with regard to $\mathcal{O}_M \cap M_1$ iff its **injection** $\iota_1 \circ g : N \to M$ is **continuous** with regard to $\mathcal{O}_M$.

$$\begin{tikzcd}
 & (M;\mathcal{O}_M) \\
(N;\mathcal{O}_N) \arrow[ur, "\iota_1 \circ f"] \arrow[r, "f"] & (M_1;\mathcal{O}_M \cap M_1) \arrow[u, "\iota_1"']
\end{tikzcd}$$

20.3 Product manifolds

The **product** $(M \times N; \mathfrak{C}(M \times N))$ of the X-manifold $(M;\mathfrak{C}(M))$ and the Y-manifold $(N;\mathfrak{C}(N))$ is an $X \times Y$-manifold provided with charts $(U \times V; \varphi \times \psi)$ with the **product coordinates** $\varphi \times \psi = (\varphi;\psi) : U \times V \to X \times Y$. Obviously every such product is admissible to $\mathfrak{C}(M \times N)$ and conversely the two components $\eta_X;\eta_Y$ of every admissible chart $\eta = (\eta_X;\eta_Y) : U \times V \to X \times Y$ must have the form $\eta_X : U \to X$ resp. $\eta_Y : V \to Y$ whence follows $\mathfrak{C}(M \times N) = \mathfrak{C}(M) \times \mathfrak{C}(N)$. Consequently the **initial**

topology induced on $M \times N$ by the product coordinates coincides with its **product topology**. The standard example of a product manifold is given by the **torus** 20.16.

20.4 Quotient manifolds

For a **fixed-point-free involution** $\tau : M \to M$ on an X-manifold $(M; \mathfrak{C}(M))$, i.e. a **continuous bijection** with $\tau \circ \tau = \mathrm{id}_M$, $\tau(m) \neq m \; \forall m \in M$ and for every chart $(U; \varphi)$ of $m \in M$ exists a neighborhood $m \in U_0 \subset U$ such that $U_0 \cap \tau[U_0] = \emptyset$ the **quotient** M/R with $R = \{(m; \tau(m)) : m \in M\} \subset M \times M$ is an X-manifold. In that case its maximal atlas $\mathfrak{C}(M/R)$ is **uniquely determined** by the property that for every $m \in M$ exists a neighborhood $U(m)$ such that the **canonical projection** $\pi : U \to \pi[U] \subset M/R$ is a **local homeomorphism**. Furthermore if the **original topology** $\mathcal{O}$ on M with a restriction to **$\mathcal{O}$-continuous charts** $(U; \varphi)$ such that $\mathcal{O}_M \subset \mathcal{O}$ is **second countable** and **Hausdorff** the **final topology** $\mathcal{O}_{M/R}$ of the quotient space M/R is also **second countable** and **Hausdorff**.

$$\begin{tikzcd} & M \arrow[dl, "\pi"'] \arrow[dr, "\pi"] & \\ (M/\tau)_1 \arrow[rr, "\mathrm{id}"] & & (M/\tau)_2 \end{tikzcd}$$

Proof: The **uniqueness** of an atlas $\mathfrak{A}$ on M/R for which $\pi : (M; \mathfrak{C}(M)) \to (M/R; \mathfrak{A})$ is a local homeomorphism follows from the fact that the identity $\mathrm{id} : (M/R; \mathfrak{A}_1) \to (M/R; \mathfrak{A}_2)$ between two possible atlases with the required property regarding the projection π must be a local and hence global homeomorphism. Due to the hypothesis for every chart $(U; \varphi)$ of $m \in M$ exists a neighborhood $m \in U_0 \subset U$ such that $U_0 \cap \tau[U_0] = \emptyset$. Hence the map $\varphi_0 : \pi[U_0] \to \varphi[U_0]$ defined by $(\varphi_0 \circ \pi)(m) = \varphi(m)$ is bijective with $\varphi_0^{-1}(x) = m \in U_0 : \varphi(m) = x$ and for every other chart $(V; \psi)$ of $m \in M$ with $m \in V_0 \subset V$ such that $V_0 \cap \tau[V_0] = \emptyset$ the coordinate change $\psi_0 \circ \varphi_0^{-1} : \varphi[U_0 \cap V_0] \to \psi[U_0 \cap V_0]$ is a **homeomorphism**. Hence $(\pi[U_0]; \varphi_0)$ is a chart at $\pi(m)$.

The **Hausdorff-property** of $(M; \mathcal{O})$ extends to $\left(M/R; \mathcal{O}_{M/R}\right)$ since in that case for any two distinct points $\pi(m) \neq \pi(n)$ exist neighborhoods U of m resp. V of n such that $U \cap V = U \cap \tau[V] = \emptyset$ whence follows $\pi[U] \cap \pi[V] = \emptyset$. Likewise the **second countability** is carried over to the quotient space since for the countable basis $(U_n)_{n \geq 1}$ of $\mathcal{O}$ the sequence $(\pi[U_n])_{n \geq 1}$ is a basis of $\mathcal{O}_{M/R}$.

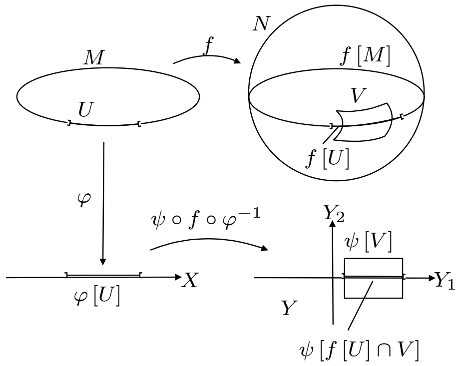
20.5 Immersions

A map $f : M \to N$ between an X-manifold M and a Y-manifold N is an **immersion at** $m \in M$ iff one of the following equivalent conditions is satisfied:

1. The vector space $Y = Y_1 \oplus Y_2$ **splits** and there are charts $(U; \varphi) \in \mathcal{C}_m M$ and $(V; \psi) \in \mathcal{C}_n N$ with $n = f(m)$ and $\varphi[U] = \psi[V] \subset Y_1$ such that the **downstairs map** $\psi \circ f \circ \varphi^{-1} : \varphi[U] \to \psi[V]$ is an **inclusion**.
2. There is a chart $(U; \varphi) \in \mathcal{C}_m M$ such that $f : U \to f[U]$ is a **homeomorphism** and its image $f[U] \subset N$ is a **Y_1-submanifold** on a **subspace** $Y_1 \subset Y$**.**

Both conditions imply that f is **continuous** and that X is **homeomorphic** to Y_1. The map $f : M \to N$ is an **immersion** iff it is an **immersion at every** $m \in M$ Every **X_1-submanifold** $M \subset N$ of an X-manifold N admits the **injection** $\iota : M \to N$ as a trivial **immersion**. The **local injectivity** does not imply **global injectivity** since two charts $(U_1; \varphi_1) \in \mathcal{C}_{m_1} M$ and $(U_2; \varphi_2) \in \mathcal{C}_{m_2} M$ with disjoint coordinate sets $U_1 \cap U_2 = \emptyset$ may contain a common crossing point $f(m_1) = f(m_2)$. The **lemniscate** C in 20.11 shows that even a **continuous bijection** $f : M \to f[M]$ may not be an **open map** such that the X-manifolds M and $f[M]$ may not be homeomorphic to each other. However due to 4.8 **every immersion is a quotient map** whence $M/R_f \cong f[M]$ via the **equivalence relation** given by $mR_f n \Leftrightarrow f(m) = f(n)$. An immersion which is also a **homeomorphism** is an **embedding**and in that case $f[M]$ is an **Y_1-submanifold**.

Proof:

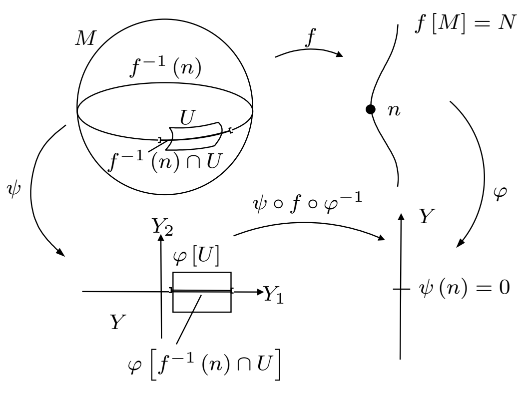


1. $\Rightarrow$ 2.: By the hypothesis there are charts $(U;\varphi) \in \mathcal{C}_m M$ and $(V;\psi) \in \mathcal{C}_n N$ with $n = f(m)$ such that the **downstairs map** $\psi \circ f \circ \varphi^{-1} : \varphi[U] \to \psi[V]$ coincides with the **inclusion** $\iota : X \to Y_1 \oplus Y_2$ and since we can assume $\psi[f[U] \cap V] \subset Y_1$ the map $\psi \circ f \circ \varphi^{-1} : \varphi[U] \to \psi[f[U] \cap V]$ coincides with the **identity** $\mathrm{id} : X \to Y_1$. Hence $f = \psi^{-1} \circ \mathrm{id} \circ \varphi = \psi^{-1} \circ \varphi : U \to f[U] \cap V$ is a **homeomorphism** such that we can assume $f[U] \subset V$ and for every chart $(W;\eta) \in \mathcal{C}_n N$ follows $\eta[f[U] \cap W] = (\eta \circ \psi^{-1} \circ \psi)[f[U] \cap W] = (\eta \circ \psi^{-1})(\psi[f[U] \cap W]) \subset Y_1$ since $\psi[f[U] \cap W] \subset Y_1$ and the transition function $\eta \circ \psi^{-1}$ provides a local homeomorphism which according to [16] extends to the entire vector space Y_1. Hence $f[U]$ is a Y_1**-submanifold**.

2. $\Rightarrow$ 1.: According to the hypothesis there are charts $(U_0;\varphi) \in \mathcal{C}_m M$ and $(V_0;\eta) \in \mathcal{C}_n N$ with $n = f(m)$ and $\eta[V_0] = V_1 \times V_2$ for some open $V_i \subset Y_i$ with $i \in \{1;2\}$ in the product $Y_1 \oplus Y_2 = Y$ such that $\eta[f[U_0] \cap V_0] \subset V_1$. The map $\varphi \circ f^{-1} : f[U_0] \to \varphi[U_0] \subset X$ is a homeomorphism into a vector space X and its restriction $\psi = (\varphi \circ f^{-1})|_V$ on the open set $V = f[U_0] \cap V_0$ is a **chart** on the Y_1-submanifold $f[U_0]$ whence X is **isomorphic** to Y_1. Hence for the open set $U = U_0 \cap f^{-1}[V_0]$ the downstairs map $\psi \circ f \circ \varphi^{-1} : \varphi[U] \to \psi[V] \subset X$ coincides with the **identitiy** and due to $X \oplus Y_2 \cong Y_1 \oplus Y_2 = Y$ we can write it as an **inclusion** into Y.

20.6 Submersions

A map $f : M \to N$ is a **submersion at** $m \in M$ iff there are charts $(U;\varphi) \in \mathcal{C}_m M$ and $(V;\psi) \in \mathcal{C}_n N$ at $n = f(m)$ such that $X = X_1 \oplus X_2$ **splits** and the downstairs map $f_{U;V} : \varphi[U] \to \psi[V]$ coincides with the **projection** $\pi_1 : X_1 \oplus X_2 \to X_1$. In that case the restriction $f : U \to V$ is a **continuous open** map and the preimage $f^{-1}(n)$ is an X_2**-submanifold** of M .



Proof: By the hypothesis there are charts $(U;\varphi) \in \mathcal{C}_m M$ and $(V;\psi) \in \mathcal{C}_n N$ with $n = f(m)$ such that the **downstairs map** $\psi \circ f \circ \varphi^{-1} : \varphi[U] \to \psi[V]$ coincides with the **projection** $\pi_1 : X_1 \oplus X_2 \to X_1$. This implies $f[U] = V$ and in particular $f[U]$ is **open** in N. For every open $O \subset M$ the coordinate set $\varphi[U \cap O]$ is **open** in X, according to 4.2 its **projection** $(\psi \circ f)[U \cap O] = (\pi_1 \circ \varphi)[U \cap O]$ is **open** in Y, whence $f[U \cap O]$ is open in N so that we have shown that f is an **open map**. By 4.2 there are open $U_i \subset X_i$ for $i \in \{1;2\}$ such that $\varphi[U] = U_1 \times U_2$. For every open $W \subset N$ the coordinate set $\psi[V \cap W]$ is **open** in Y, its homeomorphic preimage $\varphi[f^{-1}[V \cap W]] \subset U_1$ is **open** in X_1 whence $\varphi[f^{-1}[V \cap W]] \times U_2 = \varphi[f^{-1}[V \cap W] \cap U]$ is **open** in X and finally $f^{-1}[V \cap W] \cap U$ is **open** in M with $f[f^{-1}[V \cap W] \cap U] \subset V \cap W$ which proves that $f : U \to V$ is **continuous**. By a simple **translation** we can assume $\psi(n) = \mathbf{0}$ such that the bijective character of the coordinate functions entails $\varphi[U \cap f^{-1}(n)] = \ker f_{U;V} \subset U_2 \subset X_2$ whence $f^{-1}(n)$ is an X_2**-submanifold** of M.

20.7 Euclidean manifolds

Every **Euclidean manifold** over $\mathbb{R}^n$ is **locally compact** with a **countable** basis of **regular coordinate balls** , i.e. **precompact open** sets $U \cong \mathbb{B}^n$ with a neighborhood $V \supset \bar{U}$ such that $V \cong (1+\epsilon)\mathbb{B}^n$ and **compact closure** $\bar{U} \cong \bar{\mathbb{B}}^n$ since the coordinate vector $\varphi(m)$ of every point $m \in M$ in a chart $(U;\varphi)$ has an open neighborhood $\varphi^{-1}[B_\epsilon(\boldsymbol{x})]$ with rational radius $\epsilon \in \mathbb{Q}$ and rational centre $\boldsymbol{x} \in \mathbb{Q}^n$.

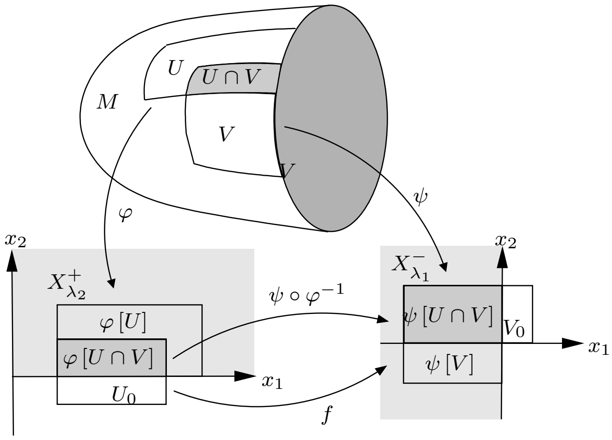
20.8 Compact manifolds

For every **compact Euclidean manifold** M over $\mathbb{R}^n$ exists a $k \in \mathbb{N}$ such that M is homeomorphic to a compact subset $K \subset \mathbb{R}^{nk+k}$.

Proof: By the hypothesis there is a finite cover $(U_i; \varphi_i)_{1 \le i \le k}$ of charts with open coordinate sets $\varphi[U_i] \subset \mathbb{R}^n$ and due to 8.5 and 9.5 exists a subordinate **partition of unity** $(\psi_i)_{1 \le i \le k}$, such that the composition $F = (\varphi_1 \cdot \psi_1; ...; \varphi_k \cdot \psi_k; \psi_1; ...; \psi_k) : M \to \mathbb{R}^{nk+k}$ is **continuous**. Due to $\sum_{1 \le i \le n} \psi_i = 1$ or every $m, n \in M$ with $F(m) = F(n)$ there is a $1 \le j \le k$ with $\psi_j(n) = \psi_j(m) > 0$ whence follows $m; n \in U_j$. Since we also have $\varphi_j(m) = \varphi_j(n)$ and the coordinate functions are bijective this implies $m = n$. Hence F is injective and the **closed map lemma** enu:9.8.3 follows that F is a topological embedding onto the compact subset $F[M] \subset \mathbb{R}^{nk+k}$.

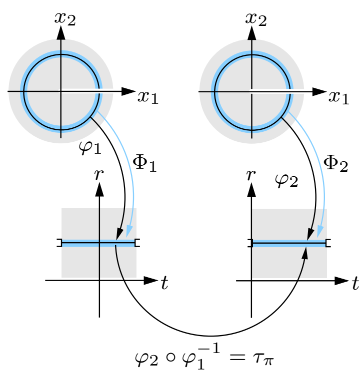
20.9 Manifolds with boundary

The charts $(U; \varphi)$ of an X-**manifold with boundary** M over a Banach space X consist of subsets $U \subset M$ and coordinate functions $\varphi : U \to \varphi[U] \subset X^+_\lambda$ into **positive halfspaces** $X^+_\lambda = \{\mathrm{Re}\lambda \ge 0\}$ of some **functional** $\lambda \in X^*$ such that the **transition map** $\psi \circ \varphi^{-1} : \varphi[U \cap V] \to \psi[U \cap V]$ has a **homeomorphic extension** $f : U_0 \to V_0$ between **open** sets $U_0; V_0 \subset X$ satisfying $U_0 \cap X^+_\lambda = \varphi[U \cap V]$ and $V_0 \cap X^+_\lambda = \psi[U \cap V]$ with $f|_{\varphi[U \cap V]} = \psi \circ \varphi^{-1}$ for every pair of charts $(U; \varphi)$ and $(V; \psi) \in \mathfrak{A}$. These charts include **interior charts** as defined above with $\varphi[U]$ **open** in X and **boundary charts** with $\varphi[U]$ **open** in X^+_λ with $\varphi[U] \cap X^0_\lambda \neq \emptyset$ for the **kernel** $X^0_\lambda = \{\lambda = 0\}$ of some functional $\lambda \in X^*$. Note that the hyperplane X^+_λ is a **closed vector subspace** of X and that the case $\lambda = 0$ results in $X^+_\lambda = X$ and hence an X-**manifold** without boundary. The **interior** Int M consists of all points $m \in M$ with an interior chart at m while the remaining set is called the **manifold boundary** $\partial M = M \setminus \mathrm{Int}\, M$. Hence all boundary points $m \in \partial M$ have a boundary chart $(U; \varphi)$ at m such that $\varphi(m) \in X^0_\lambda$ for some hyperplane X^0_λ. The converse is far from obvious but still true as will be shown in 26.2. Note that the hyperplane $X^0_\lambda = \delta X^\pm_\lambda$ is the topological boundary of the corresponding half planes $X^\pm_\lambda$ but that the existence of a **topological boundary** δM of a manifold with a boundary itself and generally its possible topology depend on the choice of the space of which it may be a subset. For example the **closed disk** $\bar{B}^2$ may be considered as an $\mathbb{R}^2$-manifold with the **manifold boundary** $\partial \bar{B}^2 = S^1$ but its topological boundary may vary between $\delta \bar{B}^2 = \emptyset$ if $\bar{B}^2 \subset \bar{B}^2$ is regarded as a subset of itself, $\delta \bar{B}^2 = S^1 = \partial \bar{B}^2$ in the case of $\bar{B}^2 \subset \mathbb{R}^2$ and even $\partial \bar{B}^2 = \bar{B}^2$ for $\bar{B}^2 \subset \mathbb{R}^3$.



20.10 The circle

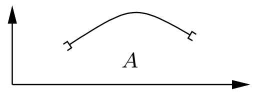
The **circle** $\mathbb{S}^1 = \partial \mathbb{B}^2$ can be obtaind by **gluing** together any number of open intervals, e.g. **both ends of the interval** $I = \mathbb{B}^1 =]-1; 1[$ in the **quotient space** I/R with $R = \{(t; t) : -1 < t < 1\} \cup \{(t-1; t) : 0 < t < 1\}$. Since due to its construction from equivalence classes containing up to **two pairs** of elements the quotient space needs at least **two charts** for covering. For easier parametrization by $R = \left\{(t; t) \in (I_1 \cup I_2)^2\right\} \cup \left\{(t - 2\pi; t) \in (I_1 \Delta I_2)^2\right\}$ we **glue** together the **two intervals** $I_1 =]-\pi; \pi[$ and $I_2 =]0; 2\pi[$ so that the **two charts** $(\pi[I_i]; \pi^{-1})$ given by the restrictions of the **canonical projection** $\pi : I_i \to (I_1 \cup I_2)/R$ result in the $\mathbb{R}$-**manifold**



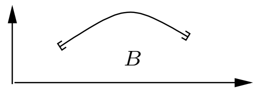
$(I_1 \cup I_2)/R$. Alternatively the **two charts** $(\varphi[I_i]; \varphi^{-1})$ with the **parametrizations** $\varphi(t): I_i \to \mathbb{S}^1 \subset \mathbb{R}^2$ given by $\varphi(t) = (\cos t; \sin t)$ on I_i for $i \in \{1; 2\}$ produce the **circle** $\mathbb{S}^1$ as an $\mathbb{R}$**-manifold** in $\mathbb{R}^2$. It can also be represented as a **closed** $\mathbb{R}$**-submanifold** of the **manifold** $\mathbb{R}^2$ given by $(\mathbb{R}^2; \mathrm{id})$. By the additional **charts** $\left(U_\epsilon(\mathbb{S}^1) \setminus \left(\{0\} \times \mathbb{R}_0^\pm\right); \psi_i^{-1}\right)$ with the ϵ**-neighbourhood** $U_\epsilon(\mathbb{S}^1) = \{\boldsymbol{x} \in \mathbb{R}^2 : d(\boldsymbol{x}; \mathbb{S}^1) < \epsilon\}$ for any $0 < \epsilon < 1$ and $\psi : B_\epsilon(1) \times I_i \to \mathbb{R}^2$ given by $\psi(r; t) = (r \cos t; r \sin t)$ we obtain $\psi^{-1}[\mathbb{S}^1 \cap U_\epsilon(\mathbb{S}^1)] = \{0\} \times I_i \subset \{0\} \times \mathbb{R}$. The **initial topology** $\mathcal{O}_{\mathbb{S}^1}$ generated by the **subbasis** sets $\varphi[U_i \cap I_i]$ for **open** $U_i \subset \mathbb{R}$ and $i \in \{1; 2\}$ with respect to the **charts** $(\varphi[I_i]; \varphi^{-1})$ **into** $\mathbb{R}$ coincides with the **trace topology** $\mathcal{O}_{\mathbb{R}^2} \cap \mathbb{S}^1$ on $\mathbb{S}^1 \cap \mathbb{R}^2$, i.e. with the **initial topology** with regard to the **injection** $\iota : \mathbb{S}^1 \to \mathbb{R}^2$. The existence of the $\mathbb{R}$-submanifold $\mathbb{S}^1 \subset \mathbb{R}^2$ also implies the trivial **embedding** $\mathrm{id}_{\mathbb{S}^1} : \mathbb{S}^1 \to \mathbb{S}^1 \subset \mathbb{R}^2$ of the $\mathbb{R}$**-manifold** S **onto** the $\mathbb{R}$**-submanifold** $\mathbb{S}^1$. Finally $\mathbb{S}^1$ can be obtained as an $\mathbb{R}$-$\mathbb{R}^2$**-embedding** $f : (I_1 \cup I_2)/R \to \mathbb{S}^1$ given by $\varphi_i^{-1} \circ f \circ \pi_i = \mathrm{id}_{I_i}$. resp. $f \circ \pi_i = \varphi_i$. In a geometric sense the equivalence classes can be made visible by expanding them into two dimensions; conversely the circle can be folded into a quotient space.

20.11 Lines in space

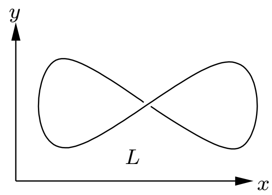
An **atom** $\{\boldsymbol{a}\} \subset X$ is an $\mathbb{R}^0$**-manifold** and a **closed** $\mathbb{R}^0$**-submanifold** of $\mathbb{R}$.

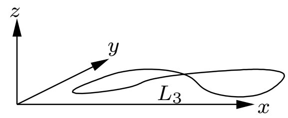


Every **open interval** $I \subset \mathbb{R}$ is an $\mathbb{R}$**-manifold** and an $\mathbb{R}$**-submanifold** of $\mathbb{R}^2$ whereas a **closed interval** is a **closed** $\mathbb{R}$**-submanifold** of $\mathbb{R}^2$. In its own right a closed interval is an $\mathbb{R}$**-manifold with a boundary** as defined in 20.9.



Every **curve segment** $\boldsymbol{a} + I\boldsymbol{b} = \{\boldsymbol{a} + \boldsymbol{b}(t) : t \in I\}$ with $\boldsymbol{a} \in X$ and $\boldsymbol{b} \in \mathcal{C}^r(I; X)$ is an $\mathbb{R}$**-submanifold** of X and in the case of an **open** interval $I \subset \mathbb{R}$ as in example A it is an $\mathbb{R}$**-manifold**. Example B shows an $\mathbb{R}$**-manifold with a boundary** resp. a **closed** $\mathbb{R}$**-submanifold** of $\mathbb{R}^2$.



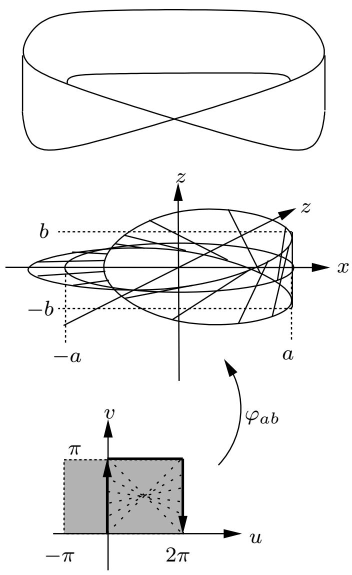


By the single chart $(I; \varphi)$ with $I =]0; 2\pi[$ and parametrization $\varphi_2^{-1}(t) = (\sin t; \sin 2t)$ we obtain the **lemniscate** L from 20.2 as an $\mathbb{R}$**-manifold** with the topology $\mathcal{O}_L$ consisting of all parametrizations $\varphi_2^{-1}[V] \subset L$ of **open coordinate sets** $V \subset I$. In this topology the neighborhoods $\varphi_2^{-1}[U_\epsilon(\pi)]$ of the origin $\varphi_2^{-1}(\pi) = \mathbf{0} \in L$ are isomorphic to **open intervals** in $\mathbb{R}^2$ so that they are **locally closed** but neither **open** nor **closed** in $\mathbb{R}^2$. Every neighborhood $U_\epsilon(\mathbf{0}) \subset \mathbb{R}^2$ contains **disjoint** sections $\varphi_2[U_\epsilon(0)]$ resp. $\varphi_2[U_\epsilon(2\pi)]$ of the **tails** approaching $\mathbf{0}$ such that there is no open $V \subset \mathbb{R}^2$ with $\varphi_2^{-1}[U_\epsilon(\pi)] = V \cap L$. Hence the neighborhood $\varphi_2^{-1}[U_\epsilon(\pi)]$ is not open in the **trace topology** $\mathcal{O}_{\mathbb{R}^2} \cap L$ of the **closed** subset $L \subset \mathbb{R}^2$ such that φ_2 is **not continuous** with regard to $\mathcal{O}_{\mathbb{R}^2} \cap L \subsetneq \mathcal{O}_L$. Consequently the parametrization $\varphi_2^{-1} : I \to L$ from the $\mathbb{R}$-manifold I with the chart $(I; \mathrm{id})$ into the $\mathbb{R}$-manifold L with the chart $(L; \varphi_2)$ and **downstairs map** $\varphi_2 \circ \varphi_2^{-1} \circ \mathrm{id}_I = \mathrm{id}_I$ is **globally injective** and **continuous** but **not open** at π. It is an **immersion** but not an **embedding**. Therefore resp. due to $\mathrm{id}[U_\epsilon(\mathbf{0}) \cap L] \subsetneq \mathbb{R}$ the set L is **not** an $\mathbb{R}$**-submanifold** of $\mathbb{R}^2$ with the chart $(\mathbb{R}^2; \mathrm{id}) \in \mathfrak{C}(\mathbb{R}^2)$ and hence generally for $\mathfrak{C}(\mathbb{R}^2)$. By extending the parametrization to $\varphi_3^{-1} : I \to \mathbb{R}^3$ with $\varphi_3^{-1}(t) = (\sin t; \sin 2t; \cos t)$ the limit point $\varphi_2(\pi)$ is **removed into the third dimension** such that we obtain $\mathbb{R}$**-submanifold** L_3 of $\mathbb{R}^3$ with the **embedding** $\varphi_3^{-1} : I \to L$.

Similarly the union of two **coordinate axes** $X = \mathbb{R} \times \{0\} \cup \{0\} \times \mathbb{R} \subset \mathbb{R}^2$ is a **disconnected** $\mathbb{R}$**-manifold** with coordinate functions $\psi^x : \mathbb{R} \times \{0\} \to I_x =]-1; 1[$ given by $\psi^x(x) = \frac{x}{|x|+1}$ and $\psi^{\pm y} : \{0\} \times \mathbb{R}^\pm \to I_y^\pm =]\pm 2; \pm 3[$ with $\psi^{\pm y}(x) = \psi^x(x) \pm 2$. It has **three connected components** and the corresponding **immersion** $\psi^{-1} : I_x \cup I_y^+ \cup I_y^- \to X$ is **globally injective** and **continuous** but **not open** at 0 whence it is **not** an **embedding** and X is **not** an $\mathbb{R}$**-submanifold** of $\mathbb{R}^2$. Adding a **third dimension** we obtain a **disconnected** $\mathbb{R}$**-manifold** with coordinate functions $\eta^\pm : X' = \mathbb{R} \times \{0\} \times \{\pm 1\} \to \mathbb{R}^\pm$ given by $\eta^\pm(x) = \pm e^x$ on **two connected components** resp. an **embedding** $\eta^{-1} : \mathbb{R} \setminus \{0\} \to X'$ onto the corresponding $\mathbb{R}^2$**-submanifold** of $\mathbb{R}^3$.

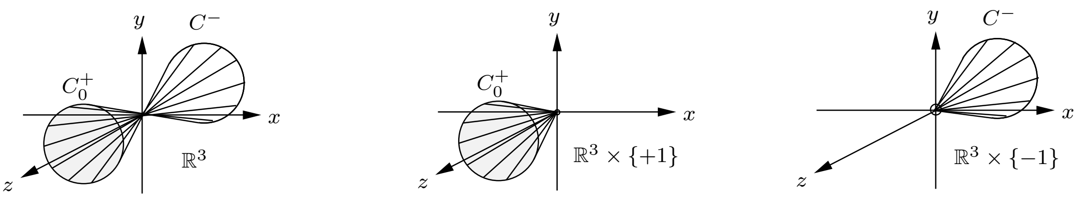
20.12 The Möbius strip

The following examples are conveniently parametrized in **polar coordinates** taken from rectangular products of the intervals $I =]0;1[$; $I_1 =]-\pi;\pi[$; $I_2 =]0;2\pi[$ and $J = \frac{1}{2}I_2$. The **Möbius strip** can be described as the **compact** set $\bar{M}_{ab} = \varphi_{1ab}\left[\bar{I}_1 \times \bar{J}\right] \subset \mathbb{R}^3$ parametrized by $\varphi_{wab}(u;v) = \varphi_{ab}(u;v;w) = ((wa + vb\sin u)\cdot\cos 2u; (wa - vb\sin u)\cdot\sin 2u; vb\cos u)$ with radii $a > b > 0$ and the additional parameter $1-\epsilon < w < 1+\epsilon$ which will be used later for the embedding. According to enu:4.8.2 this parametrization is a **quotient map** whence $\bar{M}_{ab}$ is homeomorphic to the **quotient space** $\left(\bar{I}_1 \times \bar{J}\right)/R_\varphi$ identifying the **two vertical sides** of the **open rectangle** $\bar{I}_1 \times \bar{J}$ with each other in **inverted sense** since $(0;v)\,R_\varphi\,(2\pi;1-v)$ for $v \in \bar{J}$. The set $M_{ab} = \varphi_{1ab}\left[\bar{I}_1 \times J\right]$ is also an $\mathbb{R}^2$**-manifold** with **two charts** $\left(\varphi_{ab}[I_i \times J];\varphi_{ab}^{-1}\right)$ for $i \in \{1;2\}$ which is neither open nor closed but **locally closed** in $\mathbb{R}^3$. Similarly to 20.14 it can be **embedded** as an $\mathbb{R}^2$**-submanifold** in $\mathbb{R}^3$ with the canonical chart $(\mathbb{R}^3;\mathrm{id})$ and the **additional two charts** $\left(\varphi_{ab}[I_i \times J \times B_\epsilon(1)];\varphi_{ab}^{-1}\right)$ of $\mathbb{R}^3$ such that $\varphi_{ab}^{-1}[\varphi_{ab}[I_i \times J \times B_\epsilon(1)] \cap M_{ab}] = I_i \times J \times \{1\} \subset \mathbb{R}\times\mathbb{R}\times\{1\} \cong \mathbb{R}^2$**.** Its **closure** $\bar{M}_{ab}$ is an $\mathbb{R}^2$**-manifold with boundary** with the **interior charts** $\left(\varphi_{ab}[I_i \times J];\varphi_{ab}^{-1}\right)$ and **four boundary charts** $\left(\varphi_{ab}[I_i \times]\frac{\pi}{2};\pi]];\varphi_{ab}^{-1}\right)$ resp. $\left(\varphi_{ab}[I_i \times [0;\frac{\pi}{2}[];\varphi_{ab}^{-1}\right)$.



20.13 The double cone

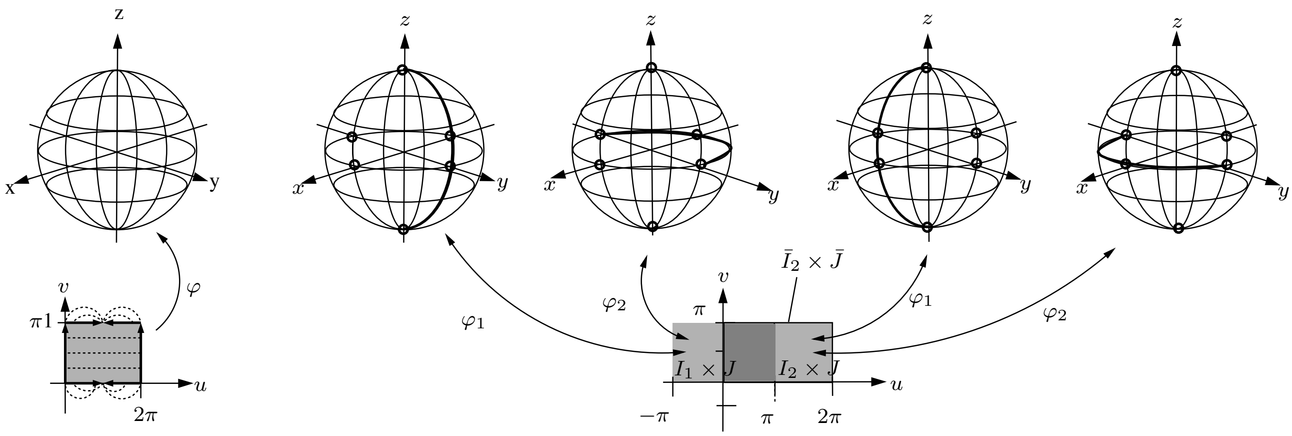
The **double cone** $C = \{z^2 = x^2 + y^2\} = C^+ \cup C_*^- \subset \mathbb{R}^3$ with $C^+ = \{(x;y;z) : z \geq 0\}$ and $C_*^- = \{(x;y;z) : z < 0\}$ by the charts $\left(C^+;\eta^+ \circ \pi_{xy}^+\right)$ and $\left(C_*^-;\eta^- \circ \pi_{xy}^-\right)$ with the **projections** $\pi_{xy}^\pm : C^\pm \to \mathbb{R}^2$ defined by $\pi_{xy}^\pm(x;y;z) = (x;y)$, the **contraction** $\eta^+ : \mathbb{R}^2 \to \mathbb{B}^2$ with $\eta^+(\boldsymbol{x}) = \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|+1}$ and the **dilation** $\eta^- : \mathbb{R}_*^2 \to \mathbb{R}^2 \setminus \bar{\mathbb{B}}^2$ with $\eta^-(\boldsymbol{x}) = (\|\boldsymbol{x}\|+1)\cdot\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}$ can be represented as a **disconnected** $\mathbb{R}^2$**-manifold** with **two connected components**. Note that the two cones C^+ and C_*^- are disconnected with regard to $\mathcal{O}_C$ but not in $\mathcal{O}_{\mathbb{R}^3} \cap C$.



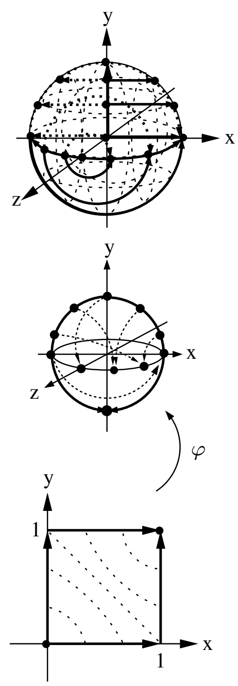
The corresponding **immersion** $\left(\eta^\pm \circ \pi_{xy}^\pm\right)^{-1} : \mathbb{R}^2 \setminus \mathbb{S}^1 \to C$ is **injective** and **continuous** but **not open** since it maps the open disk $\epsilon\mathbb{B}^2$ onto the subset $\left(\eta^+ \circ \pi_{xy}^+\right)^{-1}[\epsilon\mathbb{B}^2] = (\delta\mathbb{B}^2 \times [0;\delta[) \cap C$ with $\delta = \frac{\epsilon}{1-\epsilon}$ which is not open in C. By removing the two parts from each other in the fourth (temporal) dimension we may construct another **disconnected**$\mathbb{R}^2$**-manifold** $C_4 = C_*^- \times \{-1\} \cup C^+ \times \{1\}$ with the two charts $\left(C^\pm;\eta^\pm \circ \pi_{xy}^\pm\right)$ into the same connected components as above. In this space the cones are disconnected with respect to both topologies $\mathcal{O}_C = \mathcal{O}_{\mathbb{R}^4} \cap C$ and the map $\left(\eta^\pm \circ \pi_{xy}^\pm\right)^{-1} : \mathbb{R}^2 \setminus \mathbb{S}^1 \to C_4$ is an **embedding** since the image of the critical neighborhood $\left(\eta^+ \circ \pi_{xy}^+\right)^{-1}[\epsilon\mathbb{B}^2] = (\delta\mathbb{B}^2 \times \mathbb{R}^2) \cap C_4$ with $\delta = \frac{\epsilon}{1-\epsilon}$ is **open** in C_4.

20.14 The two-dimensional sphere

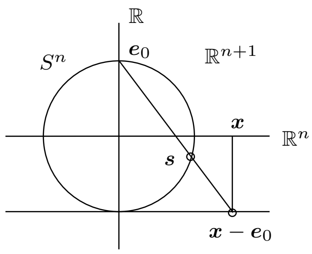
Due to the **closed map lemma** 2 the **polar coordinates** $\varphi_1 : \bar{I}_2 \times \bar{J} \to \mathbb{S}^2$ given by $\varphi_1(u;v)$ $(\cos u \cdot \sin v; \sin u \cdot \sin v; \cos v)$ define a **quotient map** whence the two-dimensional **sphere** $\mathbb{S}^2$ is homeomorphic to the **quotient space** $\left(\bar{I}_2 \times \bar{J}\right)/R_{\varphi_1}$ with $(0;v)\,R_{\varphi_1}\,(2\pi;v)$ and $(u;0)\,R_{\varphi_1}\,(w;\pi)$ for $0 < u < 2\pi$ and $0 \leq u \leq w \leq 0$ identifying the opposite points on the vertical sides and collapsing the horizontal sides each into one pole $(0;0;\pm 1)$. Since the corresponding equivalence classes contain infinitely many points there is no finite cover $\bar{I}_2 \times \bar{J} = \bigcup_{1 \leq i \leq k} U_k$ with **injective** restrictions of the canonical projections $\pi|_{U_i} \to \mathbb{S}^2$ and consequently no direct representation of the quotient space $\left(\bar{I}_2 \times \bar{J}\right)/R_{\varphi_1}$ as a manifold. The corresponding $\mathbb{R}^2$**-manifold** $\mathbb{S}^2$ can be covered by **four charts** $\left(\varphi_j\left[I_i \times J\right]; \varphi^{-1}\right)$ with $\varphi_2(u;v)$ $(\cos u \cdot \sin v; \cos v; \sin u \cdot \sin v)$ and the **closed** $\mathbb{R}^2$**-submanifold** $\mathbb{S}^2$of $\mathbb{R}^3$ is realized by the additional chart $\left(\psi_j\left[I_i \times J \times B_\epsilon(1)\right]; \psi^{-1}\right)$ of $\mathbb{R}^3$ with $\psi_j(u;v;r) = r \cdot \varphi_j(u;v)$ such that $\psi_j^{-1}\left[\psi_j\left[I_i \times J \times B_\epsilon(1)\right] \cap \mathbb{S}^2\right] = I_i \times J \times \{1\} \subset \mathbb{R} \times \mathbb{R} \times \{1\} \cong \mathbb{R}^2$.



Alternatively the quotient map $\eta : \bar{\mathbb{B}}^2 \to \mathbb{S}^2$ given by $\eta(x;y) = \left(-r_y \cdot \cos\left(\frac{\pi x}{r_y}\right); -r_y \cdot \sin\left(\frac{\pi x}{r_y}\right); y\right)$ with $r_y = \sqrt{1-y^2}$ for $y \neq 1$ and $\eta(0;1) = (0;0;1)$ provides the homeomorphic quotient space $\bar{\mathbb{B}}^2/R_\eta$. Since η maps the **upper** left and right quarter circles $\varphi_\pm(u) = (\pm\cos u; \sin u; 0)$ with $0 \leq u \leq \frac{\pi}{2}$ of $\bar{\mathbb{B}}^2$ onto the front **right** quarter circle $\eta \circ \varphi(u) = (\sin u; 0, \cos u)$ of $\mathbb{S}^2$ while the corresponding **lower** left and right quarter circles with $-\frac{\pi}{2} \leq u \leq 0$ are folded onto the back **right** quarter circle $\eta \circ \varphi(u) = (\cos u; 0; \sin u;)$ we conclude that $-\boldsymbol{x} R_\eta \boldsymbol{x}$ for every $\boldsymbol{x} \in \mathbb{S}^1$. Since due to 9.11 the **closed square** $\bar{I}^2$ is homeomorphic to the **closed unit ball** $\bar{\mathbb{B}}^2$ with $\partial\bar{I}^2 \cong \partial\bar{\mathbb{B}}^2 = \mathbb{S}^1$ the quotient space $\bar{\mathbb{B}}^2/R_\eta$ by some $\bar{\vartheta} : \bar{I}^2/R \cong \bar{\mathbb{B}}^2/R_\eta$ is also homeomorphic to the corresponding quotient space $\bar{I}^2/R$ with $(u;0)\,R\,(0;u)$ and $(u;1)\,R\,(1;u)$ for $0 \leq u \leq 1$ identifying **adjacent sides** in a **corresponding sense**. The equivalence classes of this quotient space comprise at most a pair of two opposing points such that this time we can find a finite cover $\bar{I}^2 = \bigcup_{1 \leq i \leq 4} U_i$ with **injective** restrictions of the canonical projections $\pi|_{U_i} \to \mathbb{S}^2$. The subsets $U_1 =]0;1[\times]0;1]$; $U_2 = [0;1[\times]0;1[$; $U_3 =]0;1[\times[0;1[$ and $U_4 =]0;1]\times]0;1[$ are open in $\bar{I}^2$ and also open in the **halfspaces** extending the corresponding boundaries but not in $\mathbb{R}^2$ such that we arrive at a $\mathbb{R}^2$**-manifold with boundary** with the **four charts** $\left(\left(\bar{\eta} \circ \bar{\vartheta} \circ \pi\right)[U_i]; \left(\bar{\eta} \circ \bar{\vartheta} \circ \pi|_{U_i}\right)^{-1}\right)$.



20.15 The n-dimensional sphere

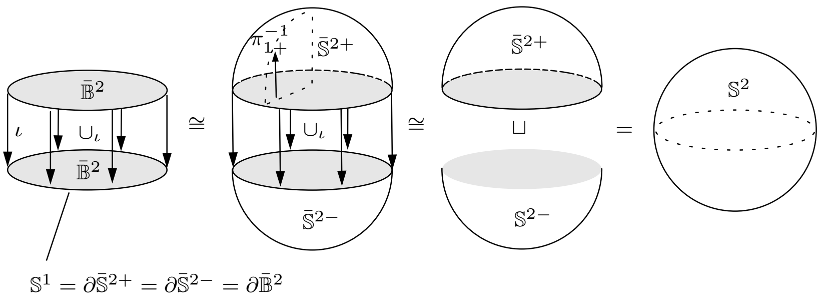


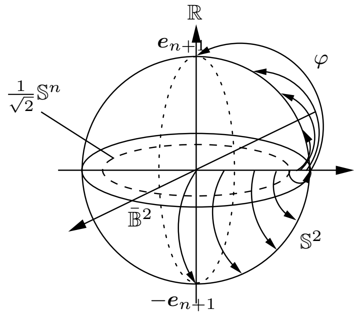
The n**-dimensional unit sphere** $\mathbb{S}^n = \partial\bar{\mathbb{B}}^{n+1}$ can be represented as an $\mathbb{R}^n$-manifold by the $2n$ charts $(\mathbb{S}^{i\pm}; \pi_i)$ with the **projection** $\pi_i : \mathbb{S}^{i\pm} \to \mathbb{B}^n$ given by $\pi_i(x_1; ...; x_i; ...; x_{n+1}) = (x_1; ...; 0; ...; x_{n+1})$ on the **upper half-spheres** $\mathbb{S}^{i+} = \mathbb{S}^n \cap \mathbb{H}^{i+}$ with the **upper half-spaces** $\mathbb{H}^{i+} = \{x_i > 0\}$ and the corresponding **lower half-spheres** $\mathbb{S}^{i-}$. Since in this case the projections are continuous bijections with continuous inverses $\pi_i^{-1}(x_1; ...; 0; ...; x_{n+1}) = \left(x_1; ...; \sqrt{1 - \|\boldsymbol{x}\|^2}; ...; x_{n+1}\right)$ for $\boldsymbol{x} = (x_1; ...; x_{n+1})$ the **half-spheres** $\mathbb{S}^{i\pm}$ are **homeomorphic** to the **open balls** $\mathbb{B}^n$. The sphere can also be **embedded** onto the identical $\mathbb{R}^n$**-submanifold** $\mathbb{S}^n$ of $\mathbb{R}^{n+1}$ with the additional charts $(\mathbb{S}_\epsilon^{i\pm}; \pi_i)$ on **open segments** $\mathbb{S}_\epsilon^{i\pm} = \{r\boldsymbol{x} : r \in B_\epsilon(1) \wedge x \in \mathbb{S}^{i\pm}\}$ such that $\pi_i\left[\mathbb{S}^n \cap \mathbb{S}_\epsilon^{i\pm}\right] = \mathbb{B}^n \subset \mathbb{R}^n$. Moreover $\mathbb{S}^n$ is homeomorphic to

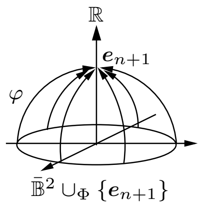
1. the **quotient space** $\bar{\mathbb{B}}^n/\mathbb{S}^n$
2. the **adjunction space** $\bar{\mathbb{B}}^n \cup_\iota \bar{\mathbb{B}}^n$ given by the **injection** $\iota : \mathbb{S}^{n-1} \to \bar{\mathbb{B}}^n$
3. the **adjunction space** $\bar{\mathbb{B}}^n \cup_\pi \{\boldsymbol{e}_{n+1}\}$ given by the **projection** $\pi : \mathbb{S}^{n-1} \to \{\boldsymbol{e}_n\}$.
4. the **one-point-compactification** $\bar{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$
5. the **projective completion** $\Phi[\mathbb{RP}^n] = \boldsymbol{e}_{n+1} + \bar{\mathbb{R}}^n$ of the **affine space** $\boldsymbol{e}_{n+1} + \mathbb{R}^n$.

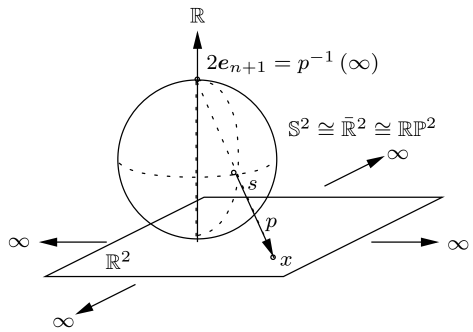
Proof:

1. $\mathbb{S}^n \cong \bar{\mathbb{B}}^n/\mathbb{S}^n$: According to enu:4.8.2 the **continuous**, **open** and **surjective** map $\varphi : \bar{\mathbb{B}}^n \to \mathbb{S}^n$ given by $\varphi(\boldsymbol{x}) = \left(2\sqrt{1 - \|\boldsymbol{x}\|^2} \cdot \boldsymbol{x}; 2\|\boldsymbol{x}\|^2 - 1\right)$ with **continuous partial inverse** $\varphi^{-1} : \mathbb{S}^n \setminus \{\boldsymbol{e}_{n+1}\} \to \mathbb{B}^n$ given by $\varphi^{-1}(\boldsymbol{y}; y_{n+1}) = \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|} \cdot \sqrt{\frac{1}{2} + \frac{1}{2}y_{n+1}} = \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|} \cdot \sqrt{\frac{1}{2} \pm \frac{1}{2}\sqrt{1 - \|\boldsymbol{y}\|^2}}$ for $\|\boldsymbol{y}\|^2 + y_{n+1}^2 = 1$ is a **quotient map** whence its **canonical bijection** $\bar{\varphi} : \bar{\mathbb{B}}^n/\mathbb{S}^n \to \mathbb{S}^n$ is a **homeomorphism**.
2. $\mathbb{S}^n \cong \bar{\mathbb{B}}^n \cup_\pi \{\boldsymbol{e}_{n+1}\}$: The **surjective** and **continuous** map $\varphi : \bar{\mathbb{B}}^n \to \mathbb{S}^n$ from 1. is **injective** on $\mathbb{B}^n \cong \mathbb{S}^n \setminus \{\boldsymbol{e}_{n+1}\}$ with **continuous inverse** and maps its complete boundary onto the **north pole** $\{\boldsymbol{e}_n\} = \varphi\left[\mathbb{S}^{n-1}\right]$ of the sphere. According to 4.12.3 follows $\bar{\mathbb{B}}^n \cup_\pi \{\boldsymbol{e}_n\} \cong \bar{\mathbb{B}}^n \sqcup \{\boldsymbol{e}_n\} \overset{\varphi}{\cong} \mathbb{S}^n \setminus \{\boldsymbol{e}_{n+1}\} \sqcup \{\boldsymbol{e}_{n+1}\} = \mathbb{S}^n$.
3. $\mathbb{S}^n \cong \bar{\mathbb{B}}^n \cup_\iota \bar{\mathbb{B}}^n$: The **projection** $\pi_{1+} : \bar{\mathbb{S}}^{n+} \to \bar{\mathbb{B}}^n$ given by $\pi_{1+}(\boldsymbol{x}; x_{n+1}) = (\boldsymbol{x}; 0)$ for $\boldsymbol{x} = (x_1; ...; x_n)$on the **closed upper half-sphere** $\bar{\mathbb{S}}^{n+} = \mathbb{S}^n \cap \bar{\mathbb{H}}^{n+1}$ with the **closed upper half-space** $\bar{\mathbb{H}}^{n+1} = \{x_{n+1} \geq 0\}$ is a continuous bijection with the continuous inverse $\pi_{1+}^{-1}(\boldsymbol{x}; 0) = \left(\boldsymbol{x}; \sqrt{1 - \|\boldsymbol{x}\|^2}\right)$ whence $\bar{\mathbb{S}}^{n+}$ is homeomorphic to $\bar{\mathbb{B}}^n$ and by the corresponding projection $\pi_{1+}^- : \bar{\mathbb{S}}^{n-} \to \bar{\mathbb{B}}^n$ the lower open half-sphere $\bar{\mathbb{S}}^{n-}$ is also homeomorphic to $\bar{\mathbb{B}}^n$. Both the projections and their inverses are continuous maps into $\bar{\mathbb{B}}^n \cup_\iota \bar{\mathbb{B}}^n$ coinciding on the boundary $\partial\bar{\mathbb{S}}^{n+} = \partial\bar{\mathbb{S}}^{n-} = \mathbb{S}^{n-1}$ with $\pi_{1+}|_{\mathbb{S}^{n-1}} = \pi_{1-}|_{S^{n-1}} = \mathrm{id}|_{\mathbb{S}^{n-1}}$ such that the **attaching lemma** 4.11 applies in both directions to yield a **continuous extension** $\pi : \mathbb{S}^n = \bar{\mathbb{S}}^{n+} \cup \bar{\mathbb{S}}^{n-} \to \bar{\mathbb{B}}^n \cup_\iota \bar{\mathbb{B}}^n$ which is a **homeomorphism.**
4. $\mathbb{S}^n \cong \bar{\mathbb{R}}^n$: The desired homeomorphism is provided by the **continuous stereographic projection** $p : \mathbb{S}^n \to \bar{\mathbb{R}}^n$ given by $p(\boldsymbol{s}; s_{n+1}) = \frac{2\boldsymbol{s}}{1 - s_{n+1}}$ for $(\boldsymbol{s}; s_{n+1}) \in \mathbb{S}^n \setminus \{\boldsymbol{e}_{n+1}\}$ with $\|\boldsymbol{s}\|^2 + s_{n+1}^2 = 1$ resp. $p(\boldsymbol{e}_{n+1}) = \infty$ and its **continuous inverse** defined by $p^{-1}(\boldsymbol{x}) = \frac{1}{1 + \frac{1}{4}\|\boldsymbol{x}\|^2}\left(\boldsymbol{x}; \frac{1}{2}\|\boldsymbol{x}\|^2\right)$ resp. $p^{-1}(\infty) = \boldsymbol{e}_{n+1}$.
5. $\bar{\mathbb{R}}^n \cong \boldsymbol{e}_{n+1} + \bar{\mathbb{R}}^n$: Obvious according to [17, th. 9.3] 9.1.

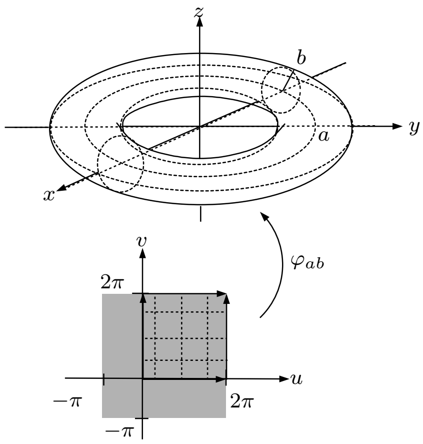








20.16 The torus



Due to the **closed map lemma** 2 the map $\varphi_{ab} : \bar{I_2}^2 \to \mathbb{T}^2$ given by $\varphi_{ab}(u;v) = \psi_{ab}(u;v;1)$ with $\psi_{ab}(u;v;r) = ((a+br\cdot\cos v)\cdot\cos u;\ (a+br\cdot\cos v)\cdot\sin u;\ br\cdot\sin v)$ is a **quotient map** whence the two-dimensional **torus** $\mathbb{T}^2_{ab}$ is homeomorphic to the quotient space $\bar{I_2}^2/R_\varphi$ with $(0;u)\,R_\varphi\,(1;u)$ and $(u;0)\,R_\varphi\,(u;1)$ for $0 \le u \le 2\pi$ identifying **opposed sides** of the **closed rectangle** $\bar{I_2}^2$ in **corresponding sense**. Each equivalence class comprises at most two distinct points such that a parametrization on the **open coordinate square** I_2^2 of an $\mathbb{R}^2$**-manifold** can be obtained by the **four charts** $\left(\varphi_{ab}\left[I_i \times I_j\right];\varphi^{-1}\right)$.

Hence by the canonical bijection $\bar{\varphi}_{ab} : \bar{I_2}^2/R_\varphi \to \mathbb{T}^2_{ab}$ the torus is **embedded** onto a **closed** $\mathbb{R}^2$**-submanifold** $\mathbb{T}^2_{ab}$ in $\mathbb{R}^3$ with the **additional four charts** $\left(\psi_{ab}\left[I_i \times I_j \times B_\epsilon(1)\right];\psi_{ab}^{-1}\right)$ of $\mathbb{R}^3$. The **open neighborhoods** $U_\pm^+\left(\mathbb{T}^2_{ab}\right) = U_\epsilon\left(\mathbb{T}^2_{ab}\right) \setminus \left(E_y^\pm \cup \bar{\mathbb{B}}_z^a\right)$ resp. $U_\pm^-\left(\mathbb{T}^2_{ab}\right) = U_\epsilon\left(\mathbb{T}^2_{ab}\right) \setminus \left(E_y^\pm \cup \mathbb{B}_z^{ac}\right)$ satisfy the submanifold condition $\psi_{ab}^{-1}\left[\psi_{ab}\left[I_i \times I_j \times B_\epsilon(1)\right] \cap \mathbb{T}^2_{ab}\right] = I_i \times I_j \times \{1\} \subset \mathbb{R} \times \mathbb{R} \times \{1\} \cong \mathbb{R}^2$.

Finally the torus can be represented as a **closed** $\mathbb{R}^2$**-product manifold** $\mathbb{T}^2_{ab} \cong a\mathbb{S}^1 \times b\mathbb{S}^1$ according to 20.3 and parametrized by the **four charts** $\left((a\epsilon \times b\epsilon)\left[I_i \times I_j\right];(a\epsilon \times b\epsilon)^{-1}\right)$ with the **exponential quotient map** $\epsilon : \mathbb{R} \to \mathbb{S}^1 \subset \mathbb{C} \cong \mathbb{R}^2$ defined by $\epsilon(u) = e^{iu}$. It describes the rotation of n coupled bodies around fixed axes, e.g. for $n = 2$ a **double pendulum**. The restriction $\epsilon|_{\bar{I}_2}$ satisfies the conditions of the closed map lemma with $R_\epsilon = R_\varphi$ whence the compositiopn $\varphi_{ab} \circ \left(a\epsilon|_{\bar{I}_2} \times b\epsilon|_{\bar{I}_2}\right)^{-1}$: $a\mathbb{S}^1 \times b\mathbb{S}^1 \to \mathbb{T}^2_{ab}$ is an **embedding**.

The n**-dimensional torus** $\mathbb{T}^n = \left(\mathbb{S}^1\right)^n$ is homeomorphic to the quotient space $\mathbb{R}^n/2\pi\mathbb{Z}^n = (\mathbb{R}/2\pi\mathbb{Z})^n$ since the **exponential quotient map** is **continuous** and **surjective** with **continuous inverse** $\epsilon^{-1}(s) = \ln s$ whence its canonical bijection $\bar{\epsilon} = \epsilon \circ \pi^{-1} : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{S}^1$ is a **homeomorphism**. According to 4.2 the continuity of the components in both directions extends to the corresponding products of the exponential quotient map.

The product representation also admits the embedding of the **space-filling** curve α on the torus $\mathbb{T}^2_{ab} \cong \mathbb{S}^1_a \times \mathbb{S}^1_b$ with the parametrizations $(\varphi_a \times \varphi_b) \circ \alpha : \mathbb{R} \to \mathbb{S}^1 \times \mathbb{S}^1$ given by the **line** $\alpha : \mathbb{R} \to (\mathbb{R}/\mathbb{Z})^2 \simeq$

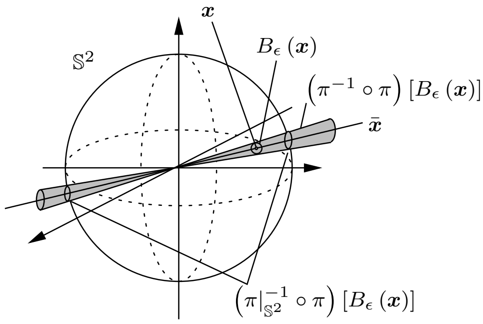
$\bar{I}_2{}^2/R$ with $\alpha(t) = (\bar{t}; \overline{\alpha t})$ for an **irrational** $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.

20.17 Affine and projective spaces

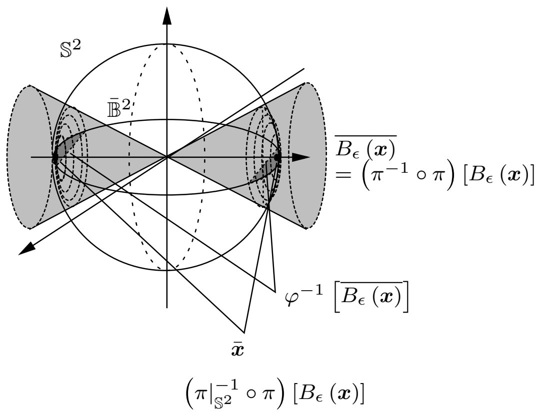
Every **affine space** $(A; X_A; \vec{\ })$ over a **Banach space** X_A as defined in [17] 8.1 is an X_A**-manifold** by the single chart $(\varphi; A)$ with **coordinates** given by $\varphi_o(x) = \overrightarrow{ox}$ for some arbitrary **origin** $o \in A$. In the case of $A = \boldsymbol{a} + X_A \subset X$ with $\boldsymbol{a} = \overrightarrow{oa} \in X$ it is an X_A**-submanifold** of X with identical coordinates which are usually translated to $\varphi_a(x) = \overrightarrow{ax}$.

The **projective spaces** $\mathbb{RP}^n = \mathbb{PR}^{n+1} = \mathbb{R}_*^{n+1}/R \cong \mathbb{S}^n/R$ with $R \subset \mathbb{R}_*^{n+1} \times \mathbb{R}_*^{n+1}$ resp. $R \subset \mathbb{S}^n \times \mathbb{S}^n$ defined by $\boldsymbol{x}R\boldsymbol{y} \Leftrightarrow \exists \alpha \in \mathbb{R}_* : \alpha\boldsymbol{x} = \boldsymbol{y}$ in [17] 9.1 are **quotient spaces** of **cosets** $\mathbb{R}_*\boldsymbol{x}$ obtained by the action of the multiplicative group $\mathbb{R}_*$ on the **punctured vector space** $\mathbb{R}_*^{n+1} = \mathbb{R}^{n+1} \setminus \{\boldsymbol{0}\}$ or **equivalently** on $\mathbb{S}^n$.

Indeed the quotient space $\mathbb{RP}^n = \mathbb{PR}^{n+1} = \mathbb{R}_*^{n+1}/R$ of all **open double rays** $(\pi^{-1} \circ \pi)(\boldsymbol{x}) = \{t\boldsymbol{x} : t \in \mathbb{R}_*\}$ given by $\boldsymbol{x}R\boldsymbol{y} \Leftrightarrow \exists t \in \mathbb{R}_* : t\boldsymbol{x} = \boldsymbol{y}$ as defined in [17] 9.1 is homeomorphic to the quotient space $\mathbb{S}^n/R$ of all **antipodal** pairs $\left(\pi|_{\mathbb{S}^n}^{-1} \circ \pi|_{\mathbb{S}^n}\right)(\boldsymbol{x}) = \{\pm\boldsymbol{x}_e\}$ $= \mathbb{S}^n \cap (\pi^{-1} \circ \pi)(\boldsymbol{x})$ on the n-dimensional sphere with the homeomorphism given by $\pi \circ \pi|_{\mathbb{S}^n}^{-1} : \mathbb{S}^n/R \to \mathbb{R}_*^n/R$ since the preimage $\pi^{-1}\left[B_{\|\boldsymbol{x}\|\epsilon}(\pi|_{\mathbb{S}^n}(\boldsymbol{x}_s))\right] = \{\boldsymbol{y} \in \mathbb{R}_*^{n+1} : |\boldsymbol{x}_s - \boldsymbol{y}_s| < \epsilon\}$ with $\boldsymbol{x}_s = \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}$ of an open neighborhood is an **open double cone** in $\mathbb{R}_*^{n+1}$ whereas $\left(\pi|_{\mathbb{S}^n}^{-1} \circ \pi\right)\left[B_{\|\boldsymbol{x}\|\epsilon}(\boldsymbol{x})\right] = (B_\epsilon(\boldsymbol{x}_e) \cup B_\epsilon(-\boldsymbol{x}_e)) \cap \mathbb{S}^n$ is the union of the intersections of the balls around the two corresponding antipodal representants $\pm\boldsymbol{x}_e$ with the sphere $\mathbb{S}^n$ which is an open subset of $\mathbb{S}^n$.



By the **closed map theorem** enu:9.8.2 the function $\psi : \bar{\mathbb{B}}^n \to \mathbb{RP}^n$ defined by $\psi(\boldsymbol{x}) = \left[x_1 : ... : x_n : \sqrt{1 - \|\boldsymbol{x}\|}\right]$ for $\boldsymbol{x} = (x_1; ...; x_n)$ is a **quotient map** whence the projective space $\mathbb{RP}^n$ is homeomorphic to the quotient space $\bar{\mathbb{B}}^n/R_\psi$ with $-\boldsymbol{x}R_\psi\boldsymbol{x}$ for $\boldsymbol{x} \in \partial\bar{\mathbb{B}}^n$ identifying the **antipodal pairs** on the boundary or equivalently gluing the **two halves of the boundary together in opposing sense**. This is equivalent to gluing the **disk** $\mathbb{B}^2$ to the **Moebius strip** M from 20.16. Finally the homeomorphism $\vartheta : \bar{\mathbb{B}}^n \to [-1; 1]^n$ given by $\vartheta(\boldsymbol{x})$ $= \alpha_{\boldsymbol{x}} \cdot \boldsymbol{x}$ with $\alpha_{\boldsymbol{x}} = \min\left\{\frac{1}{x_i} : 1 \le i \le n\right\}$ for $\boldsymbol{x} = (x_1; ...; x_n)$ shows that in particular the projective space $\mathbb{RP}^2$ is homeomorphic to the quotient space $\bar{I}^2/R$ with $I = [0; 1]$ and the equivalence relation given by $(0; u)\, R\, (1; -u)$ resp. $(u; 0)\, R\, (-u; 1)$ for $0 \le u \le 1$ identifying **opposite sides in antiparallel sense**.



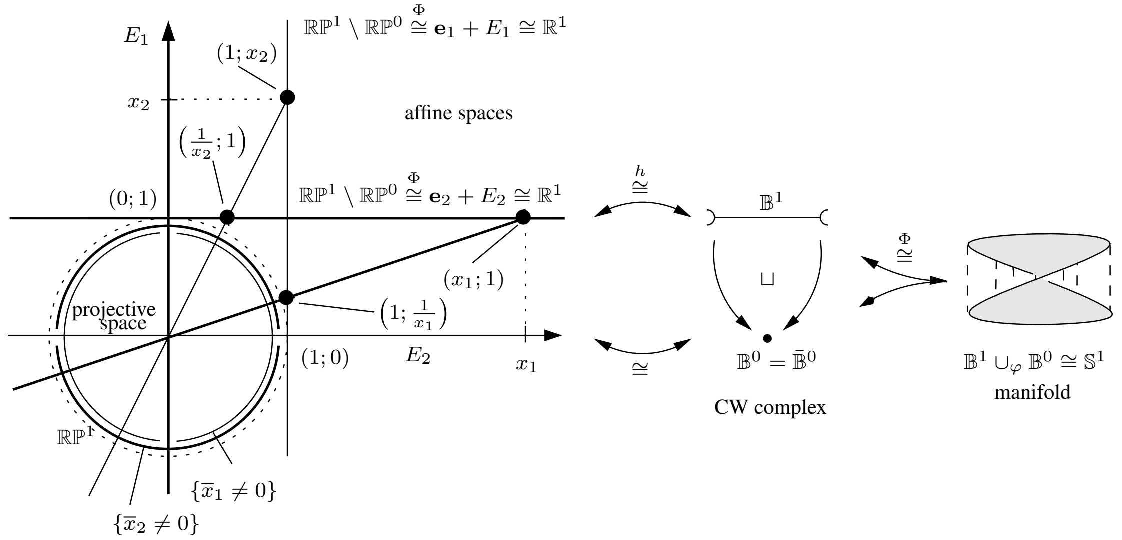
According to [17] 9.3 the projective space $\mathbb{RP}^n$ is **isomorphic** to the **projective completion** of the **affine spaces** $(\boldsymbol{e}_i + E_i; \mathbb{R}^n; \vec{\ })$ over the **infinitely distant hyperplane** $\mathbb{R}^n \cong \boldsymbol{e}_i + E_i$ with $E_i = \{x_i = 0\}$ and $\mathbb{RP}^n \setminus \mathbb{P}E_{i*} \cong \boldsymbol{e}_i + E_i$ and $\dim \mathbb{RP}^n = \dim \mathbb{R}^n + 1 = \dim \mathbb{PR}^n + 2 = n$ with the **bijections** $\Phi_i : \mathbb{RP}^n \setminus \mathbb{P}E_{i*} \to \boldsymbol{e}_i + E_i$ given by $\Phi_i[x_1 : ... : x_{n+1}] = \left(\frac{x_1}{x_i}; ...; \frac{x_{i-1}}{x_i}; 1; \frac{x_{i+1}}{x_i}; ...; \frac{x_{n+1}}{x_i}\right)$ and **inverses** $\Phi_i^{-1}(x_1; ...; 1; ...; x_{n+1}) = [x_1 : ... : 1 : ... : x_{n+1}]$.

The projective space $\mathbb{RP}^n$ is also an $\mathbb{R}^n$**-manifold** covered by $n + 1$ charts $(\pi\{x_i \neq 0\}; \varphi_i)$ with **coordinates** given by the **stereographic projections** $\varphi_i : \pi\{x_i \neq 0\} \to \mathbb{R}^n$ with $\varphi_i[x_1 : ... : x_{n+1}]$ $= \left(\frac{x_1}{x_i}; ...; \frac{x_{i-1}}{x_i}; \frac{x_{i+1}}{x_i}; ...; \frac{x_{n+1}}{x_i}\right)$ and **parametrizations** $\varphi_i^{-1}(x_1; ...; x_n) = [x_1 : ... : 1 : ... : x_n]$. Note that the projections $\pi\{x_i \neq 0\} = \mathbb{RP}^n \setminus \mathbb{P}E_{i*}$ of the **saturated open** sets $\{x_i \neq 0\} = \mathbb{R}_*^{n+1} \setminus E_{i*}$ are **open** in the **quotient topology** of $\mathbb{RP}^n$ as described in 4.7. An argument analogous to the reasoning for the homeomorphy of $\mathbb{R}_*^{n+1}/R \cong \mathbb{S}^n/R$ shows that both the **bijections** $\Phi_i : \mathbb{RP}^n \setminus \mathbb{P}E_{i*} \to \boldsymbol{e}_i + E_i$ and the **charts** $\varphi_i : \mathbb{RP}^n \setminus \mathbb{P}E_{i*} \to E_i$ are **homeomorphisms**. According to 9.8 and 9.10 the **quotient space** $\mathbb{RP}^n = \mathbb{R}_*^{n+1}/R \cong \pi[\mathbb{S}^n]$ is **compact** while every hyperplane E_i is a **locally compact**

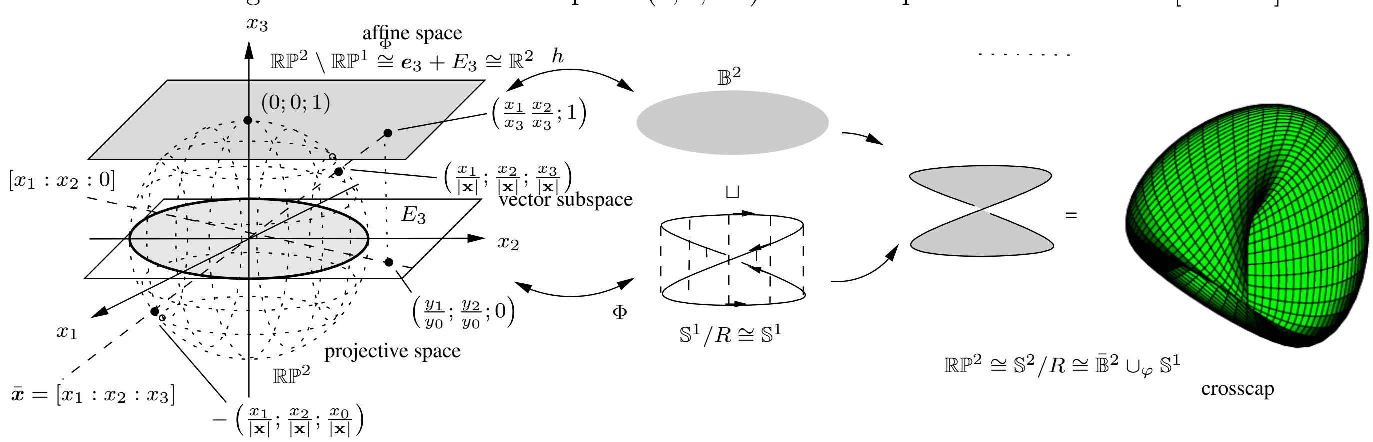
closed vector subspace of $\mathbb{R}^n$. Since the parametrization $\varphi_i^{-1}[K]$ of every **compact** set $K \subset E_i$ is **compact** and hence **closed** in $\mathbb{RP}^n$ due to 9.4 the extension $\bar{\Phi}_i : \mathbb{RP}^n \to \bar{E}_i$ onto the **one-point-compactification** 10.2 $\bar{E}_i = E_i \cup \{\infty\}$ defined by $\bar{\Phi}_i[\mathbb{P}E_{i*}] = \{\infty\}$ is **continuous**. Likewise we may extend the **homeomorphism** $h : \mathbb{B}^n \to \mathbb{R}^n$ given by $h(\boldsymbol{x}) = \frac{\boldsymbol{x}}{1-\|\boldsymbol{x}\|}$ with inverse $h^{-1}(y) = \frac{\boldsymbol{y}}{1+\|\boldsymbol{y}\|}$ to a **continuous** map $\bar{h} : \bar{\mathbb{B}}^n \to \bar{\mathbb{R}}^n$ from the **compact closure** $\bar{\mathbb{B}}^n$ of the **locally compact** unit ball onto the **one-point-compactification** $\bar{\mathbb{R}}^n = \mathbb{R}^n \cup \infty$ by defining $\bar{h}[\mathbb{S}^{n-1}] = \{\infty\}$.

Hence the $\mathbb{R}^n$**-manifold** $\mathbb{RP}^n$ can be **immersed** into $\mathbb{R}^{n+1}$ or **embedded** onto $\mathbb{R}^n$**-submanifolds** of $\mathbb{R}^{n+2}$. The **projective subspaces** $\{x_i = 0\} \cong \mathbb{S}^{n-1}/R \cong \mathbb{RP}^{n-1} \subset \mathbb{RP}^n$ admit an **immersion** in $\mathbb{R}^n$ but an **embedding** is only possible in $\mathbb{R}^{n+1}$.

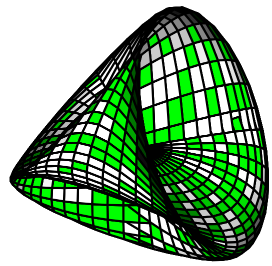
In the case of $n = 1$ the **projective line** $\mathbb{RP} = \mathbb{R}^2_*/R \cong \mathbb{S}^1/R_{\mathbb{S}^1}$ can be represented as an $\mathbb{R}$**-manifold** with **two charts** using **coordinates** $\varphi_i : \pi\{x_i \neq 0\} \to \mathbb{R}$ given by the **stereographic projections** $\varphi_1[1 : x_2] = x_2$ and $\varphi_2[x_1 : 1] = x_1$. These result in a **change of coordinates** $\varphi_2 \circ \varphi_1^{-1} : E_{1*} \to E_{2*}$ given by $\left(\varphi_2 \circ \varphi_1^{-1}\right)(x_1) = \frac{1}{x_1}$. It can also be **embedded** as a **compact** $\mathbb{R}$**-submanifold** into $\mathbb{R}^2$ which is **homeomorphic** to the **circle** $\mathbb{S}^1$.



For $n = 2$ the **projective plane** $\mathbb{RP}^2 = \mathbb{R}^3_*/R \cong \mathbb{S}^2/R_{\mathbb{S}^2}$ is an $\mathbb{R}^2$**-manifold** with **three charts** for coordinates $\varphi_i : \pi\{x_i \neq 0\} \to E_i \cong \mathbb{R}^2 \cong \mathbb{B}^2$ given by $\varphi_1[1 : x_2 : x_3] = (x_2; x_3)$, $\varphi_2[x_1 : 1 : x_3] = (x_1; x_3)$ and $\varphi_3[x_1 : x_2 : 1] = (x_1; x_2)$ where every chart describes the **projection** of an **open halfsphere** onto the **plane** $\mathbb{R}^2$. An **embedding** onto an $\mathbb{R}^2$**-submanifold in** $\mathbb{R}^3$ **is not possible** but there are several **immersions** $f : \mathbb{RP}^2 \to \mathbb{R}^3$ onto **compact** $\mathbb{R}^2$**-manifolds** in $\mathbb{R}^3$ which are determined by the **symmetry** $f[x_1 : x_2 : x_3] = f[-x_1 : -x_2 : -x_3]$. In $\mathbb{R}^3$ every such immersion has **additional self intersecting points** $f[x_1 : x_2 : x_3] = f[y_1 : y_2 : y_3]$ such that $R \subsetneq R_f$ and consequently $\mathbb{RP}^2 = \mathbb{R}^3_*/R \supsetneq \mathbb{R}^3_*/R_f \cong f\left[\mathbb{RP}^2\right]$. Usually these immersions are expressed in **polar coordinates** $(u; v)$ from 20.14 with **three charts** being sufficient since the two poles $(0; 0; \pm 1)$ are now equivalent in the class $[0 : 0 : 1]$.

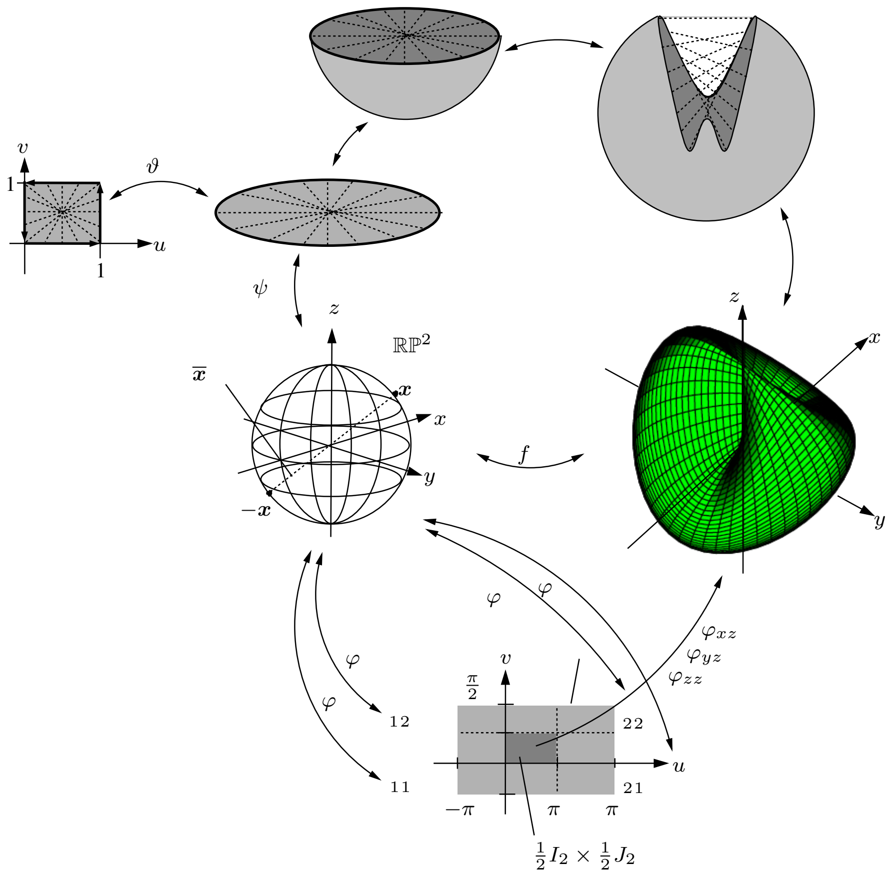


20.18 The roman surface



The simplest representant of the projective plane in an **algebraic** sense is the **roman surface** $R = f_R\left[\mathbb{RP}^2\right]$ with $f_R\,[x:y:z] = (yz; xz; xy)$ in **cartesian coordinates** resp. $(\mathrm{id}\circ f_R\circ\varphi)\,(u;v) = (\cos u\cdot\sin 2v;\ \sin u\cdot\sin 2v;\ \sin 2u\cdot\sin^2 v)$ due to $\sin 2\alpha = 2\cos\alpha\cdot\sin\alpha$ in **polar coordinates** restricted to half-spheres for $0 < u < \pi$ and $0 < v < \frac{\pi}{2}$ to retain **injectivity** on the equivalence pairs of **antipodal points**. The boundaries are $(f_R\circ\varphi)\,(u;0) = (0;0;0)$, $(f_R\circ\varphi)\left(u;\frac{\pi}{2}\right) = (0;0;\sin 2u)$ and $(f_R\circ\varphi)\left(\frac{\pi}{2}\pm\frac{\pi}{2};v\right) = (\pm\sin 2v;0;0)$ as well as the **self-intersecting line** $(f_R\circ\varphi)\left(\frac{\pi}{2};v\right) = (0;\sin 2v;0)$ which is crossed e.g. by the undulating circle $(f_R\circ\varphi)\left(u;\frac{\pi}{6}\right) = \left(\frac{\sqrt{3}}{2}\cos u;\frac{\sqrt{3}}{2}\sin u;\frac{1}{4}\sin 2u\right)$. Exchanging the coordinates by $\sigma_{xz}\,(x;y;z) = (z;y;x)$ and $\sigma_{yz}\,(x;y;z) = (x;z;y)$ we obtain further parametrizations $\eta_{ij} = \mathrm{id}\circ f_R\circ\sigma_{ij}\circ\varphi = \mathrm{id}\circ\sigma_{ij}\circ f_R\circ\varphi$ approaching the self-intersecting line along the y-axis from above and providing the missing boundary lines on the other two axes with corresponding intersecting lines. Thus the three charts $\left(\eta_{ij}\left[\frac{1}{2}I_2\times\frac{1}{2}J_2\right];\eta_{ij}^{-1}\right)$ with $ij\in\{xz;yz;zz\}$ define a **compact $\mathbb{R}^2$-manifold** in $\mathbb{R}^3$ which on account of the intersecting surfaces is **not an $\mathbb{R}^2$-submanifold in $\mathbb{R}^3$**. Geometrically it resembles a football which is squeezed together along the coordinate axes.

20.19 The crosscap



The **crosscap** $C = f_C\left[\mathbb{RP}^2\right]$ given by $f_C\,[x:y:z] = (xz; yz; x^2 - z^2)$ is the simplest representation of the projective plane in a **geometric** sense. With $\eta_{ij} = \mathrm{id}\circ\sigma_{ij}\circ f_C\circ\varphi$ we obtain the **three charts** $\left(\eta_{ij}\left[\frac{1}{2}I_2\times\frac{1}{2}J_2\right];\eta_{ij}^{-1}\right)$ on the coordinate sets from 20.14 defining a **compact $\mathbb{R}^2$-manifold** in $\mathbb{R}^3$ which on account of the intersecting surfaces is **not an $\mathbb{R}^2$-submanifold in $\mathbb{R}^3$**:

The boundaries of the **first** chart with $\eta_{zz} = (\cos u\cdot\sin 2v;\ \sin u\cdot\sin 2v;\ \cos^2 u\cdot\sin^2 v - \cos^2 v)$ run along the curve $\eta_{zz}\left(\frac{\pi}{2}\pm\frac{\pi}{2};v\right) = (\pm\sin 2v;\ 0;\ \sin^2 v - \cos^2 v)$, fuse into the point $\eta_{zz}\,(u;0) = (0;0;-1)$ and approach the **self-intersecting line** $\eta_{zz}\left(u;\frac{\pi}{2}\right) = (0;0;\cos^2 u)$ cut e.g. by $\eta_{zz}\left(\frac{\pi}{4};v\right) = \left(\frac{\sqrt{2}}{2}\sin 2v;\ \frac{\sqrt{2}}{2}\sin 2v;\ \frac{1}{2}\sin^2 v - \cos^2 v\right)$ at $\eta_{zz}\left(\frac{\pi}{4};\frac{\pi}{2}\right) = \left(0;0;\frac{1}{2}\right)$ and meeting $\eta_{zz}\left(\frac{\pi}{2};v\right) = (0;\sin 2v;-\cos^2 v)$ at $\eta_{zz}\left(\frac{\pi}{2};\frac{\pi}{2}\right) = (0;0;0)$.

The boundaries of the **second** chart with $\eta_{yz} = (\sin 2u\cdot\sin^2 v;\ \sin u\cdot\sin 2v;\ (\cos^2 u - \sin^2 u)\sin^2 v)$

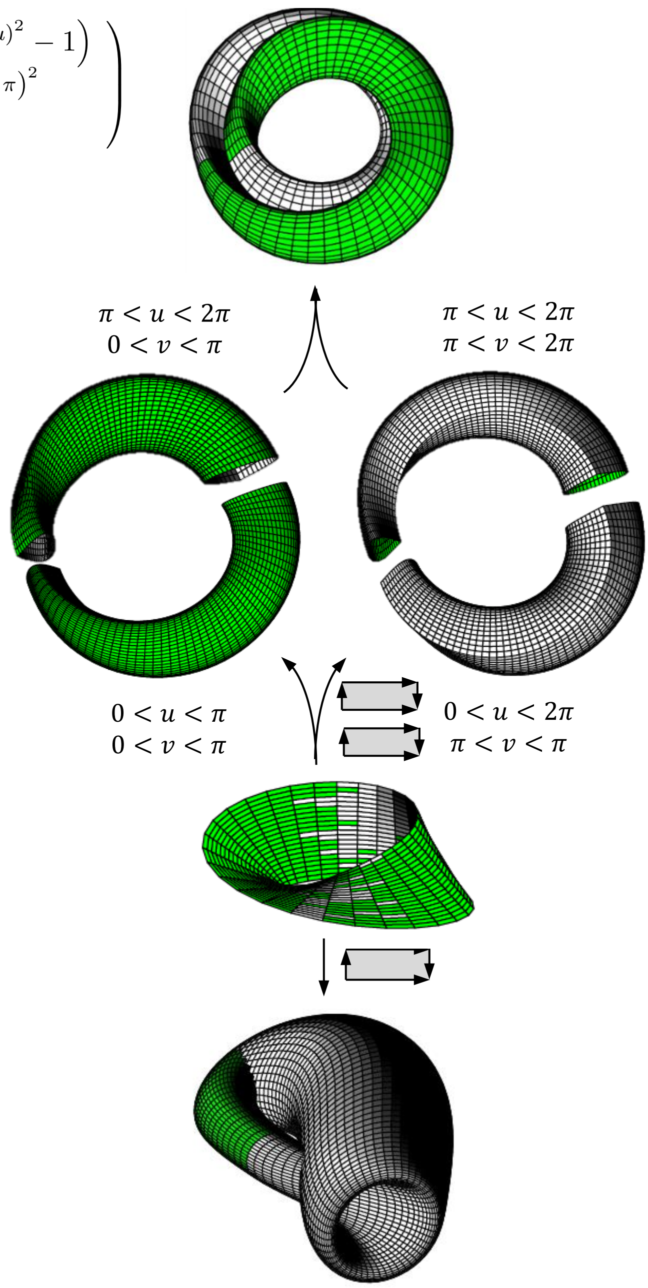
run along the **self-intersecting line** $\eta_{yz}\left(\frac{\pi}{2} \pm \frac{\pi}{2}; v\right) = \left(0; 0; \sin^2 v\right)$ between $\eta_{yz}\left(\frac{\pi}{2} \pm \frac{\pi}{2}; 0\right) = (0; 0; 0)$ and $\eta_{yz}\left(\frac{\pi}{2} \pm \frac{\pi}{2}; \frac{\pi}{2}\right) = (0; 0; 1)$. The self-intersecting line is not part of the second chart but the transverse orientation of the adjacent surfaces is visible e.g. in the **undulating eight-line** $\eta_{yz}\left(u; \frac{\pi}{4}\right) = \frac{1}{2}\left(\sin 2u; \sin u; \cos^2 u - \sin^2 u\right)$ approaching it from both sides at the point $\eta_{yz}\left(\frac{\pi}{2} \pm \frac{\pi}{2}; \frac{\pi}{4}\right) = \left(0; 0; \frac{1}{2}\right)$. The other boundary line is $\eta_{yz}\left(u; \frac{\pi}{2}\right) = \left(\sin 2u; 0; \cos^2 u - \sin^2 u\right)$ passing two poles at $\eta_{yz}\left(\frac{\pi}{2} \pm \frac{\pi}{4}; \frac{\pi}{2}\right) = (\pm 1; 0; 0)$. Also of interest is $\eta_{yz}\left(\frac{\pi}{2}; v\right) = \left(0; \sin 2v; -\sin^2 v\right)$ crossing the line $\eta_{yz}\left(u; \frac{\pi}{4}\right)$ from above at $\eta_{yz}\left(\frac{\pi}{2}; \frac{\pi}{4}\right) = \left(0; 1; -\frac{1}{2}\right)$

The boundaries of the **third** chart with $\eta_{xz} = \left(\cos u \cdot \sin 2v; \sin 2u \cdot \sin^2 v; \cos^2 v - \cos^2 u \cdot \sin^2 v\right)$ are $\eta_{xz}\left(\frac{\pi}{2} \pm \frac{\pi}{2}; v\right) = \left(\pm \sin 2v; 0; \cos^2 v - \sin^2 v\right)$, $\eta_{xz}(u; 0) = (0; 0; 1)$ and $\eta_{xz}\left(u; \frac{\pi}{2}\right) = \left(0; \sin 2u; -\cos^2 u\right)$. The **self-intersecting line** runs along $\eta_{xz}\left(\frac{\pi}{2}; v\right) = \left(0; 0; \cos^2 v\right)$ and is crossed e.g. by $\eta_{xz}\left(u; \frac{\pi}{4}\right) = \left(\cos u; \frac{1}{2}\sin 2u; \frac{1}{2} - \frac{1}{2}\cos^2 u\right)$ at $\eta_{xz}\left(\frac{\pi}{2}; \frac{\pi}{4}\right) = \left(0; 0; \frac{1}{2}\right)$.

20.20 The Klein bottle

The **Klein bottle** is parametrized by $\kappa(u; v) =$

$$\begin{pmatrix} 2 \cdot \cos 2u \cdot (1 - \sin 2u) + (2 - \cos 2u) \cdot \cos v \cdot \left(2 \cdot e^{-(\pi u)^2} - 1\right) \\ 6 \cdot \sin 2u + \frac{1}{2} \cdot (2 - \cos 2u) \cdot \sin 2u \cdot \cos v \cdot e^{-\left(2u - \frac{3}{2}\pi\right)^2} \\ (2 - \cos 2u) \cdot \sin v \end{pmatrix}$$



resulting from gluing the opposite sides of a rectangle pairwise together in parallel sense for one pair and in antiparallel sense for the other pair. If we start with the parallel pair we obtain a **tube** which can be closed to a **sinusoidal torus** parametrized by $\varphi(u; v) =$

$$\begin{pmatrix} (a + b \cdot \cos v) \cdot \cos 2u \\ (a + b \cdot \cos v) \cdot \sin 2u \\ b \cdot \sin v \cdot \cos u \end{pmatrix}$$

for $0 < u < \pi$ resp. $0 < v < 2\pi$ with the self-intersecting line along the **diagonal** $\varphi\left(\frac{\pi}{4}; \pi \pm w\right) = \left(\frac{1}{\sqrt{2}}(a + b \cdot \cos v); \frac{1}{\sqrt{2}}(a + b \cdot \cos v); 0\right)$ for $0 < w < \pi$. The same figure results if we start with the Möbius strip and curl it together so that one half of the open boundary meets the other half with the self-intersecting line at the horizontal section of the surface. By gluing together **two Möbius strips** we obtain the **eight-figure** given by $\psi(u; v) =$

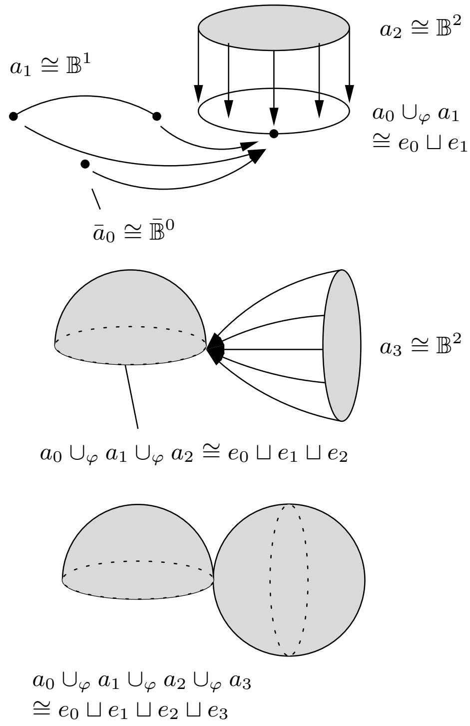
$$\begin{pmatrix} (a + b \cdot (\cos u \cdot \sin v - \sin u \cdot \sin 2v)) \cdot \cos 2u \\ (a + b \cdot (\cos u \cdot \sin v - \sin u \cdot \sin 2v)) \cdot \sin 2u \\ b \cdot (\cos u \cdot \sin v - \cos u \cdot \sin 2v) \end{pmatrix}$$

for $0 < u < \pi$ resp. $0 < v < 2\pi$ with the self-intersecting line along the **circle** $\psi\left(\frac{\pi}{2} \pm w; \pi\right) = (a \cdot \cos u; a \cdot \sin u; 0)$ for $0 < w < \frac{\pi}{2}$ as another homotopic version of the Klein bottle with a very different geometric aspect but still consisting of a one-sided surface with a single closed line of self-intersection.

21 Cell complexes

21.1 Cell complexes

An **open** resp. **closed** n**-cell** is a topological space homeomorphic to the **open** resp. **closed** n**-dimensional unit ball** $\mathbb{B}^n = B_1^n(\mathbf{0})$ resp. $\bar{\mathbb{B}}^n$. A **cell complex** $(X;\mathcal{E})$ is a **cell decomposition**, i.e. a **partition** $\mathcal{E}$ of a **Hausdorff space** $X = \bigsqcup_{e\in\mathcal{E}} e$ into **open cells** e such that for every $e \in \mathcal{E}$ of dimension $n \geq 1$ exists a **homeomorphism** $\varphi_e : a \to e$ with an n-cell $\boldsymbol{a}$ extending to a **continuous characteristic map** $\varphi_e : \bar{a} \to X$ such that $\varphi_e[\partial\bar{a}] \subset X_{n-1}$ with the n**-skeleton** $= \bigsqcup\{e \in \mathcal{E} : \dim e \leq n\}$. Hence we have $X_n \subset X_{n+1}$ and $X = \bigcup_{n\geq 0} X_n$. From the **closed map lemma** 9.8 follows $\varphi_e[\bar{a}] = \bar{e}$ whence by 9.1 the e are **precompact** but the closure $\bar{e}$ might **not be a closed cell** in X as shown by e_3 of the example since the characteristic map φ_e need not be **injective** on the boundary $\partial\bar{a}$. Conversely the open cells $e = \varphi_e[a]$ are **open** in $\bar{e} = \varphi_e[\bar{a}]$ but not necessarily open in X. If the n-skeletons are endowed with the **adjunction space** $X_n = X_{n-1} \cup_\Phi \left(\bigsqcup_{i\in I} \bar{a}_i\right) \cong X_{n-1} \sqcup \left(\bigsqcup_{i\in I_n} e_i\right)$ defined according to 4.12 with regard to a family of n-cells $(a_i)_{i\in I_n}$ and the **continuous extension** $\Phi : X_{n-1} \sqcup \left(\bigsqcup_{i\in I} \bar{a}_i\right) \to X_{n-1} \sqcup \left(\bigsqcup_{i\in I} a_i\right)$ of the **characteristic maps** $\varphi_i : \bar{a}_i \to X_{n-1}$ the attached open n-cells $e_i = \varphi_e[a_i]$ due to 4.12.2 are **open** in X_n and the subset X_{n-1} is **closed** in X_n due to 4.12.1. Consequently the open $(n-1)$-cells $e \in \mathcal{E}_{n-1}$ are open in X_{n-1} but not necessarily so in X_n. In the example the open cell e_2 is open in $a_0 \cup_\varphi a_1 \cup_\varphi a_2$ but not in $a_0 \cup_\varphi a_1 \cup_\varphi a_2 \cup_\varphi a_3$.



21.2 CW complexes

A **CW complex** is a cell complex $(X;\mathcal{E})$ with

1. **Closure finiteness**, i.e. the closure of each cell is contained in a union of finitely many cells.
2. **Weak topology** in this case meaning the **coherent topology** with respect to the family of **closures** of all cells.

Every **Hausdorff** space X with a l**ocally finite cell decomposition** $\mathcal{E}$ is a CW complex.

Note: Due to 11.4 the decomposition $\mathcal{E}$ is locally finite iff the family $\bar{\mathcal{E}} = \{\bar{e} : e \in \mathcal{E}\}$ of its **closures** is locally finite.

Proof: The **closure finiteness** is a consequence of the **compactness** of the closure of every cell. For every $x \in O \subset X$ with $O \cap \bar{e}$ open in $\bar{e}$ for every $e \in \mathcal{E}$ exists an open $x \in U \subset O \subset X$ intersecting only finitely many $\bar{e}$ and the finitely many intersections $U \cap \bar{e}$ are open in $\bar{e}$ such that in the case of $x \in U \cap \bar{e}$ exists an open $U_e \subset X$ with $x \in U_e \cap \bar{e} \subset U \cap \bar{e}$ and the finite intersection $x \in \bigcap_{x\in U\cap\bar{e}} U_e \subset U \subset O$ is an open neighborhood of x in O. Since we can find such a neighborhood for every $x \in O$ the set O is open in X and since this is true for every O with $O \cap \bar{e}$ open in every $\bar{e}$ the cell complex $(X;\mathcal{E})$ is coherent with respect to the family of **closures** of all cells.

Examples:

3. The set $X = \bigcup_{n\in\mathbb{N}} \bar{I}_n \subset \mathbb{R}^2$ with $I_0 = \{0\} \times]0;1[$ and $I_n = \left\{(x;nx) \in \mathbb{R}^2 : 0 < x < \frac{1}{n}\right\}$ from 5.9 is a cell complex comprising the disjoint open cells I_n; $\left(\frac{1}{n};1\right)$ for $n \geq 1$ and $(0;0)$ with characteristic functions $\varphi_n : \bar{\mathbb{B}}^1 \to \bar{I}_n$ and $\psi_n : \bar{\mathbb{B}}^0 \to \left\{\left(\frac{1}{n};1\right)\right\}$ for $n \geq 1$ resp. $\psi_0 = \mathrm{id}|_{\bar{\mathbb{B}}^0}$ whose trace topology is obviously coherent with that of the I_n. But since the set $A = \left(n^2;n\right)_{n\in\mathbb{N}^*}$ is closed in every I_n but due to $(0;0) \in \partial A \setminus X$ not in X this cell complex is not a CW complex.

4. The set $Y = \bar{\mathbb{B}}^2 \subset \mathbb{R}^2$ is a cell complex comprising the disjoint open cells $\mathbb{B}^2$ and $\left(\cos\frac{1}{n}; \sin\frac{1}{n}\right)$ for $n \geq 1$ with characteristic functions $\varphi_n : \bar{\mathbb{B}}^0 \to \left\{\left(\cos\frac{1}{n}; \sin\frac{1}{n}\right)\right\}$ for $n \geq 1$ resp. $\varphi_0 = \mathrm{id}|_{\bar{\mathbb{B}}^1}$ obviously satisfies 21.2.1 but since $\partial\bar{\mathbb{B}}^2 \not\subseteq \left(\cos\frac{1}{n}; \sin\frac{1}{n}\right)_{1 \leq n \leq k}$ for any $k < \infty$ this cell complex is not a CW complex.

21.3 Finite-dimensional CW complexes

A CW complex $(X; \mathcal{E})$ containing cells with maximal dimension $n \geq 0$ is called **finite dimensional** of **dimension** n. In that case every n-cell $e \in \mathcal{E}$ is **open** in X.

Proof: The **closed map lemma** 9.8 applied to its continuous and surjective **characteristic map** $\varphi_e : \bar{a} \to \bar{e}$ implies that every $e \in \mathcal{E}$ is open in $\bar{e}$ since $\varphi_e^{-1}[e] = a$ is open in $\bar{a}$. The intersection $e \cap \bar{g}$ is also open in $\bar{g}$ for every other cell $g \in \mathcal{E}$ since due to $e \cap g = \emptyset$ we have $e \cap \bar{g} \subset \partial g$, which is contained in a finite and disjoint union of cells of dimension **less** than n while e is of dimension n whence it cannot be part of this union such that $e \cap \bar{g} = \emptyset$, which is an open set. The assertion then follows from 21.2.2.

Examples:

1. A **graph** is a CW complex exclusively composed of 0-cells or **vertices** and 1-cells called **edges**. The **bouquet of circles** $\bigvee_{1 \leq i \leq n} \mathbb{S}^1$ from 4.13 is a **graph** composed from one 0-cell for the **basis point** p and a 1-cell for each of the original circles. The characteristic maps of these are the compositions $\pi_f \circ \iota \circ \pi_g$ of the projection π_f from 4.8.4 and projection π_g resp. the projection from 4.13 in the sequence $[0; 2\pi] \xrightarrow{\pi_f} \mathbb{S}^1 \xrightarrow{\iota} \bigsqcup_{1 \leq i \leq n} \mathbb{S}^1 \xrightarrow{\pi_g} \bigvee_{1 \leq i \leq n} \mathbb{S}^1$.
2. The decomposition of $\mathbb{S}^n \cong \bar{\mathbb{B}}^n \cup_\pi \{\boldsymbol{e}_{n+1}\} \cong \mathbb{B}^n \sqcup \mathbb{B}^0$ from enu:20.12.3 is an n-complex.

21.4 Subcomplexes

A **subcomplex** $Y \subset X$ of a CW complex $X = \bigsqcup_{i \in I} e_i$ is a union of cells from X containing also the **closures** of its members such that we have the form $Y = \bigsqcup_{j \in J} e_j = \bigcup_{j \in J} \bar{e}_j$ with $J \subset I$. Every such subcomplex is itself a **closed** subset of X and also a **CW complex** with regard to the subspace topology in X.

Proof: Condition 21.2.1 directly follows from the definition. Concerning 21.2.2 we consider a subset $S \subset Y$ such that every intersection $S \cap \bar{e}$ is closed in $\bar{e}$ for every $e \subset Y$. For every $e \subset X \setminus Y$ follows $S \cap e = \emptyset$ whence $S \cap \bar{e} \subset \partial e$ which is contained in a finite union of cells with some of them contained in Y such that we have $S \cap \bar{e} \subset \bigcup_{i=1}^k e_i \subset Y$. This implies $S \cap \bar{e} = \left(\bigcup_{i=1}^k S \cap \bar{e}_i\right) \cap \bar{e}$ with every $S \cap \bar{e}_i$ closed in $\bar{e}_i$ and consequently closed in the closed subset $\bigcup_{i=1}^k \bar{e}_i \subset X$ whence $\bigcup_{i=1}^k S \cap \bar{e}_i$ is closed in X and $S \cap \bar{e}$ is closed in $\bar{e}$. Since this is true for every $e \in \mathcal{E}$ we conclude that S is closed in X and thus closed in Y. Since this argument also works with $S = Y$ it implies that Y is closed in X.

21.5 n-skeletons

For every $n \geq 0$ the **n-skeleton** X_n of a CW complex X is a **subcomplex** and X is **coherent** with the family $(X_n)_{n \geq 0}$ of n-skeletons.

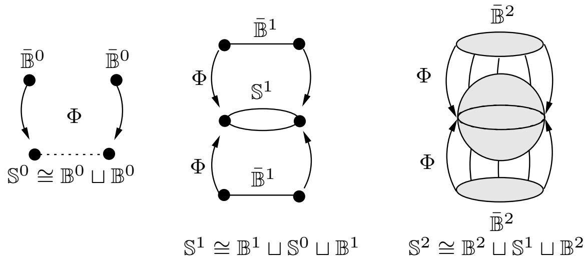
Proof: For any cell $e \in \mathcal{E}$ exists an $n \geq 0$ such that $\bar{e} \subset X_n$ and for any subset $S \subset X$ whose intersections $S \cap X_n$ are closed in X_n for every $n \geq 0$ the intersection $S \cap \bar{e}$ is closed in the closed subset $X_n \subset X$ and hence closed in X. Since this is true for every $e \in \mathcal{E}$ the set S is closed in X.

21.6 Regular cells

A cell is called **regular** iff its characteristic map $\varphi_e : \bar{a} \to \bar{e}$ is a **homeomorphism** in particular preserving the boundary by the **injective** restriction $\varphi_e|_{\partial a}$. Hence the closure $\bar{e}$ of every regular cell is a closed cell but the converse is not true as the example $\mathbb{B}^2 \setminus \{(x;0) : 0 < yx < 1\}$ shows. A CW complex is **regular** iff each of its cells is regular.

Hence every **discrete space** is a **regular** 0-dimensional CW complex.

The decomposition of $\mathbb{S}^n$ from enu:20.12.2 can be extended to a **regular** cell decomposition as follows: By an **induction** starting with $\mathbb{S}^0 = \mathbb{B}^0 \sqcup \mathbb{B}^0 \cong \bar{\mathbb{B}}^0 \cup_\iota \bar{\mathbb{B}}^0$ we assume a regular decomposition of $\mathbb{S}^{n-1} \cong \bigsqcup_{1 \le i \le n} \left(\mathbb{B}^{i-1} \sqcup \mathbb{B}^{i-1}\right)$ into n cell pairs of dimensions $0; ...; n-1$ and obtain $\mathbb{S}^n \cong \bar{\mathbb{B}}^n \cup_\iota \bar{\mathbb{B}}^n \cong \mathbb{B}^n \sqcup \mathbb{S}^{n-1} \sqcup \mathbb{B}^n$ according to 4.12.3.



21.7 Connected CW complexes

For a CW complex X the following conditions are equivalent:

1. X is **path-connected**
2. X is **connected**
3. Every n**-skeleton** X_n of X is connected
4. Some n**-skeleton** X_n of X is connected

Proof:

1. ⇒ 2. : follows from 5.8

2. ⇒ 3. : Assuming a partition $X_n = X'_n \sqcup X''_n$ and since the boundary $\partial\bar{a} \cong \mathbb{S}^n$ of every $(n+1)$-cell a is connected, its continuous image $\varphi_e[\partial\bar{a}] \subset X_n$ lies in either of the two components. Hence by $X'_{n+1} = X'_n \cup \bigcup_{\varphi_e[\partial\bar{a}] \in X'_n}$ and X'_{n+1}’ $= X''_n \cup \bigcup_{\varphi_e[\partial\bar{a}] \in X''_n}$ we obtain a partition of $X_{n+1} = X'_{n+1} \sqcup X''_{n+1}$. Then $X' = \bigcup_{k \ge n} X'_k$ and $X'' = \bigcup_{k \ge n} X''_k$ provide the desired partition $X = X' \cup X''$.

3. ⇒ 4. : obvious

4. ⇒ 1. : According to 5.8 for any $x \in X_n$ and every n-cell $e \subset X_n$ the continuous image $\bar{e} = \varphi_e[\bar{a}]$ is path-connected and so lies in the **path component** $P_n(x)$ of x in X_n . Consequently $P(x)$ is open and closed in the closure $\bar{e}$ of every $e \subset X_n$ and hence open and closed in X_n. Since X_n is connected we infer $P_n(x) = X_n$, i.e. X_n is path-connected. Then the closure $\bar{e}$ of every $(n+1)$-cell $e \subset X_{n+1}$ is a path-connected subset of X_{n+1} with $\bar{e} \cap X_n \ne \emptyset$ whence follows $\bar{e} \subset X_n \subset P_{n+1}(x)$ and therefore $P_{n+1}(x) = X_{n+1}$, i.e. X_{n+1} is also path-connected. The assertion then follows by induction.

21.8 Compact CW complexes

In a CW complex X

1. The closure $\bar{e}$ of each cell $e \subset X$ is contained in a **finite subcomplex**.
2. A subset $S \subset X$ is **discrete** iff its intersection with every cell is **finite**.
3. A subset $S \subset X$ is **compact** iff it is **closed** in X and contained in a **finite subcomplex**. In particular X is **compact** iff it is **finite**.

Proof:

1. For $n = 0$ the proposition is obvious and assuming it for all dimensions up to $n \geq 0$ condition 21.2.1 implies that the boundary $\partial\bar{e}$ of a cell $\bar{e} \subset X$ with dimension $n+1$ is contained in the union of finitely many cells of dimension up to n each of which is contained in a finite subcomplex due to the assumption. The finite union of these finite subcomplexes together with the cell e is again a finite subcomplex.
2. The intersection of the closed discrete set $\bar{S}= S$ with the compact set $\bar{e}$ is compact and discrete, hence finite. Conversely the hypothesis together with 1. imply that the intersection $s \cap \bar{e}$ is finite and hence closed in $\bar{e}$ for every subset $s \subset S$ and every cell $e \subset X$. Due to 21.2.2 every subset $s \subset S$ is closed in X whence follows the assertion.
3. Due to 9.4 every intersection $S \cap \bar{e}$ of a compact $S \subset X$ with the compact closure of a cell $e \subset X$ is closed in $\bar{e}$ whence S is closed in X. Also S is covered by finitely many cells since every point $x_e \in S \cap e \neq \emptyset$ is closed in $\bar{e}$ and consequently closed in X such that the complements $X \setminus x_e$ are open and cover S. Hence due to 1. it muss be contained in a finite subcomplex. Conversely we conclude from 21.4 that **every finite subcomplex is itself compact** such that the assertion follows from 9.4.

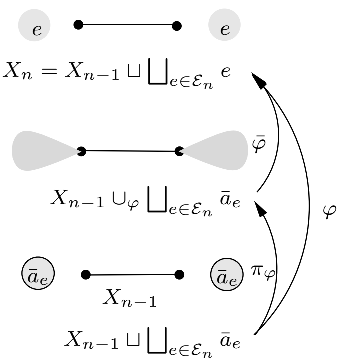
21.9 Locally compact CW complexes

A CW complex is **locally compact** iff it is **locally finite**.

Proof: $\Rightarrow$ follows from 10.7 and 21.8.3. $\Leftarrow$ is a consequence of the **local finiteness** of the closures $\bar{\mathcal{E}}$ mentioned in 21.2 combined with 21.8.1 and 21.8.3 together with the closed character of the finite subcomplex containing the open neighborhood of the chosen point x as explained in 21.4.

21.10 The structure of n-skeletons in a CW complex

Every n-**skeleton** $X_n = X_{n-1} \sqcup \bigsqcup_{e\in\mathcal{E}_n} e$ of a CW complex X is homeomorphic to the **adjunction space** $X_{n-1} \cup_\varphi \bigsqcup_{e\in\mathcal{E}_n} \bar{a}_e$ formed by attaching all n-cells a_e with $\varphi_e[a_e] = e \subset X_n \setminus X_{n-1}$ to X_{n-1} by the extension $\varphi : \bigsqcup_{e\in\mathcal{E}_n} \partial a_e \to X_{n-1}$ of the characteristic functions $\varphi|_{\partial\bar{a}_e} = \varphi_e|_{\partial\bar{a}_e} : \partial\bar{a}_e \to X_{n-1}$.



Proof: According to the **attaching lemma** 4.11 the further extension $\varphi : X_{n-1} \sqcup \bigsqcup_{e\in\mathcal{E}_n} \bar{a}_e \to X_n$ defined by $\varphi|_{X_{n-1}} = \mathrm{id}$ and $\varphi|_{\bar{a}_e} = \varphi_e$ is continuous and for every closed saturated set $\varphi^{-1}[A]$ the intersections $\varphi^{-1}[A] \cap X_{n-1}$ resp. $\varphi^{-1}[A] \cap \bar{a}_e$ are closed in X_{n-1} resp. $\bar{a}_e$ whence their homeomorphic images $A \cap X_{n-1}$ resp. $A \cap \bar{e}$ are closed in X_{n-1} resp. $\bar{e}$ for $e \in \mathcal{E}_n$ hence closed in all $e \in \mathcal{E}$ and consequently closed in X_n. Hence according to 4.8 the map φ is an **identifying map** which proves the assertion.

21.11 The CW construction theorem

The union $X = \bigcup_{n\geq 0} X_n$ of any ascending sequence $(X_n)_{n\geq 0}$ of topological spaces $X_n = X_{n-1} \cup_{\varphi_n} \bigsqcup_{e\in\mathcal{E}_n} \bar{a}_e$ for $n \geq 1$ obtained by subsequently **attaching** n**-cells** $\bar{a}_e$ to X_{n-1} according to 21.1and starting with a nonempty discrete space X_0 has a **unique topology coherent** with $(X_n)_{n\geq 0}$ and a **unique cell decomposition** $\mathcal{E}$ such that $(X; \mathcal{E})$ is a **CW complex** with n**-skeletons** X_n for $n \geq 0$.

Example: By continuation of the process from 21.6 we obtain an infinite-dimensional CW-complex $S^\infty = \bigcup_{n\geq 0} \mathbb{S}^n \cong \bigsqcup_{n\geq 0} \mathbb{B}^n \sqcup \mathbb{B}^n$ with two cells in every dimension containing every sphere $\mathbb{S}^n$as a subcomplex resp. n-skeleton.

Proof: By declaring a set $A \subset X$ as closed in X iff $A \cap X_n$ is closed in X_n for **every** $n \geq 0$ we have defined a topology on X which is obviously the only one **coherent** with $(X_n)_{n\geq 0}$. According to 4.12.1

every X_{n-1} is a closed subspace of X_n whence for any set $A \subset X_n$ being closed in X_n its intersection $A \cap X_m$ is also closed in every other X_m and hence A is closed in X. Consequently every X_n is also a **closed subspace** of X. For each 0-cell $e \in X_0$ and every n-cell defined as $e := \varphi_n [a_e] \subset X_n \setminus X_{n-1}$ for $n \geq 1$ by a composition of the **identifying map** φ_n with the **injection** $\iota_e : \bar{a}_e \to X_{n-1} \sqcup \bigsqcup_{e \in \mathcal{E}_n} \bar{a}_e$ and the **projection** $\pi_e : X_n \to \bar{e}$ we obtain a **characteristic map** $\varphi_e = \pi_e \circ \varphi_n|_{\bar{a}_e} \circ \iota_e : \bar{a}_e \to \bar{e}$. According to 21.1 the restriction $\varphi_n|_{a_e}$is a homeomorphism and this property extends to the composition $\varphi_e|_{a_e} = \mathrm{id} \circ \varphi_n|_{a_e} \circ \mathrm{id} : a_e \to e$. Thus we obtain a **cell decomposition** of X for which X_n is the n**-skeleton** for each $n \geq 0$.

According to the second case in 9.12 for every $x \in e \subset X$ in the uniquely determined n-cell $e = \varphi_e [a_e] \subset X_n \setminus X_{n-1}$ exists a continuous function $\psi : \bar{a}_e \to [0;1]$ with $\psi^{-1}(0) = \{\varphi_e^{-1}(x)\}$ and $\psi[\partial \bar{a}_e] = \{1\}$. The map $\Psi_n : X_{n-1} \sqcup \bigsqcup_{e \in \mathcal{E}_n} \bar{a}_e \to [0;1]$ defined by $\Psi_n|_{\bar{a}_e} = \psi$ and $\Psi \equiv 1$ everywhere else for $n \geq 0$ with $X_{-1} := \emptyset$ in the case of $n = 0$ by 4.10 is continuous on the topological sum with $\Psi_n^{-1}(0) = \{\varphi_e^{-1}(x)\}$. Due to the **universal property** from 4.7 the continuity extends to the composition $f_n = \Psi_n \circ \pi_{\varphi_n}^{-1} : X_n \to [0;1]$ on the quotient space with with $f_n^{-1}(0) = \{x\}$. Assuming a continuous $f_{n-1} : X_{n-1} \to [0;1]$ with $f_{n-1}^{-1}(0) = \{x\}$ for $x \in X_{n-1}$ for every $e \subset X_n \setminus X_{n-1}$ exists an $\epsilon_e > 0$ such that the compact image $(f_{n-1} \circ \pi_{\varphi_n})[\partial \bar{a}_e] \subset [\epsilon_e; 1]$ whence by the first case in 9.12 we obtain an extension $\psi_e : \bar{a}_e \to [\epsilon_e; 1]$ of $\Psi_{n-1}|_{\partial \bar{a}_e} = f_{n-1} \circ \pi_{\varphi_n}|_{\partial \bar{a}_e} : \partial \bar{a}_e \to [\epsilon_e; 1]$. In analogy to the induction start by $\Psi_n|_{\bar{a}_e} = \psi_e$ and $\Psi_n|_{X_{n-1}} = \Psi_{n-1}$ we define a continuous map $\Psi_n : X_{n-1} \sqcup \bigsqcup_{e \in \mathcal{E}_n} \bar{a}_e \to [0;1]$ with $\Psi_n^{-1}(0) = \{\varphi_e^{-1}(x)\}$ continuously descending to $f_n = \Psi_n \circ \pi_{\varphi_n}^{-1} : X_n \to [0;1]$ with $f_n^{-1}(0) = \{x\}$. The map $f : X \to [0;1]$ defined by $f|_{X_n} = f_n$ is continuous since for every open $O \subset [0;1]$ the intersection $X_n \cap f^{-1}[O] = f_n^{-1}[O]$ is open in $X_n$whence $f^{-1}[O]$ is open in X. Since we can find a such a continuous map f with $f^{-1}(0) = \{x\}$ for every $x \in X$ the space X is a **Hausdorff space**.

The **continuous** image of the **compact** boundary $\varphi_e[\partial \bar{a}] \subset \bigsqcup \{e \in \mathcal{E} : \dim e < n\} \subset X_{n-1}$ of an n-cell e is compact and hence must be included in a union of finitely many cells of dimension less than n which implies condition 21.2.1. For any set $A \subset X$ whose intersections $A \cap \bar{e}$ are closed in the closure $\bar{e}$ of every cell $e \in \mathcal{E}$ the intersection $A \cap X_0$ is obviously closed in X_0. Assuming $A \cap X_{n-1} = \varphi_n^{-1}[A] \cap X_{n-1}$ closed in X_{n-1} every intersection $\varphi_n^{-1}[A] \cap X_{n-1}$ is closed in $\bar{a}_e$ whence due to 4.10 the saturated set $\varphi_n^{-1}[A]$ is closed in $X_{n-1} \sqcup \bigsqcup_{e \in \mathcal{E}_n} \bar{a}_e$ so that by 4.12 the set A is closed in $X_n = X_{n-1} \cup_{\varphi_n} \bigsqcup_{e \in \mathcal{E}_n} \bar{a}_e$. By induction this is true for every $n \geq 0$ so that the coherence of the chosen topology with $(X_n)_{n \geq 0}$ entails condition 21.2.2.

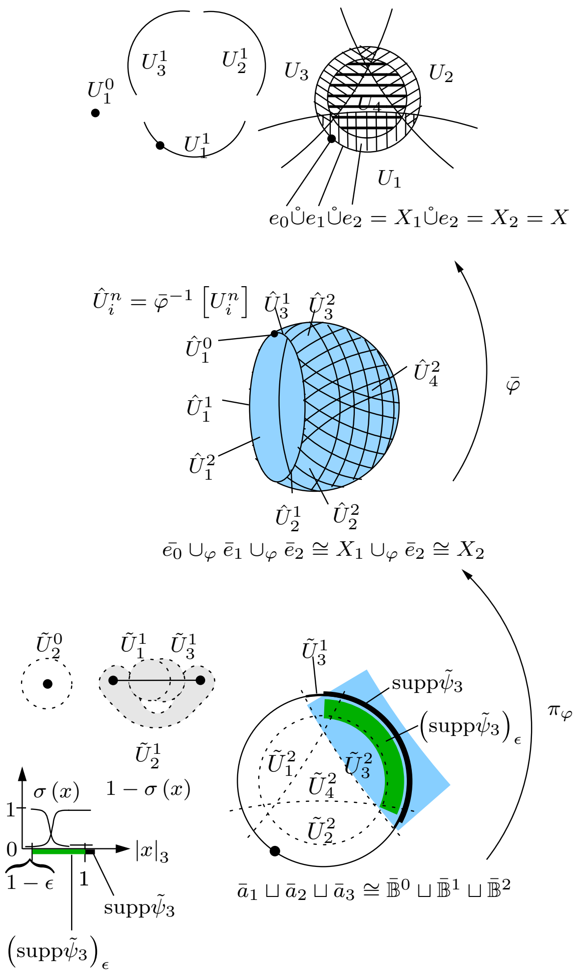
21.12 Paracompactness

Every CW-complex is paracompact.

Proof: According to 11.3 it is sufficient to show that for every open cover $\mathcal{U} = (U_i)_{i \in I}$ exists a subordinate **partition of unity**. By induction we construct a partition of unity $(\psi_i^n)_{i \in I}$ for X_n subordinate to the open cover $(U_i^n)_{i \in I}$ of the intersections $U_i^n = U_i \cap X_n$. We start with $n = 0$ by choosing an $x_i \in U_i^0$ for every nonempty U_i^0 and setting $\psi_i^0(x_j) = \delta_{ij}$.

Assuming partitions of unity $\left(\psi_i^k\right)_{i \in I}$ for X_k subordinate to the open cover $\left(U_i^k\right)_{i \in I}$for $k < n$ with $\psi_i^k|_{X_{k-1}} = \psi_i^{k-1}$ and for every open $V_{k-1} \subset X_{k-1}$ with $\psi_i^{k-1}[V_{k-1}] = \{0\}$ an open $V_{k-1} \subset V_k \subset X_k$ with $\psi_i^k[V_k] = \{0\}$ we define $\tilde{\psi}_{i;e}^{n-1} = \psi_i^{n-1} \circ \varphi|_{\partial \bar{a}_e} : \partial \bar{a}_e \to [0;1]$ and $\tilde{U}_{i;e}^n = \varphi_e^{-1}[U_i^n] \subset \bar{a}_e$ for the **identifying map** $\varphi : X_{n-1} \sqcup \bigsqcup_{e \in \mathcal{E}_n} \bar{a}_e \to X_n$ and the **characteristic map** $\varphi_e = \varphi|_{\bar{a}_e} : \bar{a}_e \to X_n$ of any n-cell $e \in \mathcal{E}_n$ and $i \in I$. Also for any subset $A \subset \partial \bar{a}_e$ and $0 < \epsilon < 1$ we define **segments** $A_\epsilon = \left\{x \in \bar{a}_e : \frac{x}{\|x\|_e} \in A \wedge 1 - \epsilon < \|x\|_e \leq 1\right\}$ for some homeomorphism $\beta_e : \bar{\mathbb{B}}^n \to \bar{a}_e$ and the **image of the euclidean norm** defined by $\|x\|_e = \left\|\beta_e^{-1}(x)\right\|$ for $x \in \bar{a}_e$. Since the **open** cover $(U_i^n)_{i \in I}$ is **locally finite** the **compact boundary** $\partial \bar{a}_e$ meets only finitely many $\tilde{U}_{j;e}^n$ with $j \in J \subset I$.

Because the sets $\operatorname{supp}\tilde{\psi}^{n-1}_{j;e} \subset \partial\bar{a}_e \cap \tilde{U}^n_j$ are **compact** there is an $\epsilon_j > 0$ such that $\left(\operatorname{supp}\tilde{\psi}^{n-1}_{j;e}\right)_{\epsilon_j} \subset \tilde{U}^n_{j;e}$ for every $j \in J$. Hence we obtain an extension $\tilde{\psi}^n_{i;e;\epsilon} : (\partial\bar{a}_e)_\epsilon \to [0;1]$ defined by $\tilde{\psi}^n_{i;e;\epsilon}(x) = \tilde{\psi}^{n-1}_{i;e}\left(\frac{x}{\|x\|_e}\right)$ for $\epsilon = \min\{\epsilon_j : j \in J\}$ on the circular strip $(\partial\bar{a}_e)_\epsilon$. In order to cover the interior of $\bar{a}_e$ we choose a **partition of unity** $\left(\tilde{\eta}^n_{i;e}\right)_{i\in I}$ subordinate to the open cover $\left(\tilde{U}^n_{i;e}\right)_{i\in I}$ on the paracompact space $\bar{a}_e \cong \bar{\mathbb{B}}^n \subset \mathbb{R}^n$ and a **bump function** $\sigma_e : \bar{a}_e \to [0;1]$ with $\operatorname{supp}\sigma_e = \bar{a}_e \backslash (\partial\bar{a}_e)_{\epsilon/2}$ and $\sigma_e^{-1}\{1\} = \bar{a}_e \backslash (\partial\bar{a}_e)_{\epsilon/2}$ according to 8.1.1. The continuous maps $\tilde{\psi}^n_{i;e} = \sigma_e \cdot \tilde{\eta}^n_{i;e} + (1-\sigma_e)\cdot\tilde{\psi}^n_{i;e;\epsilon} : \bar{a}_e \to [0;1]$ then obviously satisfy $\tilde{\psi}^n_{i;e}|_{\partial\bar{a}_e} = \tilde{\psi}^{n-1}_{i;e}$, $\operatorname{supp}\tilde{\psi}^n_{i;e} \subset \tilde{U}^n_{i;e}$ and $\sum_{i\in I}\tilde{\psi}^n_{i;e} \equiv 1$. Since these maps coincide on $\varphi^{-1}(x)$ for every $x \in X_{n-1}$ the extensions $\psi^n_i : X_n \to [0;1]$ given by $\psi^n_i|_{X_{n-1}} = \psi^{n-1}_i$ resp. $\psi^n_i|_e = \tilde{\psi}^n_{i;e} \circ \varphi_e^{-1}$ are well defined with $\operatorname{supp}\psi^n_i \subset U^n_i$. They are also **continuous** owing to the **universal property** 4.5 of the final topology on X_n with regard to the identifying map φ and the continuity of $\psi^n_i \circ \varphi : X_{n-1} \sqcup \bigsqcup_{e\in\mathcal{E}_n} \bar{a}_e \to [0;1]$.



According to the construction of the maps $\tilde{\psi}^n_{i;e}$ resp. $\tilde{\psi}^n_{i;e;\epsilon}$ as described above for every open $V_{n-1} \subset X_{n-1}$ with $\psi^{n-1}_i[V_{n-1}] = \{0\}$ and every $e \in \mathcal{E}_n$ exists an $\epsilon_i > 0$ such that $\left(\tilde{V}_{n-1}\right)_{\epsilon/2} \subset \bar{a}_e$ and $\tilde{\psi}^n_i\left[\left(\tilde{V}_{n-1}\right)_{\epsilon/2}\right] = \{0\}$. The intersections $\left(\tilde{V}_{n-1}\right)_{\epsilon/2} \cap a_e$ are open in a_e whence their homeomorphic images $\varphi_e\left[\left(\tilde{V}_{n-1}\right)_{\epsilon/2}\right] \cap a_e$ are open in e and consequently the union $V_n = V_{n-1} \cup \mathring{\bigcup}_{e\in\mathcal{E}_n}\varphi_e\left[\left(\tilde{V}_{n-1}\right)_{\epsilon/2}\right] \cap a_e$ is open in X_n with $\psi^n_i[V_n] = \{0\}$.

Since the **open** cover $(U^n_i)_{i\in I}$ is **locally finite** the **compact closure** $\bar{a}_e$ meets only finitely many $\tilde{U}^n_{i;e}$ such that for any $x \in X$ contained in some n-cell $e \subset X_n$ there are most finitely many $\operatorname{supp}\tilde{\psi}^n_{i;e} \subset \bar{a}_e$ meeting the neighborhood $\varphi_e^{-1}(x) \in a_e$ such that only finitely many $\operatorname{supp}\psi^n_i$ meet the neighborhood $x \in e$ which is open in X_n. In the case of $x \in X_{n-1}$ there is a neighborhood V^{n-1} open in X_{n-1} meeting at most finitely many $\operatorname{supp}\psi^{n-1}_i \subset X_{n-1}$ and since for each such $i \in K$ the induction hypothesis implies the existence of a $V^{n-1} \setminus \operatorname{supp}\psi^{n-1}_i \subset V^n_i$ open in X_n the finite union $V^n = \bigcup_{i\in K} V^n_i$ is open in X_n and contains the union $\bigcup_{i\in K}\left(V^{n-1} \setminus \operatorname{supp}\psi^{n-1}_i\right) = V^{n-1} \setminus \bigcap_{i\in K}\operatorname{supp}\psi^{n-1}_i = V^{n-1}$. This completes the inductive construction of a partition of unity $(\psi^n_i)_{i\in I}$ for each X_n subordinate to the open cover $(U^n_i)_{i\in I}$ of the intersections $U^n_i = U_i \cap X_n$.

Finally for each $i \in I$ we define $\psi_i : X \to [0;1]$ by $\psi_i|_e = \psi^n_i$ for $e \in \mathcal{E}_n$. This function is **continuous** on X since every ψ^n_i is continuous on $\bar{e} \subset X_n$ for $e \in \mathcal{E}_n$ and the topology on X is **coherent** with regard to $(\bar{e})_{e\in\mathcal{E}}$. For every $x \in X$ there is an $e \in \mathcal{E}_n$ with $x \in e$ whence follows $\sum_{i\in I}\psi_i(x) = \sum_{i\in I}\psi^n_i(x) = 1$. Also we have $\operatorname{supp}\psi_i \subset \bigcup_{n\geq 0}\operatorname{supp}\psi^n_i \subset \bigcup_{n\geq 0} U^n_i = U_i$. The family $(\psi_i)_{i\in I}$ is **locally finite** since for every $x \in X$ with $x \in e$ for some $e \in \mathcal{E}_n$ the local finiteness of $(\psi^n_i)_{i\in I}$ implies the existence of a finite subset $J \subset I$ and an open neighborhood $x \in V^n \subset X_n$ with $\psi^n_i[V^n] = 0$ for $i \in I \setminus J$. Due to the construction of the families $(\psi^n_i)_{i\in I}$ we also have ascending open neighborhoods $V^m \subset X_m$ with $V^{m-1} \subset V^m$ and $\psi^m_i[V^m] = 0$ for $i \in I \setminus J$ and every $m \geq n$ such that $V = \bigcup_{m\geq n} V_m$ is an open neighborhood of x in X meeting only the same finitely many $\operatorname{supp}\psi_j$ with $j \in J$.

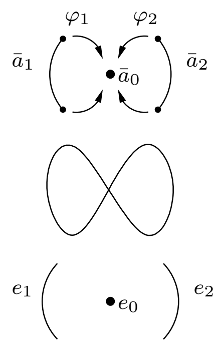
21.13 CW complexes as manifolds

Every CW complex X is a manifold. If it has **countably many cells** and every point $x \in X$ is **locally Euclidean of dimension** n with an n-cell neighborhood it is a **connected n-dimensional manifold** and in that case the **dimension** of the CW complex is also n.

Note: In [8] p. 145 th 5.27 CW complexes are used to show that every 1-**manifold** is homeomorphic to $\mathbb{S}^1$ if it is **compact** and homeomorphic to $\mathbb{B}^1$ if not. The subsequent corollary 5.28 asserts that every 1-**manifold with boundary** is homeomorphic to $\bar{\mathbb{B}}^1$ if it is **compact** and homeomorphic to $\mathbb{R}_0^+$ if not.

Examples:

1. The **sphere** due to enu:20.12.3 has a **CW decomposition** given by $\mathbb{S}^n = e_0 \sqcup e_{n-1} \cong \mathbb{B}^0 \sqcup \mathbb{B}^n$ which according to 21.6 can be extended to a **regular CW decomposition** $\mathbb{S}^n \cong \bigsqcup_{0 \le i \le n} \left(\mathbb{B}^i \sqcup \mathbb{B}^i\right)$. Apart from the trivial representation as the corresponding **disconnected manifold** it can also be described as an n-**dimensional connected manifold** with $2n$ **charts** $\left(\mathbb{S}_i^{n\pm}; \pi_i^{n+1\pm}\right)$ on the **half spheres** $\mathbb{S}^{n\pm} = \mathbb{S}^n \cap \mathbb{H}^{n+1\pm}$ in the **open halfplanes** $\mathbb{H}_i^{n+1+} = \{x_i > 0\}$ and $\mathbb{H}_i^{n+1-} = \{x_i < 0\}$ as defined in 21.12 with the **projections** $\pi_i^{n+1\pm} : \mathbb{S}_i^{n\pm} \to E_i^{n+1} \cong \mathbb{R}^n$ given by $\pi_i^{n+1\pm}(x_1; ...; x_i; ...; x_{n+1}) = (x_1; ...; 0; ...; x_{n+1})$ with **continuous inverse** given by $\left(\pi_i^{n+1\pm}\right)^{-1}(y_1; ...; 0; ...; y_{n+1}) = \left(y_1; ...; \pm\sqrt{1 - \|\boldsymbol{y}\|^2}; ...; y_{n+1}\right)$. By a simple extension of the dimension in the additional charts $\left(\mathbb{S}_{i;\epsilon}^{n+1\pm}; \pi_{i;\epsilon}^{n+1}\right)$ on the **segments** $\mathbb{S}_{i;\epsilon}^{n+1\pm} = \{1 - \epsilon < \|\boldsymbol{x}\| < 1 + \epsilon\} \cap \mathbb{H}_i^{n+1\pm} \subset \mathbb{R}^{n+1}$ with $\pi_{i;\epsilon}^{n+1}(x_1; ...; x_i; ...; x_{n+1}) = (x_1; ...; 1 - \|\boldsymbol{x}\|; ...; x_{n+1})$ we obtain the corresponding n-**dimensional connected submanifold of** $\mathbb{R}^{n+1}$**.**
2. The **closed ball** due to 1. has a **CW decomposition** given by $\bar{\mathbb{B}}^n = e_0 \sqcup e_{n-1} \sqcup e_n \cong \mathbb{S}^n \sqcup \mathbb{B}^n \cong \mathbb{B}^0 \sqcup \mathbb{B}^{n-1} \sqcup \mathbb{B}^n$ resp. a **regular CW decomposition** $\mathbb{S}^n \cong \bigsqcup_{1 \le i \le n} \left(\mathbb{B}^i \sqcup \mathbb{B}^i\right)$. Analogously to 1. it has a representation as an n-**dimensional connected manifold with boundary** with the **interior chart** $\left(\left(1 - \frac{\epsilon}{2}\right)\mathbb{B}^n; \mathrm{id}\right)$ and $2n$ **boundary charts** $\left(\mathbb{SH}_{i;\epsilon}^{n\pm}; \rho_i^{n\pm}\right)$ on the **segments** $\mathbb{SH}_{i;\epsilon}^{n\pm} = \{1 - \epsilon < \|\boldsymbol{x}\| \le 1\} \cap \mathbb{H}_i^{n\pm}$ in the **open halfplanes** $\mathbb{H}_i^{n\pm}$ with the homeomorphisms $\rho_i^{n\pm} : \bar{\mathbb{B}}_*^n \to \mathbb{H}_i^{n\pm}$ given by $\rho_i^{n\pm}(x_1; ...; x_i; ...; x_n) = \left(x_1; ...; \pm\left(1 - \frac{1}{\|\boldsymbol{x}\|}\right); ...; x_n\right)$ with **continuous inverse** given by $\left(\rho_i^{n\pm}\right)^{-1}(y_1; ...; y_i; ...; y_n) = \left(y_1; ...; \frac{\pm 1}{1 - \|\boldsymbol{y}\|}; ...; y_n\right)$. By the additional charts $\left(\left(1 - \frac{\epsilon}{2}\right)\mathbb{B}^{n+1}; \mathrm{id}\right)$ and $\left(\mathbb{SH}_{i;\epsilon}^{n+1\pm}; \rho_i^{n+1\pm}\right)$ we obtain the corresponding n-**dimensional connected submanifold with boundary of** $\mathbb{R}^{n+1}$**.** Note that according to enu:20.12.2 the **closed half sphere** $\mathbb{S}^{n\pm} = \mathbb{S}^n \cap \mathbb{H}^{n\pm} \cong \bar{\mathbb{B}}^n$ is homeomorphic to the **closed ball**.
3. he **lemniscate** $L = e_0 \sqcup e_1 \sqcup e_2$ may be constructed by attaching two equal 1-cells $\bar{a}_1 \cong \bar{\mathbb{B}}^1$ and $\bar{a}_2 \cong \bar{\mathbb{B}}^1$ to a 0-cell $a_0 \cong \bar{\mathbb{B}}^0 = \mathbb{B}^0$. The three cells may also be regarded as the **connected components** of a **disconnected manifold** with the charts $\varphi_0 : \mathbb{B}^0 \to e_0$; $\varphi_2 : \mathbb{B}^1 \to e_1$ and $\varphi_2 : \mathbb{B}^1 \to e_2$ determined by the corresponding **characteristic functions**. According to 20.11 the lemniscate may also be described as a **connected $\mathbb{R}$-manifold** in $\mathbb{R}^2$ and as a **connected $\mathbb{R}$-submanifold** in $\mathbb{R}^3$ but not in $\mathbb{R}^2$.

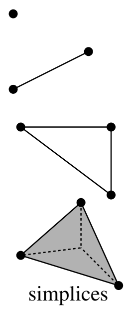


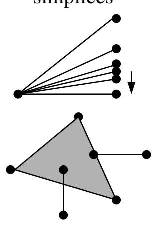
4. The **double cone** $C = \{z^2 = x^2 + y^2\} = C^+ \cup C_*^- \subset \mathbb{R}^3$ with $C^+ = \{(x; y; z) : z \ge 0\}$ and $C_*^- = \{(x; y; z) : z < 0\}$ from 20.13 can be represented as a **disconnected $\mathbb{R}^2$-manifold** and as a **disconnected $\mathbb{R}^2$-submanifold** in $\mathbb{R}^4$ but not in $\mathbb{R}^3$ each with **two connected components.** However it does not have a CW decomposition.

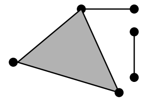
Proof: The n-cells and their homeomorphisms provide **charts** and assure the **Hausdorff** property. According to 4.6 condition 21.2.2 implies that the **injections** $\iota_e : \bar{e} \to X$ for $e \in \mathcal{E}$ into the CW complex $(X; \mathcal{E})$ are **open** maps. Definition 4.10 implies that their **continuous extension** $I : \bigsqcup_{e \in \mathcal{E}} \bar{e} \to X$ is also an **open** map and due to enu:4.8.2 it is a **quotient map**. According to the **closed map lemma** enu:9.8.2 the **continuous extension** $\Phi_I : \bigsqcup_{e \in \mathcal{E}} \bar{a}_e \to \bigsqcup_{e \in \mathcal{E}} \bar{e}$ of the **homeomorphisms** $\varphi_e : a_e \to e$ is also a **quotient map**. Obviously the composition of two quotient maps is a quotient map and so is the **continuous extension of the characteristic maps** $\Phi = I \circ \Phi_I : \bigsqcup_{e \in \mathcal{E}} \bar{a}_e \to X$. Hence X is homeomorphic to a quotient space on a countable disjoint union of closed cells which is **second countable**. Due to 4.7 the quotient space is also **second countable** and so is X. According to 20.7 and 21.9 the CW complex X is **locally finite** and due to the Note in 21.2 for every $x \in X$ exists an open neighborhood W intersecting only finitely many closures $\bar{e}$ of its cells. Then the intersection $U = W \cap e_0$ with one of these cells with maximal dimension k is nonempty and **open in** e_0 since W is open in X. Thus U is an $\mathbb{R}^k$-manifold. For every other cell $e \in \mathcal{E}$ with $e \cap W \neq \emptyset$ we have $e \cap e_0 = \emptyset$ and due to $\bar{e} \setminus e \subset X_{k-1}$ and $e_0 \cap X_{k-1} = \emptyset$ follows $\bar{e} \cap e_0 = \emptyset$ and in particular $\bar{e} \cap U = \emptyset$. Since this is true for every $e \in \mathcal{E} \setminus \{e_0\}$ condition 21.2.2 implies that U is **open in** X and therefore an $\mathbb{R}^n$-manifold. By the **invariance of dimension** 26.1 follows $k = n$. Hence every neighborhood of every $x \in X$ intersects a cell of dimension n but not larger whence follows that the dimension of the CW complex is indeed n.

22 Simplicial complexes

22.1 Definitions


simplices


not simplicial complexes


simplicial complex

A k-**simplex** $\sigma = [\boldsymbol{a}_0; ...; \boldsymbol{a}_k] = \text{co }\{\boldsymbol{a}_0; ...; \boldsymbol{a}_k\} = \left\{\sum_{0 \le i \le k} t_i \boldsymbol{a}_i : t_i \ge 0; \sum_{0 \le i \le k} t_i = 1\right\}$ generated by the **affinely independent** vectors $(\boldsymbol{a}_i)_{0 \le i \le k} \subset \mathbb{R}^n$ with **linearly independent translations** $(\boldsymbol{a}_i - \boldsymbol{a}_0)_{1 \le i \le k}$ is their **convex hull** as defined in [16] 1.6 and furnished with the **subspace topology.** The **barycentric coordinates** $(t_0; ...; t_k) \in [0; 1[^{k+1}$ of $\boldsymbol{x} = \sum_{0 \le i \le k} t_i \boldsymbol{a}_i$ represent the **mass distribution** of the **vertices** $\boldsymbol{a}_i$ if the **center of gravity** is at the **point** x with the associated **vector** $\boldsymbol{x} = \boldsymbol{a}_0 + \sum_{0 \le i \le k} t_i (\boldsymbol{a}_i - \boldsymbol{a}_0)$ in the **affine space** $\boldsymbol{a}_0 + \mathbb{R}^n$ as described in [17] 9.1. Any simplex σ spanned by a subset of the vertices of a simplex σ is called a **face** of σ. Faces generated by two resp. $k-1$ vertices are **edges** resp. **boundary faces**. Due to 9.7 and 9.11 every k-simplex is a **closed** k-**cell** and consequently a **compact** k-**manifold with boundary**. Its **boundary** is the **union of its faces** and identical to the **manifold boundary** as defined in 20.9. A k-simplex without its faces is called its **interior** resp. an **open** k-**simplex** and is defined by $\left\{\sum_{0 \le i \le k} t_i \boldsymbol{a}_i : t_i > 0; \sum_{0 \le i \le k} t_i = 1\right\}$, i.e. the barycentric coordinates are not allowed to vanish. Note that similarly to the manifolds with boundary except for $k = n$ an open k-simplex is **not open in** $\mathbb{R}^n$ and its interior and boundary **do not coincide** with its topological interior and boundary as subset of $\mathbb{R}^n$.

A **simplicial complex** K is a **locally finite** set of simplices **including every face** of every $\sigma \in K$ such that the **intersection of every two simplices is either empty or a face of each**. Its **dimension** is the maximal dimension of the simplices included in K and its union $\bigcup K = |K|$ is the **polyhedron** of K. A subset $K' \subset K$ is a **subcomplex** of K iff it includes every face of every simplex $\sigma \in K'$ such that every subcomplex is also a simplicial complex. The subcomplex comprising every simplicial complex of dimension at most k is the k-**skeleton** of K. Obviously the set of all **interiors** of the elements of a simplicial complex K is a **regular cell decomposition** of its **polyhedron** $|K|$.

Examples:

1. The set including an n-**simplex** together with all of its faces is an n-dimensional simplicial complex whose polyhedron is homeomorpic to $\bar{\mathbb{B}}^n$.

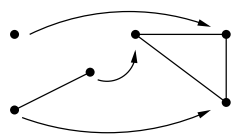
2. The set of all **proper faces** of an n-simplex constitutes an $(n-1)$-dimensional simplicial complex whose polyhedron is homeomorpic to $\mathbb{S}^{n-1}$.
3. The set of all **vertices** and **edges** of a **convex m-polygon** in $\mathbb{R}^2$ with **distinct vertices** is a 1-dimensional simplicial complex whose polyhedron is homeomorpic to $\mathbb{S}^1$.
4. The set of all 1-simplices $[n; n+1]$ and all 0-simplices $\{n\}$ for $n \in \mathbb{Z}$ is a 1-dimensional simplicial complex whose polyhedron is homeomorpic to $\mathbb{R}$.
5. The set of all 1-simplices $[n; n+1]$ and all 0-simplices $\{n\}$ for $n \in \mathbb{N}$ is a 1-dimensional simplicial complex whose polyhedron is homeomorpic to $\mathbb{R}_0^+$.

22.2 Triangulation

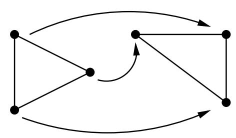
A **triangulation** is a homeomorphism between a topological space and the polyhedron of some simplicial complex. The proofs of the following theorems can be found in [9]:

1. **Triangulation theorem for $\mathbb{R}^2$-manifolds:** Every $\mathbb{R}^2$-manifold is homeomorphic to the polyhedron of a 2-dimensional simplicial complex, in which every 1-simplex is a face of exactly two 2-simplices.
2. **Triangulation theorem for $\mathbb{R}^3$-manifolds:** Every $\mathbb{R}^3$-manifold is triangulable.
3. There are $\mathbb{R}^4$-manifolds which are **not triangulable**.

22.3 Simplicial maps



simplicial map



simplicial isomorphism

According to [17] 9.3 an **affine map** $F : \boldsymbol{a} + E_a \to \boldsymbol{b} + E_b$ between **affine $\mathbb{R}^n$-spaces** with $\boldsymbol{a}; \boldsymbol{b} \in \mathbb{R}^n$ and **vector subspaces** $E_a; E_b \subset \mathbb{R}^n$ is given by $F(\boldsymbol{a}) = \boldsymbol{b}$ and $F(\boldsymbol{a}+\boldsymbol{x}) - F(\boldsymbol{a}) = \boldsymbol{f}(\boldsymbol{x})$ resp. $F(\boldsymbol{a}+\boldsymbol{x}) = \boldsymbol{b} + \boldsymbol{f}(\boldsymbol{x})$ for every $\boldsymbol{x} \in E_a$ and some **linear map** $\boldsymbol{f} : E_a \to E_b$. Due to [16] th. 1.10 the linear map $\boldsymbol{f}$ is **continuous** and so is F with regard to the **closed subsets** $\boldsymbol{a} + E_y$ and $\boldsymbol{b} + E_b$. According to [17] th. 3.7 for any two k-simplices $\sigma = [\boldsymbol{a}_0; ...; \boldsymbol{a}_k]$ and $\tau = [\boldsymbol{b}_0; ...; \boldsymbol{b}_k]$ exists a **unique affine extension** $F : \boldsymbol{a}_o + \mathbb{R}^n \to \boldsymbol{b}_o + \mathbb{R}^n$ with $F[\sigma] = \tau$ of the **vertex map** given by $F(\boldsymbol{a}_i) = \boldsymbol{b}_i$ for $0 \le i \le n$ since there is a **unique endomorphism** $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^n$ with $\boldsymbol{f}(\boldsymbol{a}_i - \boldsymbol{a}_0) = F(\boldsymbol{a}_i) - F(\boldsymbol{a}_0)$ $= \boldsymbol{b}_i - \boldsymbol{b}_0$ such that $F\left(\sum_{0 \le i \le n} t_i \boldsymbol{a}_i\right) = F(\boldsymbol{a}_0) + \boldsymbol{f}\left(\sum_{0 \le i \le n} t_i \boldsymbol{a}_i - \boldsymbol{a}_0\right) = \boldsymbol{b}_0 + \boldsymbol{f}\left(\sum_{0 \le i \le n} t_i (\boldsymbol{a}_i - \boldsymbol{a}_0)\right) = \boldsymbol{b}_0 + \sum_{0 \le i \le n} t_i \boldsymbol{f}(\boldsymbol{a}_i - \boldsymbol{a}_0) = \boldsymbol{b}_0 + \sum_{0 \le i \le n} t_i (\boldsymbol{b}_i - \boldsymbol{b}_0)$ $= \sum_{0 \le i \le n} t_i \boldsymbol{b}_i$.

A **simplicial map** between simplicial complexes K and L is a **continuous** map $F : |K| \to |L|$ whose restriction $F|_\sigma$ to any simplex $\delta \in K$ is an affine map onto a simplex $F[\sigma] \in L$. In the case of a **homeomorphism** it is called a **simplicial isomorphism** . For any given **vertex map** $F_0 : K_0 \to L_0$ between the 0-skeletons K_0 and L_0 of simplicial complexes K and L preserving every simplex $[\boldsymbol{a}_0; ...; \boldsymbol{a}_k] \in K$ such that its image $[F_0(\boldsymbol{a}_0); ...; F_0(\boldsymbol{a}_k)]$ is again a simplex in L exists a unique **simplicial map** $F : |K| \to |L|$ coinciding on K_0 with $F_0 = F|_{K_0}$. If in addition the preimage $\left[F_0^{-1}(\boldsymbol{b}_0); ...; F_0^{-1}(\boldsymbol{b}_k)\right]$ of every simplex $[\boldsymbol{b}_0; ...; \boldsymbol{b}_k] \in L$ is again a simplex in K the continuous extension is a **simplicial isomorphism**. Indeed the map F determined by the unique continuous extensions $F|_\sigma : \sigma \to F[\sigma]$ of the vertex map on every **closed cell** $\sigma \in K$ is **continuous** on the **regular CW complex** $|K| = \bigcup_{\sigma \in K} \sigma$ since every preimage $F|_\sigma^{-1}[O] = F^{-1}[O] \cap \sigma$ of an open set $O \subset |L|$ is open in σ whence due to 21.2.2 $F^{-1}[O]$ is open in $|K|$. A second application of this argument to the inverse yields the asserted homomorphism.

22.4 The Hauptvermutung

A simplicial complex K' is a **subdivision** of the simplicial complex K iff every simplex $\sigma' \in K'$ is contained in a simplex $\sigma' \subset \sigma \in K$ and every simplex $\sigma \in K$ is the union of simplices $\sigma' \in K'$. Two simplicial complexes with a common subdivision are **combinatorally equivalent** and in that case their **polyhedra** coincide: $|K| = |K'|$. The converse assertion is called the **Hauptvermutung** and was conjectured by Ernst Steinitz and Heinrich Tietze in 1908. It is known to be true for all simplicial complexes of **dimension** 2 and **triangulated compact complexes of dimension** 3 but false in all higher dimensions.

23 Compact surfaces

24 Topological groups

24.1 Group actions

A **topological group** is a **group** $(G;\circ)$ endowed with a topology such that the group operation $\circ : G \times G \to G$ is **continuous**. Obviously every **subgroup** $H \subset G$ of a topological group is a topological group with regard to the **trace topology** and every **product** $(G \times H; \circ \times \diamond)$ of topological groups $(G;\circ)$ and $(H;\diamond)$ is a topological group with regard to the **product topology**. For every **continuous left action** $G \times X \to X$ of a **group** G on a **topological space** X defined according to [17] Def. 1.13 by a map $(g;x) \mapsto g \cdot x$ with the **associative law** $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$ and the conformity with the **neutral element** $e \cdot x = x$ for every $x \in X$ the **left translations** $g \cdot X$ are **homeomorphic** to X since **all** maps $x \mapsto g \cdot x$ are obviously continuous and this implies the continuity of their inverses $g \cdot x \mapsto x = \left(g^{-1}g\right) \cdot x$. The same is true for the **right actions** defined by $x \cdot g = g^{-1} \cdot x$. The **orbits** $G \cdot x$ of all $x \in X$ defined in [17] 1.14 form a **partition** of $X = \mathring{\bigcup}_{x \in X} G \cdot x$ since $g \cdot x = h \cdot y \in (G \cdot x) \cap (G \cdot y) \Rightarrow \left(h^{-1}g\right) \cdot x = y \Rightarrow y \in G \cdot x \Rightarrow G \cdot y \subset G \cdot x$ and vice versa. The corresponding **orbit space** X/G is the quotient space with regard to $xGy \Leftrightarrow \exists g \in G : y = g \cdot x$. The action is **transitive** iff $G \cdot x = X$ for every $x \in X$ and it is **free** iff $g \cdot x = x$ implies $g = e$. According to [17] 1.7 in the case of a **subgroup** $H \subset G$ acting on a topological group the orbits gH of the **left** action are the **right cosets** and vice versa. The **orbit space** G/H is then called the **left coset space** of G by H and in the case of coinciding cosets $gH = Hg$ according to [17] 1.8 the orbit space inherits the algebraic structure of a **factor group**.
Obvious examples of topological groups are provided by the real numbers $\left(\mathbb{R}_*^+;\cdot\right) \subset (\mathbb{R}_*;\cdot) \subset (\mathbb{C}_*;\cdot)$, the **circle** $\left(\mathbb{S}^1;\cdot\right) \subset (\mathbb{C}_*;\cdot)$ with the **complex multiplication** and the **torus** $(\mathbb{T}^n;\cdot) = \left(\mathbb{S}^1 \times ... \times \mathbb{S}^1;\cdot\right)$with the **direct group structure** defined according to [17] 1.4 as componentwise multiplication $\boldsymbol{xy} = (x_i \cdot y_i)_{1 \le i \le n}$ for $x_i; y_i \in \mathbb{C}$.

24.2 The general linear group

The continuity of the multiplication and addition on $\mathbb{C}^2 \to \mathbb{C}$, the resulting continuity of polynomials on $\mathbb{C}^{2n} \to \mathbb{C}$ resp. the continuity of the components of the matrix multiplication $\mathbb{C}^{2n^2} \to \mathbb{C}$ imply the continuity of the matrix multiplication $\mathbb{C}^{2n^2} \to \mathbb{C}^{n^2}$ with regard to the corresponding product spaces while **Cramer's rule** [17] 4.3 assures the continuity of the inversion whence the **general linear groups** $GL(n;\mathbb{R}) \subset GL(n;\mathbb{C})$ are topological group with regard to the **product topology** on $\mathbb{R}^{n^2} \subset \mathbb{C}^{n^2}$. Among its subgroups we have the **orthogonal group** $O(n;\mathbb{R}) \subset GL(n;\mathbb{R})$ and the **normal subgroup** $U(n) \subset GL(n;\mathbb{C})$ of the **unitary matrices** defined in [17] 6.6.

From the argument above follows that the **general linear group** $GL(n;\mathbb{R})$ by matrix multiplication **continuously** acts on the left on $\mathbb{R}^n$. According to [17] Def. 3.10 for any pair $\boldsymbol{x};\boldsymbol{y} \in \mathbb{R}^n$ with $x_i \neq 0$ and $y_j \neq 0$ exists a $T^{\mathcal{A}}_{\mathcal{B}} = \left(T^{\mathcal{E}}_{\mathcal{B}}\right)^{-1} * C_{ij} * T^{\mathcal{E}}_{\mathcal{A}} \in GL(n;\mathbb{R})$ with regard to the bases $\mathcal{A} = (\boldsymbol{e}_1;...;\boldsymbol{e}_{i-1};\boldsymbol{x};\boldsymbol{e}_{i+1};...;\boldsymbol{e}_n)$ and $\mathcal{B} = (\boldsymbol{e}_1;...;\boldsymbol{e}_{j-1};\boldsymbol{y};\boldsymbol{e}_{j+1};...;\boldsymbol{e}_n)$ with the **coordinate systems** $T^{\mathcal{E}}_{\mathcal{A}} * \boldsymbol{x} = \boldsymbol{e}_i$ resp. $T^{\mathcal{E}}_{\mathcal{B}} * \boldsymbol{y} = \boldsymbol{e}_j$ and the **index exchange** $C_{ij} * \boldsymbol{e}_i = \boldsymbol{e}_j$ which implies $T^{\mathcal{A}}_{\mathcal{B}} * \boldsymbol{x} = \boldsymbol{y}$. Hence we have $GL(n;\mathbb{R}) * \mathbb{R}^n_* = \mathbb{R}^n_*$ and consequently **two orbits** $\mathbb{R}^n_*$ and $\{\boldsymbol{0}\}$ resp. the orbit space $\mathbb{R}^n_*/GL(n;\mathbb{R})$

$$
\begin{array}{cc}
\begin{array}{ccccccc} 1 & \cdots & i & \cdots & j & \cdots & n \end{array} & \\
E_{ij} = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 0 & & 1 & & \\ & & & \ddots & & & \\ & & 1 & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} & \begin{array}{c} 1 \\ \vdots \\ i \\ \vdots \\ j \\ \vdots \\ n \end{array}
\end{array}
$$

$= \{\pi(\boldsymbol{e}_1);\pi(\mathbf{0})\}$. Its quotient topology comprises the three sets $\{\pi(\boldsymbol{e}_1);\pi(\mathbf{0})\}$, $\{\pi(\boldsymbol{e}_1)\}$ and $\emptyset$ whence it is **not a Hausdorff** space. The corresponding orbits of the **orthogonal group** are the **spheres** $r\mathbb{S}^{n-1}$ with the orbit space $\mathbb{R}^n_*/O(n;\mathbb{R}) = \{\pi(r\boldsymbol{e}_1) : r \geq 0\}$. This space is homeomorphic to $\mathbb{R}^+$ and hence a **Hausdorff** space.

24.3 The torus

The group $(\{\pm 1\};\cdot)$ endowed with the **discrete topology** by multiplication **freely** and **continuously** acts on the **sphere** $\mathbb{S}^n$ with orbits consisting of pairs of **antipodal points** $\pm\boldsymbol{e}$ for $\boldsymbol{e} \in \mathbb{S}^n$ such that its orbit space is homeomorphic to the **projective space** $\mathbb{P}^n$ defined in [17] 9.1 as the orbit space of $(\mathbb{R}_*;\cdot)$ acting on $\mathbb{R}^{n+1}\setminus\{\mathbf{0}\}$ resulting in the orbits $\mathbb{R}_*\cdot\boldsymbol{e}$ for $\boldsymbol{e} \in \mathbb{S}^n$. The action of the additive group $(\mathbb{Z}^n;+)$ endowed with the **direct group structure**, i.e. $\boldsymbol{x}+\boldsymbol{y} = (x_i+y_i)_{1\leq i\leq n}$ by $\boldsymbol{g}\cdot\boldsymbol{x} = \boldsymbol{g}+\boldsymbol{x}$ on $\mathbb{R}^n$ results in the orbit space $\mathbb{R}^n/\mathbb{Z}^n$ which due to the **commutativity** of the componentwise addition due to [17] 1.8 inherits the algebraic structure of a **factor group**. The **exponential quotient map** $\epsilon : \mathbb{R}^n \to \mathbb{S}^{n-1}$ defined by $\epsilon(\boldsymbol{r}) = \left(e^{2\pi i r_k}\right)_{1\leq k\leq n}$ is **continuous**, **open** and **surjective** so that according to **??** it is a **quotient map** and its canonical bijection $\bar{\epsilon} : \mathbb{R}^n/\mathbb{Z}^n \to \mathbb{S}^{n-1}$ is a **homeomorphism**.

25 Homotopy

Two **continuous** maps $f;g : X \to Y$ are **homotopic** iff there is a **continuous** $F : X \times I \to Y$ with $I = [0;1]$ such that $F(t;0) = f(t)$ and $F(t;1) = g(t)$ for every $t \in I$. Obviously homotopy is an **equivalence relation** and the set X is **simply connected iff any two closed paths are homotopic**.

26 Homology

26.1 Invariance of dimension

The dimension n of a finite-dimensional manifold is **uniquely** determined.

Proof: [8] p. 379 problem 13-3

26.2 Invariance of the boundary

Every **manifold with a boundary** is the **disjoint** union of its **interior** and its **boundary**..

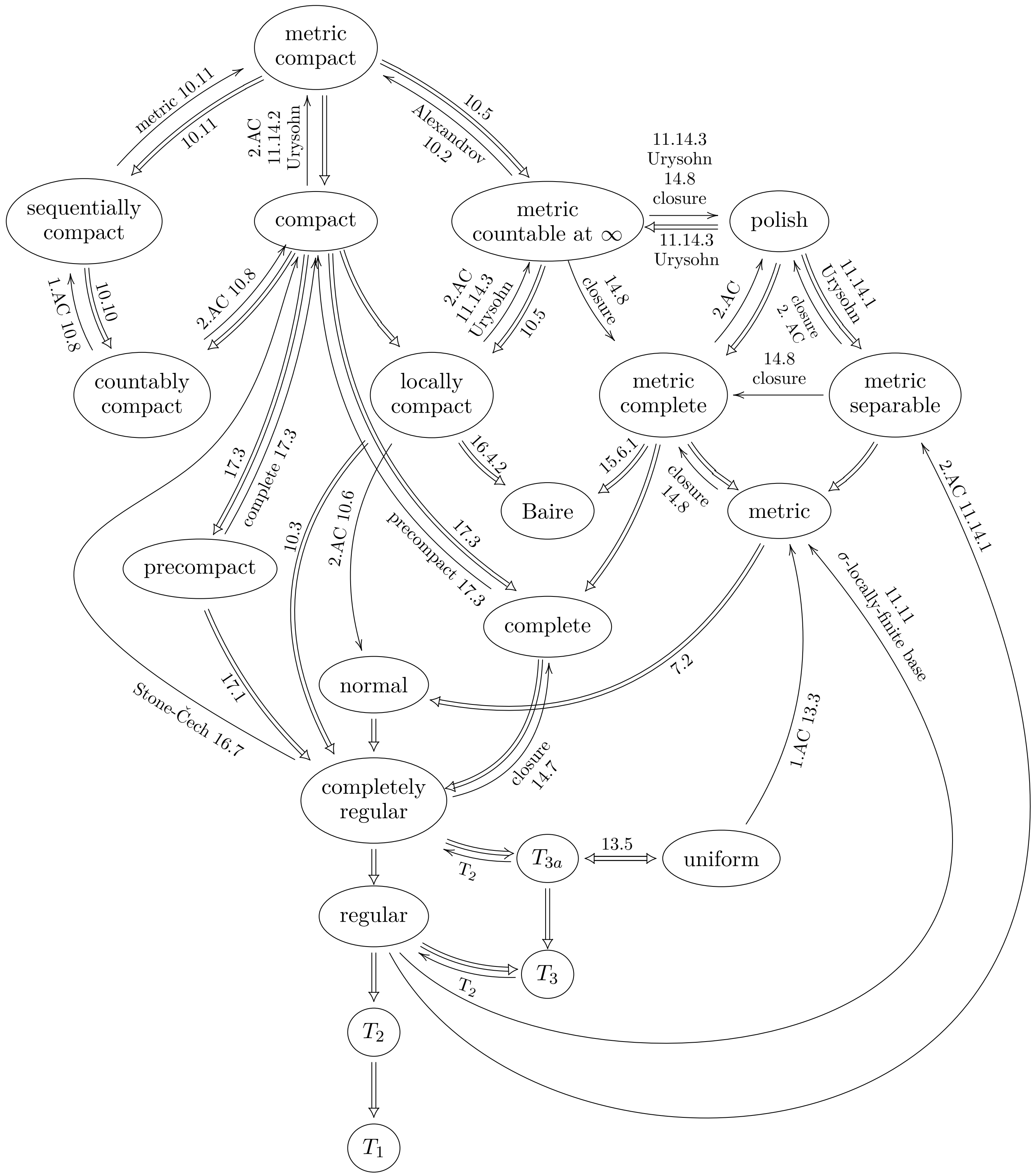
Proof: [8] p. 379 problem 13-4

26.3 Brouwer's fixed point theorem

Every continuous map $f : \bar{\mathbb{B}}^n \to \bar{\mathbb{B}}^n$ has a fixed point $\boldsymbol{x}$ with $f(\boldsymbol{x}) = \boldsymbol{x}$.

Proof: [8] p. 379 problem 13-7

27 Overview



metric compact
sequentially compact
compact
metric countable at ∞
polish
countably compact
locally compact
metric complete
metric separable
Baire
metric
precompact
complete
normal
completely regular
T_{3a}
uniform
regular
T_3
T_2
T_1
metric 10.11
10.11
2.AC 11.14.2 Urysohn
10.5
Alexandrov 10.2
11.14.3 Urysohn 14.8 closure
11.14.3 Urysohn
1.AC 10.8
10.10
2.AC 10.8
2.AC 11.14.3 Urysohn
10.5
14.8 closure
2.AC
closure 2. AC
11.14.1 Urysohn
14.8 closure
16.4.2
15.6.1
closure 14.8
17.3
complete 17.3
10.3
2.AC 10.6
precompact 17.3
17.3
2.AC 11.14.1
σ-locally-finite base
11.11
7.2
17.1
Stone-Čech 16.7
closure 14.7
1.AC 13.3
13.5
T_2
T_2

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