Algebraic Topology

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1 Homotopy

1.1 The fundamental group

Two **continuous** maps $f \simeq g : X \to Y$ between topological spaces X and Y are **homotopic** iff there is a **continuous homotopy** $H : X \times I \to Y$ on $I = [0;1]$ with $H(x;0) = H_0(x) = f(x)$ and $H(x;1) = H_1(x) = g(x)$ for $x \in X$. An $f \in \mathcal{C}(X;Y)$ is **nullhomotopic** iff it is homotopic to a **constant function** $g \equiv y_0 \in Y$ and a space X is **contractible** iff the **identity** $\mathrm{id} : X \to X$ is **nullhomotopic**. Hence a **loop** $f \in \mathcal{C}(I;Y)$ with $f(0) = f(1) = y_0$ is **nullhomotopic** iff its image $f[I]$ is **contractible**. The space Y is **convex** iff any two **paths** $f; g \in \mathcal{C}(I;Y)$ are homotopic by the **linear homotopy** $F : I^2 \to Y$ given by $F(s;t) = (1-t)\,g(s) + tf(s)$. Homotopy is an **equivalence relation** on $\mathcal{C}(X;Y)$ with the set $\pi(X;Y) = \mathcal{C}(X;Y) / \simeq$ of **homotopy classes** $[f]$ of any $f \in C(X;Y)$. The **transitivity** is assured by the **attaching lemma** [4, p. 4.11] since for homotopies $F : f \simeq g$ and $G : g \simeq h$ the extension $H : X \times I \to Y$ defined by

$$H(x;t) = \begin{cases} F(x;2t) & \text{for } 0 \leq t \leq \frac{1}{2} \\ G(x;2t-1) & \text{for } \frac{1}{2} \leq t \leq 1 \end{cases}$$

is continuous with $H_0 = f$ and $H_1 = h$. A homotopy H between maps $f \simeq g \in \mathcal{C}(X;Y)$ is **stationary** on a **closed set** $A \subset X$ iff $H_t|_A = f|_A = g|_A$. Two paths $f \sim g \in \mathcal{C}(I;Y)$ are **path homotopic** iff there is a path homotopy $H \in \mathcal{C}(I^2;Y)$ with $H_0 = f$ and $H_1 = g$ which is stationary on the **initial point** $p = H_t(0) = f(0) = g(0)$ and the **terminal point** $q = H_t(1) = f(1) = g(1)$. Path homotopy is an equivalence relation and its **path classes** $[f]$ are subsets of the **homotopy classes** of f. The path class $[f]$ includes all $f \circ \varphi$ with **continuous** $\varphi : I \to I$. Note that **none** of the hitherto defined functions are required to be **injective**, i.e. any path, loop and reparametrization may **cross** itself. According to the **attaching lemma** [4, p. 4.11] the **product** $f \cdot g : I \to Y$ defined by

$$(f \cdot g)(t) = \begin{cases} f(2t) & \text{for } 0 \leq t \leq \frac{1}{2} \\ g(2t-1) & \text{for } \frac{1}{2} \leq t \leq 1 \end{cases}$$

for **composable paths** $f, g \in \mathcal{C}(I;Y)$ with $f(1) = g(0)$ is again a path. The corresponding product $[f] \cdot [g] = [f \cdot g]$ for **path classes** is well defined since for path homotopic $f_0 \sim f_1$ and $g_0 \sim g_1$ with path homotopies F and G the map

$$H(s;t) = \begin{cases} F(2s;t) & \text{for } 0 \leq s \leq \frac{1}{2} \\ G(2s-1;t) & \text{for } \frac{1}{2} \leq s \leq 1 \end{cases}$$

is a path homotopy for $f_0 \cdot g_0 \sim f_1 \cdot g_1$. The elements of the path class $[c_p]$ of the **constant loop** $c_p \equiv p$ are called the **nullhomotopic loops** and for any path f the **reverse path** $\bar{f}$ is defined by $\bar{f}(s) = f(1-s)$. Any paths $f, g, h \in \mathcal{C}(I;Y)$ with $f(0) = p$ and $f(1) = q$ have the following properties:

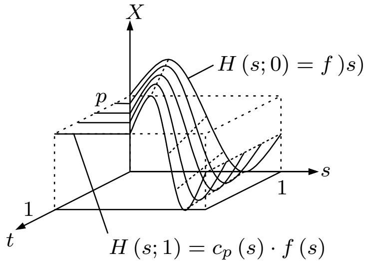
1. **Neutral element**: $[c_p] \cdot [f] = [f] \cdot [c_q] = [f]$
2. **Inverse element**: $[f] \cdot \left[\bar{f}\right] = [c_p]$ and $\left[\bar{f}\right] \cdot f = [c_q]$
3. **Associativity**$([f] \cdot [g]) \cdot [h] = [f] \cdot ([g] \cdot [h])$

In particular the path classes on the set $\Omega(Y,p)$ of all **loops** f with **base point** $p = f(0) = f(1)$ form a **group** which is called the **fundamental group** $\pi_1(Y,p)$ **of** Y **based at** p**.**

Proof:

1. $H : f \sim c_p \cdot f$ with $H(s;t) = \begin{cases} p & \text{for } 2s \le t \\ f\left(\frac{2s-t}{2-t}\right) & \text{for } 2s \ge t \end{cases}$ and

   $\tilde{H} : f \sim f \cdot c_q$ with $\tilde{H}(s;t) = \begin{cases} f\left(\frac{2s}{2-t}\right) & \text{for } 2s \le t \\ q & \text{for } 2s \ge t \end{cases}$.



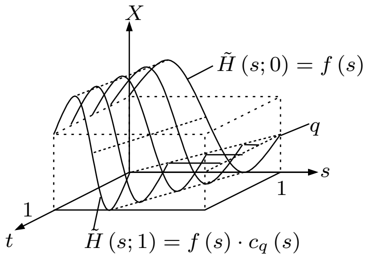
2. $H : c_p \sim f \cdot \tilde{f}$

   given by $H(s;t) = \begin{cases} f(2s) & \text{for } 2s \le t \\ f(t) = \tilde{f}(1-t) & \text{for } t \le 2s \le 1-t \\ \tilde{f}(2s) & \text{for } 1-t \le 2s \end{cases}$

   and

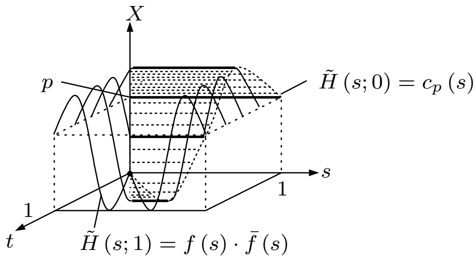
   $\tilde{H} : c_q \sim \tilde{f} \cdot f$

   given by $\tilde{H}(s;t) = \begin{cases} \tilde{f}(2s) & \text{for } 2s \le t \\ \tilde{f}(t) = f(1-t) & \text{for } t \le 2s \le 1-t \\ f(2s) & \text{for } 1-t \le 2s \end{cases}$.



3. $H : (f \cdot g) \cdot h \sim f \cdot (g \cdot h)$

   with $H(s;t) = \begin{cases} f\left(\frac{4s}{1+t}\right) & \text{for } 4s \le 1+t \\ g(4s-t) & \text{for } 1+t \le 4s \le 2+t \\ h((1+t) \cdot 2s - (1+2t)) & \text{for } 2+t \le 4s \end{cases}$



1.2 Homotopies on CW complexes

A map $F : X \times I \to Y$ on a CW complex X with cell decomposition $\mathcal{E}$ and parameter interval $I = [0;1]$ is **continuous,** iff the **restrictions** $F|_{\bar{e} \times I} : \bar{e} \times I \to Y$ for every $e \in \mathcal{E}$ are **continuous**.

Proof: According to [4, 18.10.2 and 21.8.4] the product space $X \times I$ is **compactly generated** whence according to [4, p. 10.15.3] the map F is continuous iff the restrictions $F|_K : K \to Y$ for every **compact** $K \subset X \times I$ are **continuous**. Due to [4, p. 21.1] the closed cells $\bar{e}$ are **compact** which covers $\Rightarrow$. Concerning $\Leftarrow$ we observe that $\pi_x[K] \subset X$ is **compact** and due to [4, p. 21.8.3] included in a finite union $\bigcup_{k \in J} \bar{e}_j$ of closed cells whence $K \subset \bigcup_{k \in J} \bar{e}_j \times I$. Due to the hypothesis the preimage $F^{-1}[O]$ of every open $O \subset Y$ is open in every $\bar{e}_j \times I$ we follow the proof of [4, p. 21.8.3] by concluding, that O is open in $\bigcup_{k \in J} \bar{e}_j \times I$ and finally in the subset $K \subset \bigcup_{k \in J} \bar{e}_j \times I$.

1.3 Change of base point

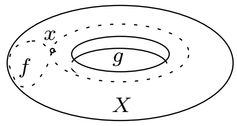
For any two **base points** $p, q \in Y$ in a **path-connected** space Y and every path $g \in \mathcal{C}(I;Y)$ with $g(0) = p$ and $g(1) = q$ the map $\Phi_g : \pi_1(Y,p) \to \pi_1(Y,q)$ defined by $\Phi_g[f] = [\bar{g}] \cdot [f] \cdot [g]$ is an **isomorphism** with **inverse** $\Phi_g^{-1} = \Phi_{\bar{g}}$.

Proof: According to the preceding result 1.3 for any $[f_1], [f_2] \in \pi_1(X,p)$ we have $\Phi_g([f_1] \cdot [f_2]) = [\bar{g}] \cdot [f_1] \cdot [c_p] \cdot [f_2] \cdot [g] = [\bar{g}] \cdot [f_1] \cdot [g] \cdot [\bar{g}] \cdot [f_2] \cdot [g] = \Phi_g[f_1] \cdot \Phi_g[f_2]$. Also for $[h] \in \pi_1(X,q)$ holds $(\Phi_g \circ \Phi_{\bar{g}})[h] = [\bar{g}] \cdot ([g] \cdot [h] \cdot [\bar{g}]) \cdot [g] = ([\bar{g}] \cdot (([g] \cdot [h]) \cdot [\bar{g}])) \cdot [g] = (([\bar{g}] \cdot ([g] \cdot [h])) \cdot [\bar{g}]) \cdot [g] = ((([\bar{g}] \cdot [g]) \cdot [h]) \cdot [\bar{g}]) \cdot [g] = (([c_q] \cdot [h]) \cdot [\bar{g}]) \cdot [g] = ([h] \cdot [\bar{g}]) \cdot [g] = [h] \cdot ([\bar{g}] \cdot [g]) = [h] \cdot [c_q] = [h]$ and analogously for for $[f] \in \pi_1(X,p)$ follows $(\Phi_{\bar{g}} \circ \Phi_g)[f] = [f]$.

1.4 Simple connectivity

A topological space X is **simply connected** iff it is **path connected** and one of the following three equivalent conditions is satisfied:

1. The **fundamental group** is $\pi_1(Y,p) = \{[c_p]\}$ for every $p \in Y$.



2. Every **loop** is path-homotopic to its base point.
3. Any two paths with coinciding initial and terminal points are path-homotopic.

Note: The latter criterion applies to every **convex** subset of $\mathbb{R}^n$ including $\mathbb{R}^n$ itself.

Proof:

1. $\Leftrightarrow$ 2. : Directly follows from the definition.

2. $\Rightarrow$ 3. : The product $f \cdot \bar{g}$ of two paths $f, g \in \mathcal{C}(I;Y)$ with $f(0) = g(0) = p$ and $f(1) = g(1) = q$ is a loop with base point p. The assertion then follows from $f \cdot \bar{g} \sim c_p \overset{25.1.2}{\Leftrightarrow} \bar{g} \sim \bar{f} \overset{\text{Def}}{\Leftrightarrow} g \sim f$.

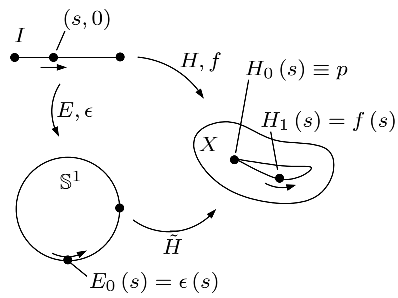
3. $\Rightarrow$ 2. : Any loop $h \in \mathcal{C}(I;Y)$ with base point $h(0) = h(1) = p$ is path-homotopic to the product $f \cdot \bar{g}$ of the two paths given by $f(s) = h\left(\frac{s}{2}\right)$ and $g(s) = h\left(\frac{2-s}{2}\right)$ with $f(0) = h(0) = h(1) = g(0) = p$ and $f(1) = g(1) = h\left(\frac{1}{2}\right) = q$. This implies $g \sim f \overset{\text{Def}}{\Leftrightarrow} \bar{g} \sim \bar{f} \overset{25.1.2}{\Leftrightarrow} f \cdot \bar{g}$.

1.5 Circle representatives

According to the **closed map lemma** [4, p. 9.8.2] every **loop** $f : I \to Y$ is a quotient map such that the **circle representative** $\tilde{f} = \bar{f} \circ \bar{\epsilon}^{-1} : \mathbb{S}^1 \to f[I]$ with the **exponential quotient map** $\epsilon : I \to \mathbb{S}^1$ given by $\epsilon(s) = e^{2\pi i s}$ is a well defined **homeomorphism**. In this case the following three conditions are equivalent:

1. f is a **nullhomotopic** loop.
2. $\tilde{f}$ is **homotopic** to a constant map.
3. $\tilde{f}$ extends to a **continuous** $F : \bar{\mathbb{B}}^2 \to Y$.

Proof:



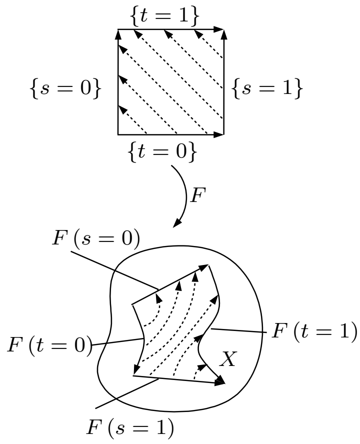
1. $\Leftrightarrow$ 2. : Consider $\tilde{H} = \bar{H} \circ \bar{E} : \mathbb{S}^1 \times I \to Y$ composed from the canonical bijections of the **nullhomotopy** $H : I^2 \to Y$ with $H_0(s) = c_p(s) = p$ and $H_1(s) = f(s)$ for the **base point** $f(0) = f(1) = p$ and the **quotient map** $E = \epsilon \times \text{id} : I^2 \to \mathbb{S}^1 \times I$.

2. $\Rightarrow$ 3. : By the hypothesis exists a **homotopy** $H : \mathbb{S}^1 \times I \to Y$ with $H_0 \equiv q$ and $H_1 = f$. Since $\beta : \left(\mathbb{S}^1 \times I\right)/R_H \to \bar{\mathbb{B}}^2$ given by $\beta\left(\overline{\epsilon; t}\right) = t \cdot \epsilon$ is a **homeomorphism** the composition $F = \bar{H} \circ \beta^{-1} : \bar{\mathbb{B}}^2 \to Y$ is **continuous** and **injective** with $F(1 \cdot \epsilon) = H(\epsilon; 1) = f(\epsilon)$, i.e. $F|_{\mathbb{S}^1} = f$.

3. $\Rightarrow$ 2. : According to the hypothesis the map $H : I^2 \to Y$ given by $H(s;t) = F\left(t \cdot e^{2\pi i s}\right)$ is **continuous** with $H(s;0) = F(0) = p \in Y$ and $H(s;1) = F(\epsilon) = \tilde{f}(\epsilon) = \left(\bar{f} \circ \bar{\epsilon}^{-1}\right)(\omega) = \left(f \circ \epsilon^{-1}\right)(\epsilon) = f(s)$ which proves the assertion.

1.6 The square lemma

For every **continuous** $F : I^2 \to Y$ the paths defined by $f_0(s) = F(s;0)$, $f_1(s) = F(s;1)$, $g_0(t) = F(0;t)$ and $g_1(t) = F(1;t)$ satisfy $f_0 \cdot g_1 \sim g_0 \cdot f_1$.



Proof: The products are defined by $(f_0 \cdot g_1)(t) = \begin{cases} F(2t;0) & \text{for } 0 \le t \le \frac{1}{2} \\ F(1;2t) & \text{for } \frac{1}{2} \le t \le 1 \end{cases}$

and by $(g_0 \cdot f_1)(t) = \begin{cases} F(0;2t) & \text{for } 0 \le t \le \frac{1}{2} \\ F(2t;1) & \text{for } \frac{1}{2} \le t \le 1 \end{cases}$.

The desired homotopy $H : (f_0 \cdot g_1)[I] \to (g_0 \cdot f_1)[I]$ is then given by

$$H(s;t) = \begin{cases} F((1-2s)\,2t; 4st) & \text{for } 0 \le t \le \frac{1}{2} \\ F(2-2s+(2s-1)\,2t; 2s-1+(2-2s)\,2t) & \text{for } \frac{1}{2} \le s \le 1 \end{cases}$$

1.7 Fundamental groups of spheres

Any path $f : I \to M$ on an n-dimensional manifold M with $n \ge 2$ from $f(0) = p_1$ to $f(1) = p_2$ for any given $q \in M \setminus \{p_1;p_2\}$ is path-homotopic to a path g that does not pass through q.

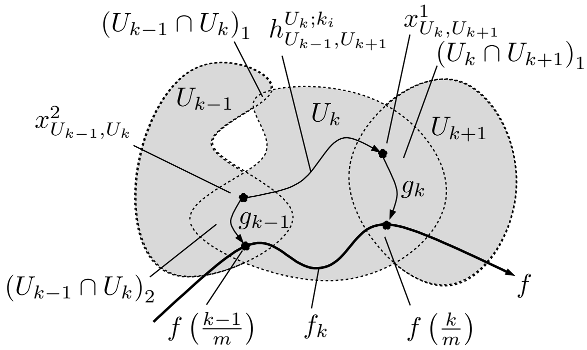
Corollary: For $n \ge 2$ the **sphere** $\mathbb{S}^n$ is **simply connected** since according to 1.4 the Euclidean space $\mathbb{R}^n$ is simply connected and due to [4, p. 20.16.4] resp. [4, p. 20.18] this property carries over to the punctated sphere $\mathbb{S}^n \setminus \{1\} \cong \mathbb{R}^n$ whence the corollary follows from the application of the proposition to $q = 1$.

Proof: For any coordinate ball $q \in U \subset M$ consider the open cover $\{f^{-1}[U]; f^{-1}[M \setminus \{q\}]\}$ of I. According to **Lebesgue's lemma** [4, p. 9.15] exists a w.l.o.g. rational **Lebesgue number** $\frac{1}{m} > 0$ such that for every $0 \le k < m$ holds either $\left[\frac{k}{m}; \frac{k+1}{m}\right] \subset f^{-1}[U] \Leftrightarrow f\left[\frac{k}{m}; \frac{k+1}{m}\right] \subset U$ or $\left[\frac{k}{m}; \frac{k+1}{m}\right] \subset f^{-1}[M \setminus \{q\}] \Leftrightarrow f\left[\frac{k}{m}; \frac{k+1}{m}\right] \subset M \setminus \{q\}$. In the case of $f\left(\frac{k}{m}\right) = q$ follows $f\left[\frac{k-1}{m}; \frac{k+1}{m}\right] \subset U$. Thus we obtain a sequence of closed intervals $I_i = [a_i; a_{i+1}]$ with $0 \le k_i < m$ such that $a_i = \frac{k_i}{m} \ne q$ for all $1 \le i \le r$ and and $\bigcup_{1 \le i \le r} I_i = I$ such that the curve segments $f[I_i]$ lie either in U or in $M \setminus \{q\}$. Due to $n \ge 2$ the set $U \setminus \{q\} \cong \mathbb{B}^n \setminus \{0\}$ is **path connected**, whence for every path segment $f|_{I_i} : [a_i; a_i + 1] \to U$ exists another path $g|_{I_i} : [a_i; a_i + 1] \to U \setminus \{q\}$ and since U is **simply connected** we have $g|_{I_i} \sim f|_{I_i}$ in U and thus in M. Leaving the path segments $f|_{I_i} : [a_i; a_i + 1] \to M \setminus \{q\}$ unchanged we obtain a path $g \sim f$ that does avoid q.

1.8 Fundamental groups of Euclidean manifolds

The fundamental group of an Euclidean manifold is countable.

Proof: According to [4, p. 20.7] the manifold M has a **countable cover** $\mathcal{U}$ of **coordinate balls**. For every **connected component** $(U \cap V)_j$ of the intersection $U \cap V = \bigsqcup_{i \in I_{U,V}} (U \cap V)_i$ of every pair of coordinate balls $U, V \in \mathcal{U}$ we choose a point $x^i_{U,V} \in (U \cap V)_i$ and for every $U, V, W \in \mathcal{U}$ and $x^i_{U,V}, x^j_{U,W} \in U$ we choose a path $h^{U,i,j}_{V,W}$ from $x^i_{U,V}$ to $x^j_{U,W}$ in U. Now choose any point $p \in \mathcal{X} = \left\{x^i_{U,V} : U, V \in \mathcal{U}; i \in I_{U,V}\right\}$ as **base point** and denote a loop based at p as **special** if it is a **finite product** of paths of the form $h^{U,i,j}_{V,W}$. Because both $\mathcal{U}$ and $\mathcal{X}$ are countable, there are only countably many special loops and all of them are element of an equivalence class in $\pi_1(M,p)$. For any other loop f with $[f] \in \pi_1(M,p)$ an application of **Lebesgue's lemma** [4, p. 9.15] as in the proof of 1.7 implies the existence of an $n \in \mathbb{N}$ and for every



$1 \leq k \leq n$ a coordinate ball $U_k \in \mathcal{U}$ such that $f\left[\frac{k-1}{n}; \frac{k}{n}\right] \subset U_k$. The reparametrized paths $f_k : I \to U_k$ defined by $f_k(t) = f|_{\left[\frac{k-1}{n}; \frac{k}{n}\right]}(nt - k + 1)$ then form the product $[f] = [f_1] \cdot ... \cdot [f_n]$. For every $1 \leq k < n$ exists a path $g_k : I \to (U_k \cap U_{k+1})_i$ from $g_k(0) = x^i_{U_k, U_{k+1}}$ to $g_k(1) = f\left(\frac{k}{n}\right) = f_k(1) \in (U_k \cap U_{k+1})_i$ with the base point $f_1(0) = f(0) = f(1) = f_n(1) = p \in (U_1 \cap U_n)_i$. Due to [4, p. 23.5.3] and since every coordinate ball U_k is **simply connected** the products $\tilde{f}_k = g_{k-1} \cdot f_k \cdot \bar{g}_k$ with $g_0 = g_n = c_p$ are path-homotopic to $h^{U_k, i, j}_{U_{k-1}, U_{k+1}}$. Since for every $0 \leq k \leq n$ the paths g_k and $\bar{g}_k$ cancel out we also conclude $f \sim \tilde{f}_1 \cdot ... \cdot \tilde{f}_n$, i.e. **every** loop based at p is **special**.

1.9 Fundamental groups of product spaces

For a **finite product** $X = X_1 \times ... \times X_n$ of topological spaces X_i with **fundamental groups** $\pi_i(X_i; x_i)$ and **projections** $p_i : X \to X_i$ the map $P : \pi_1(X; x) \to \pi_1(X_1; x_1) \times ... \times \pi_1(X_n; x_n)$ defined by $P[f] = (p_{1*}[f]; ...; p_{n*}[f]) = ([p_1 \circ f]; ...; [p_n \circ f]) = ([f_1]; ...; [f_n])$ is an **isomorphism**.

Proof: $P([f] \cdot [g]) = P[f] \cdot P[g]$ is obvious. P is **injective** because $P[f] = ([c_{x_1}]; ...; [c_{x_n}])$ implies **continuous** $H_i : I \times I \to X_i$ with $H_i(s; 0) \equiv x_i$ and $H_i(s; 1) = f_i(s)$ whence due to [4, p. 4.2] $H : I \times I \to X$ with $p_i \circ H = H_i$ is continuous with $H(s; 0) \equiv x$ and $H(s; 1) = (f_1(s); ...; f_n(s)) = f(s)$ which means $f \sim c_x$ with $x = (x_1; ...; x_n)$. By a similar argument P is **surjective** since for every $[f_1]; ...; [f_n] \in \pi_1(X_1; x_1) \times ... \times \pi_1(X_n; x_n)$ the continuous map $f : I \to X$ defined by $p_i \circ f = f_i$ satisfies $P[f] = [f_1]; ...; [f_n]$.

1.10 Homomorphisms induced by continuous maps

The **homomorphism** $f_* : \pi_1(X, p) \to \pi_1(Y, f(p))$ **induced by a continuous map** $f : X \to Y$ via $f_*([\varphi]) = [f \circ \varphi]$ has the following properties:

1. It is **well defined** since $\varphi \sim \psi \Rightarrow f_*(\varphi) \sim f_*(\psi)$
2. It is a **homomorphism** since $f_*([\varphi] \cdot [\psi]) = f_*([\varphi]) \cdot f_*([\psi])$
3. For **continuous** $f : X \to Y$ and $g : Y \to Z$ holds $(g \circ f)_* = g_* \circ f_*$.
4. For every $p \in X$ holds $(\mathrm{id}_X)_* = \mathrm{id}_{\pi_1(X,p)}$.
5. Homeomorphic spaces have isomorphic fundamental groups: For every **homeomorphism** $f : X \cong Y$ the **induced homomorphism** $f_* : \pi_1(X, p) \to \pi_1(Y, f(p))$ is an **isomorphism**.

Proof:

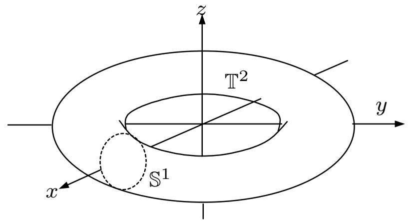
1. obvious
2. $f_*([\varphi] \cdot [\psi]) = f_*([\varphi \cdot \psi]) = [f \circ (\varphi \cdot \psi)] = f_*([\varphi]) \cdot f_*([\psi])$.
3. obvious
4. For every path $\varphi \in \mathcal{C}(I, X)$ we have $(\mathrm{id}_X)_*([\varphi]) = [\mathrm{id}_X \circ \varphi] = [\varphi]$.
5. **Injectivity** follows from $f \circ \varphi \sim f \circ \psi \Leftrightarrow \varphi = f^{-1} \circ f \circ \varphi \sim f^{-1} \circ f \circ \psi = \psi$ and **surjectivity** is a consequence of $f \circ \varphi \sim \psi \Leftrightarrow \varphi \sim f^{-1} \circ \psi$.

1.11 Retractions

Since neither paths nor homotopies are required to be **injective** the properties of **injectivity** and **surjectivity** do not extend independently from φ to the induced map φ_*. For example the **inclusion** $\iota : \mathbb{S}^1 \to \mathbb{R}^2$ is **injective** but ι_* is not since according to 4.9 the **fundamental group of the circle** $\pi_1(\mathbb{S}^1) = (\omega_n)_{n \in \mathbb{N}}$ with the loops from 1.5 given by $\omega_n(s) = e^{2\pi i n s}$ is **infinite cyclic** while $\pi_1(\mathbb{R}^2) = \{[c_o]\}$ is **trivial**. Similarly the loop $\omega_1 : I \to \mathbb{S}^1$ is surjective but $(\omega_1)_*$ is not since I is **convex**, hence **simply connected** with the trivial fundamental group $\pi_1(I) = \{[c_0]\}$.

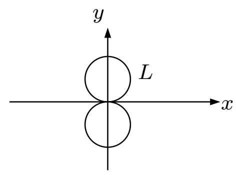
A subset $A \subset X$ is a **retract** of X iff there is a **continuous retraction** $r : X \to A$ such that $r \circ \iota_A = \mathrm{id}_A$ with the **inclusion** $\iota_A : A \to X$. In the case of a retract A according to 1.10.3 we have $(r)_* \circ (\iota_A)_* = (r \circ \iota_A)_* = (\mathrm{id}_A)_* = \mathrm{id}_{\pi_1(A;p)}$ whence $(\iota_A)_* : \pi_1(A;p) \to \pi_1(X;p)$ is **injective** and $(r)_* : \pi_1(X;p) \to \pi_1(A;p)$ is **surjective** for every $p \in A$. In particular **simple connectedness** of X extends to every **retract** $A \subset X$, since the injectivity of $(\iota_A)_*$ means that a trivial $\pi_1(X;p)$ implies a trivial $\pi_1(A;p)$. Conversely in the following three examples the nontrivial nature of $\pi_1(\mathbb{S}^1;1)$ as shown in 4.9 is used to deduce the corresponding character of $\pi_1(X;p)$ from a retract $\mathbb{S}^1 \cong A \subset X$:

Examples:



1. **The retraction $r_1 : \mathbb{R}^2 \setminus \{0\} \to \mathbb{S}^1$ given by $r_1(\boldsymbol{x}) = \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}$ implies that the punctated plane $\mathbb{R}^2 \setminus \{\mathbf{0}\}$ is not simply connected and in particular not homeomorphic to $\mathbb{R}^2$.**

2. **The retraction $r_2 : \mathbb{T}^2 \cong \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^1$ defined by $r_2(x;y) = (x;1)$ proves that the torus $\mathbb{T}^2$ from [4, p. 20.17] is not simply connected.**

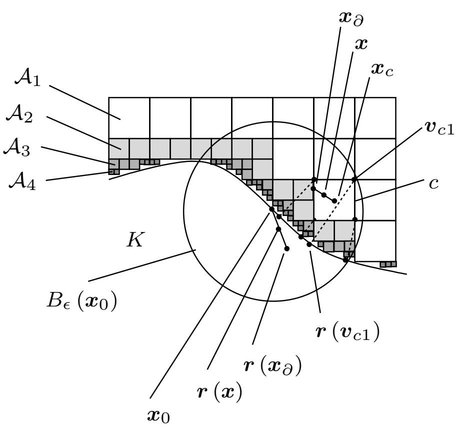


3. **The retraction $r_3 : L \to \mathbb{S}^1$ with $r_3(x;y) = (x;|y|)$ shows that the lemniscate L from [4, p. 20.12] is not simply connected.**

1.12 Retracts in Euclidean space

A compact subset $K \subset \mathbb{R}^n$ is a **retract** of some neighborhood O iff K is **locally contractible** in the **weak sense** that for each $x \in K$ and each neighborhood U of x in K exists a neighborhood $V \subset U$ of x such that the inclusion $\iota : V \to U$ is **nullhomotopic**.

Proof: The collection $\mathcal{A} = (\mathcal{A}_k)_{k \geq 0}$ formed of the families $\mathcal{A}_k$ of ever smaller n-dimensional **closed cubes** towards K in the open space $X = \mathbb{R}^n \setminus K$ with vertices of the form $z \cdot 2^{-k}$ for $z \in \mathbb{Z}$ and sidelength 2^{-k} **not contained** in cubes from the families $(\mathcal{A}_i)_{i<k}$ of larger cubes further outside from K defines a **CW complex of dimension** n since the intersection of any two faces of two cubes is a face of one of them such that the open faces of cubes that are minimal with respect to inclusion among such faces form **open cells** whose boundaries meet a finite number of cells of lower dimensions. This CW complex is **locally finite** since every $\boldsymbol{x} \in X$ has a positive distance from the closed set K whence it meets at most (in the case of a vertex) 2^n closed cells with a minimal sidelength $2^{-k} > 0$ such that $B_{2^{-k-1}}(\boldsymbol{x})$ meets only finitely many adjacent cells.



Next we define a **continuous map** $\boldsymbol{r} : Z \to K$ on a **subcomplex** $Z \subset X$ starting with its 0-skeleton $Z_0 = X_0$ and choosing for each $\boldsymbol{v} \in X_0$ some $\boldsymbol{r}(\boldsymbol{v}) \in \partial K$ with minimal distance $\|\boldsymbol{r}(\boldsymbol{v}) - \boldsymbol{v}\|$. Hence for every $\boldsymbol{x}_0 \in \partial K$ and every $\epsilon > 0$ there is a k with $2^{-k} < \frac{\epsilon}{2}$ and cubes $c \in \mathcal{A}_k \cap B_{\epsilon/2}(\boldsymbol{x}_0)$ such that $r(\boldsymbol{v}_c) \subset B_\epsilon(\boldsymbol{x}_0) \cap K$ for every vertex $\boldsymbol{v}_c$ of c. For the induction step we assume a continuous $\boldsymbol{r}_k : Z_k \to K$ for every cube $c \in X_{k+1} \cap B_{\epsilon/2}(\boldsymbol{x}_0)$ with centre $\boldsymbol{x}_c$ satisfying $\partial c \subset Z_k$ and $\boldsymbol{r}_k[\partial c] \subset B = B_\epsilon(\boldsymbol{x}_0) \cap K$ for some $\boldsymbol{x}_0 \in \partial K$ and $\epsilon > 0$ small enough such that we have a **homotopy** $H : B \times I \to B$ with $H_0 = \mathrm{id}_B$ and $H_1 \equiv \boldsymbol{x}_0$. Then for every $\boldsymbol{x} \in c$ exists a unique $t_x \geq 0$ and an $\boldsymbol{x}_\partial \in \partial c$ such that $(\boldsymbol{x}_\partial - \boldsymbol{x}_c) = t_x(\boldsymbol{x} - \boldsymbol{x}_c)$ whence $\boldsymbol{r}_k : \partial c \to K$ extends to $\boldsymbol{r}_{k+1} : c \to K$ by $\boldsymbol{r}_{k+1}(\boldsymbol{x}) = H(\boldsymbol{r}_k(\boldsymbol{x}_\partial); t_x) \subset B$ have defined an extension of $\boldsymbol{r}_k$ to the union Z_{k+1} of all $(k+1)$-dimensional closed cubes allowing such an extension. The induction terminates with the union of the n-dimensional cubes $Z = Z_n$ and the corresponding restriction $\boldsymbol{r}_n : Z \to K$. The final extension to $\boldsymbol{r} : Z \sqcup K \to K$ defined by $\boldsymbol{r}|_Z = \boldsymbol{r}_n$ and $\boldsymbol{r}|_K = \mathrm{id}_K$ is **continuous** since for every $\boldsymbol{x}_0 \in \partial K$ exists a $\epsilon > 0$ such that $\boldsymbol{r}\left[B_{\epsilon/2}(\boldsymbol{x}_0)\right] \subset B_\epsilon(\boldsymbol{x}_0) \cap K$. Due to the **compactness** of K it is covered by a finite collection of open balls $B_{\epsilon/2}(\boldsymbol{x}_0)$ whose union is the desired neighborhood $O \supset K$.

Conversely it suffices to show that a retract $A = r[X]$ of a locally contractible space X is again locally contractible since every open set in $\mathbb{R}^n$ is locally contractible. Assuming this hypothesis for every neighborhood U of every $\boldsymbol{x}_0 \in A$ exists a neighborhood $\boldsymbol{x}_0 \in V \subset U$ such that $r^{-1}[V]$ is contractible in $r^{-1}[U]$ by some continuous $H : r^{-1}[V] \times I \to r^{-1}[V]$ with H_0= id and $H_1 \equiv r^{-1}(\boldsymbol{x}_0)$ whence $V \cap A$ is contractible in $U \cap A$ by $r \circ H : V \times I \to V$.

1.13 Euclidean neighborhood retracts

A **compact** space X is an **Euclidean neighborhood retract** with a topological **embedding** $\iota : X \to \mathbb{R}^n$ such that $\iota[X]$ is a **retract** of some neighborhood in $\mathbb{R}^n$, iff it can be embedded as a retract of a **finite simplicial complex**.

Proof: According to [4, p. 22.3] the affine extension of the **vertex map** F assigning every vertex v_i of a finite simplicial complex K with $n+1$ vertices to the point $F(v_0) = 0$ resp. $F(v_i) = \mathbf{0} + \boldsymbol{e}_{i-1}$ in the affine space $\mathbf{0} + \mathbb{R}^n$ is a **simplicial isomorphism** to a subcomplex of the simplicial complex Δ^n generated by the faces of an n-simplex $\sigma \subset \mathbb{R}^n$. Hence the map $F : |K| \to \mathbb{R}^n$ is an **embedding** onto the obviously **locally contractible subset** $F[|K|] \subset |\Delta^n| \subset \mathbb{R}^n$ which by the preceding theorem 1.12 it is a retract of some neighborhood $U \subset \mathbb{R}^n$. By the composition of the two retracts every compact retract $\varphi[X] \subset |K|$ of the **embedding** $\varphi[X]$ into the **polyhedron** $|K|$ of the finite simplicial complex K is homeomorphic to the retract $\iota[X] = (F \circ \varphi)[X] \subset F[|K|] \subset U \subset \mathbb{R}^n$.

Conversely any compact retract $K \subset U \subset \mathbb{R}^n$ of a neighborhood U in $\mathbb{R}^n$ is **bounded** whence it is the subset of some large simplex Δ^n. By successive barycentric subdivision we obtain a simplicial complex L with equal polyhedron $|L| = |\Delta^n|$ whose simpleces have a diameter less than half of the distance between $|\Delta^n|$ and $\mathbb{R}^n \setminus U$. The restriction $r|_{|L_K|}$ of the given retraction $r : U \to K$ to the polyhedron of th subcomplex $L_K \subset L$ of all simplices in L meeting K then provides the desired image of K as a retract of $|L_K|$.

1.14 Compact manifolds and finite CW complexes

Every **compact manifold**, with or without boundary, and in particular every **finite CW complex** is an Euclidean neighborhood retract.

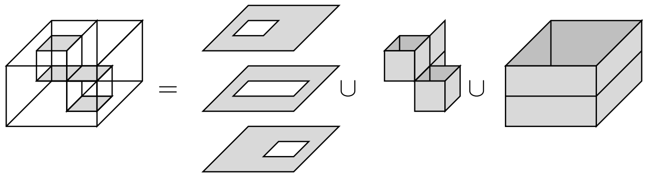
Proof: According to [4, 20.10 and 4.12.2]every **compact manifold** M **with boundary** embeds into the closed subspace M of its **double** $D(M) = M \cup_{\mathrm{id}} M$ which is a **compact manifold without boundary**. Due to [4, 21.8.3 and 21.13] **and** every **finite CW complex** is is also a **compact manifold without boundary**. Every such compact manifold of dimension n is covered by finitely many **open cells** $U_i = \psi_i^{-1}[\mathbb{B}^n]$ with **coordinate functions** ψ_i for $1 \leq i \leq m$. The **quotient maps** $f_i : M \to \mathbb{S}^n$ defined by $f_i|_{U_i} = \varphi \circ \psi_i : U_i \to \mathbb{S}^n \setminus \{\boldsymbol{e}_{n+1}\}$ resp. $f_i|_{M \setminus U_i} \equiv \boldsymbol{e}_{n+1}$ with the **quotient maps** $\varphi : \bar{\mathbb{B}}^n \to \mathbb{S}^n$ with $\varphi[\mathbb{S}^{n-1}] = \{\boldsymbol{e}_{n+1}\}$ from [4, p. 20.16.1] are the components of an **injective**, **continuous** and **open** map $f : M \to (\mathbb{S}^n)^m$. By composition with the **canonical injection** $\iota : (\mathbb{S}^n)^m \to \mathbb{R}^{n \cdot m}$ we obtain a continuous injection $\iota \circ f : M \to \mathbb{R}^{n \cdot m}$ which by the **closed map lemma** [4, p. 9.8.3] is an **embedding**. Since M is **locally contractible** by its **charts** the assertion follows from 1.12.

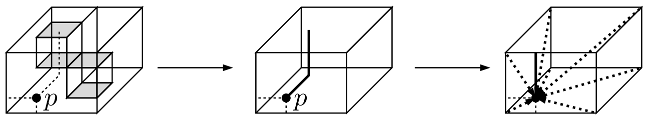
1.15 Deformation retractions

A retraction $r : X \to A$ is a **deformation retraction** iff $\iota_A \circ r \simeq \mathrm{id}_X$, i.e. there is a homotopy $H : X \times I \to X$ with $H_0 = \mathrm{id}_X$ and $H_1 : X \to A$ with $H_1|_A = \mathrm{id}_A$. Since every retraction satisfies $r \circ \iota_A = \mathrm{id}_A$ this implies $A \simeq X$ and A is then called a **deformation retract** of X. A retraction $r : X \to A$ is a **strong deformation retraction** iff $\iota_A \circ r \simeq \mathrm{id}_X$ **and** the homotopy is **stationary** on A with $H_t|_A = \mathrm{id}_A$ for every $t \in I$. From $\iota_A \circ r \simeq \mathrm{id}_X$ due to 1.10.3 then follows $(\iota_A)_* \circ r_* = (\iota_A \circ r)_*$ $= (\mathrm{id}_X)_*$ whence $r_* : \pi_1(X; p) \to \pi_1(A; p)$ is **injective**.

In the case of a single point $A = \{p\}$ the **contraction** $H_{\boldsymbol{p}}$ to **one** $\boldsymbol{p} \in X$ yields deformation retractions to **any** $\boldsymbol{q} \in X$ by $H_{\boldsymbol{q}}(\boldsymbol{x};t) = \begin{cases} H_{\boldsymbol{p}}(\boldsymbol{x};2t) & \text{for } 0 \le t \le \frac{1}{2} \\ H_{\boldsymbol{p}}(\boldsymbol{q};2-2t) & \text{for } \frac{1}{2} \le t \le 1 \end{cases}$. Hence a set $X \subset \mathbb{R}^n$ is **contractible** in the sense of 1.1 iff **one or equivalently every point** of X is a **deformation retract** of X.

Examples:





1. The **house with two rooms** R depicted above has a **regular CW decomposition** with 29 0-cells, 51 1-cells and 23 2-cells resulting in the **Euler characteristic** $\chi = 29 - 51 + 23 = 1$. The lower floor is accessible from above via the upper room and vice versa. **Every point** $p \in R$ **is a strong deformation retract**:
   In a first stage the interior sides are moved linearly inwards until the two convexities are filled in with the exception of a small access tunnel ending in p. In the second move a simple translation may be used to straighten the tunnel and finally the cubicle is linearly contracted towards the centre p.

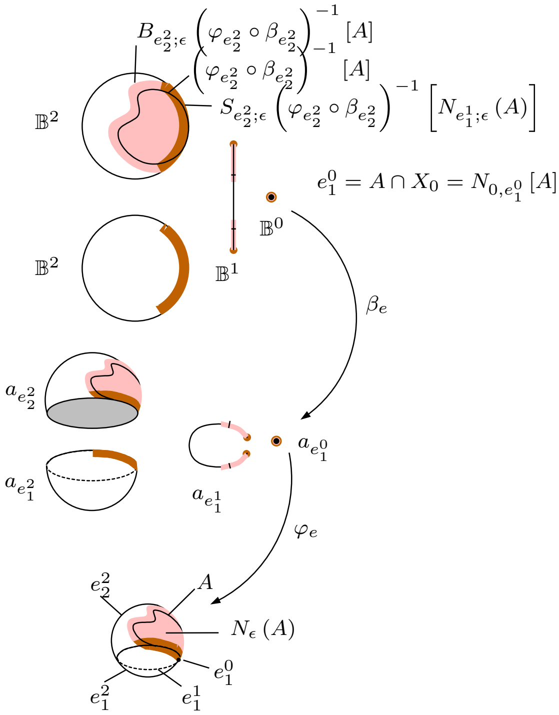
6. The central accumulation point $\{\boldsymbol{0}\} \subset X = \bigcup_{n\in\mathbb{N}} \bar{I}_n \subset \mathbb{R}^2$ from [4, p. 21.2.3] is a **strong deformation retract** by $H(\boldsymbol{x};t) = (1-t)\,\boldsymbol{x}$ while the peripheral accumulation point $\{\boldsymbol{e}\}$ with the coordinate $\boldsymbol{e} = (0;1)$ is a **deformation retract** by $G(\boldsymbol{x};t) = \begin{cases} (1-2t)\,\boldsymbol{x} & \text{for } 0 \le t \le \frac{1}{2} \\ (0;2t-1) & \text{for } \frac{1}{2} \le t \le 1 \end{cases}$.
   If we suppose a **strong** deformation retraction $F : X \times I \to X$ by **continuity** the preimage $F^{-1}[B_\epsilon(\boldsymbol{e})]$ must be an open set whence the condition $F_t(\boldsymbol{e}) = \boldsymbol{e}$ for $0 \le t \le 1$ implies that it includes an open neighborhood $B_\delta(\boldsymbol{e};0)$ such that $F_t(\boldsymbol{e}_n) \in B_\epsilon(\boldsymbol{e})$ for $t < \delta$ contrary to $F_0 = \mathrm{id}_X$. Hence a point of a **contractible** X is **not necessarily a strong deformation retract** of X. This phenomenon is typical for **self-approaching curves** like the **lemniscate** L from [4, p. 20.12.4] which is an **immersion** but not an **embedding** with regard to the **trace topology** in $\mathbb{R}^2$.

1.16 Deformation retracts in CW complexes

Every subset $A \subset X$ of a CW complex X has **open neighborhoods** $N_\epsilon(A)$ for arbitrarily small $\epsilon > 0$. Every **subcomplex** A is a **deformation retract** of its open neighborhood $N_\epsilon(A)$. Also every CW complex has a **basis of contractible neighborhoods** $N_\epsilon(x)$ with $x \in X$.

Proof: By induction starting with $N_\epsilon^0(A) = A \cap X_0$ we define **neighborhoods** $N_\epsilon^n(A) = \bigcup_{e\in\mathcal{E}_n} N_{e;\epsilon}^n(A)$ with $N_{e;\epsilon}^{n+1}(A) = \varphi_e\left[B_{e;\epsilon}\left(\varphi_e^{-1}[A]\right) \cup \bigcup_{f\in\mathcal{E}_n} S_{e;\epsilon}\left(\varphi_e^{-1}\left[N_{f;\epsilon}^n(A)\right]\right)\right]$ using **segments** $S_{e;\epsilon}(B) = \left\{x \in \bar{a}_e : \frac{\tilde{x}}{\|\tilde{x}\|} \in \tilde{B} \wedge 1 - \epsilon < \|\tilde{x}\| \le 1\right\}$ as defined in [4, p. 21.12] of subsets $B \subset \partial\bar{a}_e$ with $\tilde{x} = \beta_e^{-1}(x)$ for the **homeomorphism** $\beta_e : \bar{\mathbb{B}}^n \to \bar{a}_e$ and **balls** $B_{e;\epsilon}(C) = \beta_e\left[B_\epsilon\left(\tilde{C}\right)\right]$ for subsets $C \subset a_e$ and $0 < \epsilon < 1$. The $N_\epsilon^n(A)$ are **open in** X_n but since the n-cells $d \in \mathcal{E}_n$ are open in X_n but not necessarily in X the segments $S_{e;\epsilon}\left(\varphi_e^{-1}\left[N_{d;\epsilon}^n(A)\right]\right)$ in the $(n+1)$-cell $e \in \mathcal{E}_{n+1}$ of the union of the preimages of all neighborhoods $N_{d;\epsilon}^n(A)$ of n-cells $d \in \mathcal{E}_n$ meeting its closure $\bar{e}$ are needed to provide an open buffer around each boundary $\bar{e} \setminus e$ into the **next dimension** such that the set $N_\epsilon(A) = \bigcup_{n\ge 0} N_\epsilon^n(A)$ is **open** in X.

By $H: I \times S_\epsilon(B) \to S_\epsilon(B)$ with $H(t;y) = \frac{\boldsymbol{y}}{1+t(\|\boldsymbol{y}\|-1)}$ each subset $B \subset \partial\bar{\mathbb{B}}^n$ is a **deformation retract** of its **segment** $S_\epsilon(B)$ in $\bar{\mathbb{B}}^n$. In the case of a **subcomplex** A consisting of closures $\bar{e}$ the **balls** $B_{e;\epsilon}\left(\varphi_e^{-1}[A]\right) \subset a_e$ are either empty or coincide with a_e. Hence a deformation retraction of the neighborhood $N_\epsilon^n(A)$ affects only the segments and can be realized by $H(t;x) = \begin{cases} H_e(t;x) & \text{for } (t;x) \in I \times e; e \in \mathcal{E}_n \\ (t;x) & \text{else} \end{cases}$ with $H_e = \varphi_e \circ \beta_e \circ H \circ (\eta_e \times \varphi_e \circ \beta_e)^{-1} : I \times N_\epsilon^{n+1}(A) \to N_\epsilon^{n+1}(A)$ for $\eta_e : I \to \left[2^{-n-1}; 2^{-n}\right]$ defined by $\eta_e(t) = 2^{-n-1} \cdot (t+1)$ for $e \in \mathcal{E}_n$ and $n \geq 0$. Note that that the retraction "starts" at infinity $n = \infty$ and proceeds to $n = 0$ such that each neighborhood $N_\epsilon^n(A)$ is a **deformation retract** of the **preceding neighborhood** $N_\epsilon^{n+1}(A)$ whence the **subcomplex** A is a **deformation retract** of its open neighborhood $N_\epsilon(A)$.

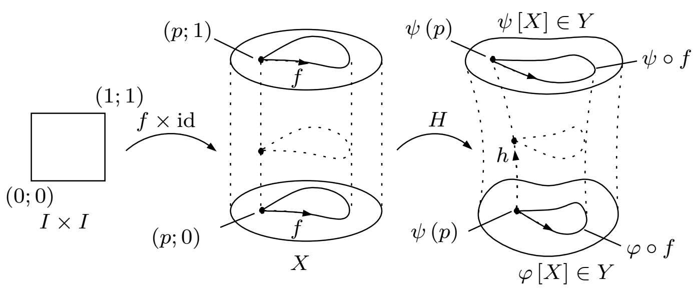


For any $x \in X$ exists an $n \geq 0$ such that $x \in e \subset X_n \setminus X_{n-1}$ for an open n-cell $e \in \mathcal{E}_n$. Hence $\varphi_e^{-1}(x) \in a_e$ and there is an $\epsilon > 0$ with $B_\epsilon\left((\varphi_e \circ \beta_e)^{-1}(x)\right) \subset \mathbb{B}^n$ resp. an **contractible neighborhood** $N_\epsilon^n(x) = (\varphi_e \circ \beta_e)\left[B_\epsilon\left((\varphi_e \circ \beta_e)^{-1}(x)\right)\right] \subset e$ which is **open** in X_n. The set $N_\epsilon(x) = \bigcup_{m \geq n} N_\epsilon^m(x)$ with $N_\epsilon^{m+1}(x) = \varphi_e\left[\bigcup_{d \in \mathcal{E}_m} S_{e;\epsilon}\left(\varphi_e^{-1}\left[N_{d;\epsilon}^m(x)\right]\right)\right]$ is open in X and contractible by $H(t;y) = \begin{cases} H_d(t;y) & \text{for } (t;y) \in I \times d; d \in \mathcal{E}_m; m > n \\ H_e(t;y) & \text{for } (t;y) \in I \times e \\ (t;y) & \text{else} \end{cases}$ with H_d defined as above and $H_e : \left[\frac{1}{2}; 1\right] \times N_\epsilon^n(x) \to \{x\}$ defined by $H_e(t;y) = (1-2t) \cdot \left((\varphi_e \circ \beta_e)^{-1}(x) - (\varphi_e \circ \beta_e)^{-1}(y)\right)$.

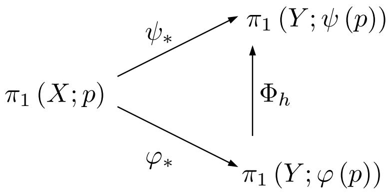
1.17 Homotopy equivalence

A **continuous** $\psi : Y \to X$ is a **homotopy inverse** for the **continuous** $\varphi : X \to Y$ iff $\psi \circ \varphi \simeq \mathrm{id}_X$ and $\varphi \circ \psi \simeq \mathrm{id}_Y$. In this case $X \simeq Y$ are **homotopy equivalent** which defines an equivalence relation on the set of all topological spaces. Note that the special case of a **deformation retract** $X \subset Y$ with the **inclusion** $\iota : X \to Y$ and the **deformation retraction** $r : Y \to X$ is **not symmetric**. Properties which are preserved by homotopy equivalence are **homotopy invariants** . For every **homotopy equivalence** $\varphi : X \simeq Y$ and any point $p \in X$ the **induced homomorphism** $\varphi_* : \pi_1(X;p) \to \pi_1(Y;\varphi(p))$ is an **isomorphism**.

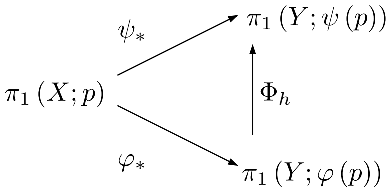
Lemma: According to 1.3 the **change of base point** $\Phi_h : \pi_1(Y;\varphi(p)) \to \pi_1(Y;\psi(p))$ for any two **homotopic** $\varphi;\psi : X \to Y$ defined by $\Phi_h[f] = \left[\bar{h}\right] \cdot [f] \cdot [h]$ for the path $h : I \to Y$ given by the restriction $h(t) = H(p;t)$ of the homotopy $H : X \times I \to Y$ with $H_0 = \varphi$ and $H_1 = \psi$ is an **isomorphism**.



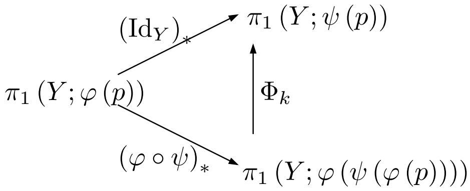
By the **square lemma** 1.6 applied to $F : I \times I \to Y$ defined by $F(x;t) = H(f(s);t)$ then follows that $h \cdot (\psi \circ f) \sim (\varphi \circ f) \cdot h$ $\Leftrightarrow \psi \circ f \sim \bar{h} \cdot (\varphi \circ f) \cdot h \Leftrightarrow \psi_* [f] = \Phi_h (\varphi_* [f]) \Leftrightarrow \psi_* = \Phi_h \circ \varphi_*$, i.e. the **first** diagram on the right **commutes.**



Proof: In the sequence $\pi_1(X;p) \xrightarrow{\varphi_*} \pi_1(Y;\varphi(p)) \xrightarrow{\psi_*} \pi_1(X;\psi(\varphi(p))) \xrightarrow{\varphi_*} \pi_1(Y;\varphi(\psi(\varphi(p))))$ the first φ_* is **injective** since due to $\psi \circ \varphi \simeq \mathrm{Id}_X$ the lemma applied to the situation in the **second** diagram at the right hand side shows that $\psi_* \circ \varphi_* = \Phi_h$ which is an isomorphism. Hence φ_* is **injective** and ψ_* is **surjective**.

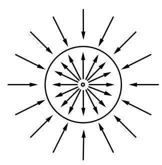


Similarly the application to the homotopy $K : \varphi \circ \psi \simeq \mathrm{Id}_Y$ in the **third** diagram yields that $\varphi_* \circ \psi_* : \psi \circ \varphi \simeq \mathrm{Id}_X \to \pi_1(Y;\varphi(\psi(\varphi(p))))$ is an **isomorphism** whence ψ_* is **injective** and consequently an isomorphism. This implies that $\varphi_* = (\psi_*)^{-1} \circ \Phi_h : \pi_1(X;p) \to \pi_1(Y;\varphi(p))$ is also an isomorphism.



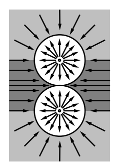
Examples:

1. By the **straight-line homotopy** $H : X \times I \to X \times I$ defined by $H(\boldsymbol{x};t) = (1-t)\boldsymbol{x} + t\frac{\boldsymbol{x}}{|\boldsymbol{x}|}$ with the strong deformation retraction $r : X \to \mathbb{S}^{n-1}$ given by $r(\boldsymbol{x}) = H(\boldsymbol{x};1) = \frac{\boldsymbol{x}}{|\boldsymbol{x}|}$ the **sphere** $\mathbb{S}^{n-1}$ is a strong deformation retract of the punctated set $X^* = X \setminus \{\mathbf{0}\}$ of any **star-shaped** set $\mathbf{0} \in X \subset \mathbb{R}^n$ with a **linear contraction** $H_{\boldsymbol{p}} : X \times I \to \boldsymbol{p}$ given by $H(\boldsymbol{x};t) = \{(1-t)\boldsymbol{x} + t\boldsymbol{p}\}$ for some $\boldsymbol{p} \in X$. From 1.7 and 1.15 follows that in the case of $n \geq 3$ the punctated space X^* is **simply connected**. In particular the **punctated balls** $\bar{\mathbb{B}}^n \setminus \{\mathbf{0}\}$; $\mathbb{B}^n \setminus \{\mathbf{0}\}$ and the **punctated space** $\mathbb{R}^n \setminus \{\mathbf{0}\}$ are **simply connected**.



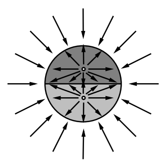
$\mathbb{S}^1 \simeq \mathbb{B}^2 \setminus \{\mathbf{0}\}$

6. The theorem 1.17 above shows that every **deformation retract** $r : X \to A \subset X$ with its homotopy inverse $\iota_A : A \to X$ implies a homotopy equivalence $A \simeq X$. Hence the converse reasoning in the case of $n = 2$ allows the application of 4.9 to the **punctated disks** $\bar{\mathbb{B}}^2 \setminus \{\mathbf{0}\}$; $\mathbb{B}^2 \setminus \{\mathbf{0}\}$ as well as the **punctated plane** $\mathbb{R}^2 \setminus \{\mathbf{0}\}$ whose **fundamental groups are consequently infinite and cyclic**.



$\mathcal{E} \simeq \mathbb{B}^2 \setminus \{(0;\pm 1)\}$

6. The **straight-line homotopy** can also be applied to different sections of the plane with common boundaries whence the continuity of the composition is assured by the **attaching lemma** [4, p. 4.11]. Hence the **figure-eight-space** $\mathcal{E} = \left\{x^2 + (y \pm 1)^2 = 1\right\}$ and the **theta space** $\Theta = \{x^2 + y^2 = 4\} \cup \{(x;0) : -2 \leq x \leq 2\}$ are both strong deformation retracts of $\mathbb{R}^2 \setminus \{(0;\pm 1)\}$. Since homotopy equivalence is **transitive** we conclude $\mathcal{E} \simeq \Theta$. This example will be generalized in the following theorem 1.18.

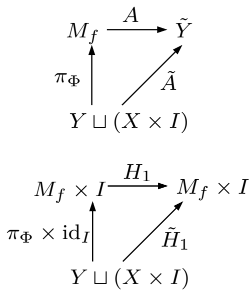


$\Theta \simeq \mathbb{B}^2 \setminus \{(0;\pm 1)\}$

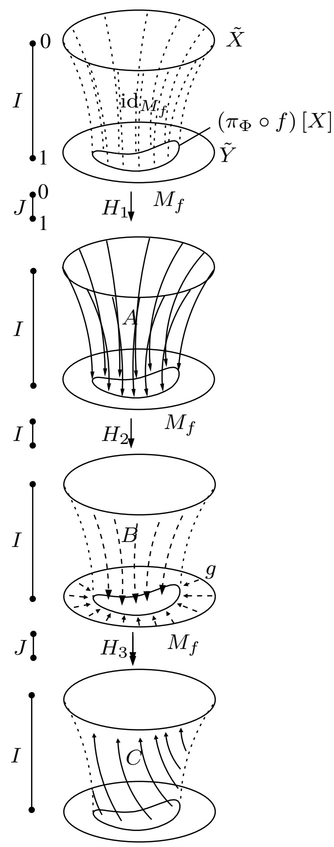
7. According to 1.15 a topological space X is **contractible** iff it is **homotopy equivalent** to a **one-point-space**.

1.18 Homotopy equivalence and deformation retractions

Two spaces X and Y are **homotopy equivalent** via $f : X \to Y$ iff they are both homeomorphic to **deformation retracts** of their **mapping cylinder** M_f**.**



Proof: From [4, p. 4.13] we recall the definition of the **mapping cylinder** $M_f = Y \cup_\varphi (X \times I)$ with the **identifying map** $\varphi : X \times \{1\} \to Y$ given by $\varphi(x;1) = f(x)$. For every $0 \leq s < 1$ the **saturated closed** subsets $X \times \{s\} \subset Y \sqcup (X \times I)$ are homeomorphic to X whence follows $\tilde{X} = \pi_\Phi[X \times \{0\}] \cong X$ while [4, p. 4.12.2] implies $\tilde{Y} = \pi_\Phi[Y] \cong Y$. The map $\tilde{A} : Y \sqcup (X \times I) \to \tilde{Y}$ defined by $\tilde{A}(x;s) = \pi_\Phi(x;1) = (\pi_\Phi \circ f)(x) = (\pi_\Phi \circ f \circ \pi_X)(x;1)$ and $\tilde{A}(y) = \pi_\Phi(y)$ is **continuous**. Due to the **universal property** [4, p. 4.7] its **passing to the quotient** $A = \tilde{A} \circ \pi_\Phi^{-1} : M_f \to \tilde{Y}$ also is **continuous** and it satisfies $A|_{\tilde{Y}} = \mathrm{id}_{\tilde{Y}}$ whence according to 1.11 it is a **retraction.**



Similarly the map $\tilde{H}_1 : (Y \sqcup (X \times I)) \times J \to M_f$ with parameter intervals $J = I$ given by $\tilde{H}_1(x;s;t) = \pi_\Phi(x;s+t-s\cdot t)$ and $\tilde{H}_1(y;t) = \pi_\Phi(y)$ is **continuous** and so is its **passing to the quotient** $H_1 = \tilde{H}_1 \circ r^{-1} : M_f \times J \to M_f$ since due to [4, p. 10.19] the **product** $r = \pi_\Phi \times \mathrm{id}_J : (Y \sqcup (X \times I)) \times J \to M_f \times J$ is a **quotient map** whence $M_f \times J \cong ((Y \sqcup (X \times I)) \times J)/R_r$ is a **quotient space**. Due to $H_1(\zeta;0) = \zeta$ and $H_1(\zeta;1) = A(\zeta)$ it is a **homotopy** between id_{M_f} and A collapsing $H_1\left[\tilde{X} \times [0;1[\right] = \tilde{X}$ onto $H_1\left[\tilde{X} \times \{1\}\right] = (\pi_\Phi \circ f)[X] \subset \tilde{Y}$. Because of $H_1(\zeta;t) = \zeta$ for $\zeta \in \tilde{Y}$ and $1 \leq t \leq 1$ it is **stationary** on $\tilde{Y}$ whence $\tilde{Y}$ is a **strong deformation retract** of M_f. Note that in this first step only the **continuity** of f is required.

For the **homotopy inverse** $g : Y \to X$ exists a homotopy $F : Y \times I \to Y$ such that $F(y;0) = (f \circ g)(y)$ and $F(y;1) = y$. By the same argument as above the composition $H_2 = \tilde{H}_2 \circ r^{-1} : M_f \times I \to M_f$ with $\tilde{H}_2(x;s;t) = (\pi_\Phi \circ F)(f(x);1-t)$ and $\tilde{H}_2(y;t) = (\pi_\Phi \circ F)(y;1-t)$ also is a **homotopy** deforming $A : M_f \to \tilde{Y}$ into $B = \tilde{B} \circ \pi_\Phi^{-1} : M_f \to \pi_\Phi[X \times \{0\}] = (\pi_\Phi \circ f)[X] \subset \tilde{Y}$ given by $\tilde{B}(x;s) = \pi_\Phi((g \circ f)(x);0)$ and $\tilde{B}(y) = \pi_\Phi(g(y);0)$ since $\tilde{H}_2(x;s;0) = \tilde{A}(x;s)$, $\tilde{H}_2(y;0) = \tilde{A}(y)$, $\tilde{H}_2(x;s;1) = \tilde{B}(x;s) = (\pi_\Phi \circ f \circ g \circ f)(x)$ and $\tilde{H}_2(y;1) = \tilde{B}(y) = (\pi_\Phi \circ f \circ g)(y)$. Similarly to A the continuous map $B : M_f \to \tilde{Y}$ shrinks Z_f to $\tilde{Y}$ with the difference that $(A \circ \pi_\Phi)(y) = \pi_\Phi(y)$ while $(B \circ \pi_\Phi)(y) = \pi_\Phi(g(y);0)$ for $y \in Y$. Correspondingly the **homotopy** H_2 at first moves $y \in Y$ to $g(y) \in X$ and $x \in X$ to $(g \circ f)(x) \in X$ and then gradually transports them downwards along $\pi_\Phi[X \times I]$ to their equivalence classes $\pi_\Phi((g \circ f)(x);0)$ resp. $(\pi_\Phi \circ f \circ g \circ f)(x)$ in $\tilde{Y}$. Hence $B \overset{H_2}{\simeq} A \overset{H_1}{\simeq} \mathrm{id}_{M_f}$ is a **deformation retraction** and $(\pi_\Phi \circ f)[X]$ is a **deformation retract** of M_f.

Finally the homotopy $G : X \times I \to X$ with $G(x;0) = (g \circ f)(x)$ and $G(x;1) = x$ yields a further homotopy $H_3 = \tilde{H}_3 \circ r^{-1} : M_f \times I \to M_f$ with $\tilde{H}_3(x;s;t) = \pi_\Phi(G(x;st);t)$ and $\tilde{H}_3(y;t) = \pi_\Phi(g(y);t)$ deforming $B : M_f \to \tilde{Y}$ into $C = \tilde{C} \circ \pi_\Phi^{-1} : M_f \to \pi_\Phi[X \times \{1\}] = \tilde{X}$ given by $\tilde{C}(x;s) = \pi_\Phi(G(x;s);1)$ and $\tilde{C}(y) = \pi_\Phi(g(y);1)$ since $\tilde{H}_3(x;s;0) = \tilde{B}(x;s)$, $\tilde{H}_3(y;0) = \tilde{B}(y)$, $\tilde{H}_3(x;s;1) = \tilde{C}(x;s) = \pi_\Phi(G(x;s);1)$ and $\tilde{H}_3(y;1) = \tilde{C}(y) = (\pi_\Phi \circ g)(y)$. Geometrically the relation $\tilde{C}(x;1) = \pi_\Phi(G(x;1);1) = \pi_\Phi(x;1)$ shows that C is **stationary** on $\tilde{X}$ and shrinks the rest of M_f onto $\tilde{X}$ along the homotopy G. Correspondingly the **homotopy** H_3 moves $y \in Y$ to $g(y) \in X$ and $x \in X$ to $(g \circ f)(x) \in X$ and then gradually transports them back upwards along $\pi_\Phi[X \times I]$ to their equivalence classes $\pi_\Phi((g \circ f)(x);1)$ resp. $\pi_\Phi(g(y);1)$ in $\tilde{X}$.

Thus in the preceding three steps we have shown that $\mathrm{id}_{M_f} \overset{H_1}{\simeq} A \overset{H_2}{\simeq} B \overset{H_3}{\simeq} C$ whence $\tilde{X}$ is a **deformation retract** of M_f.

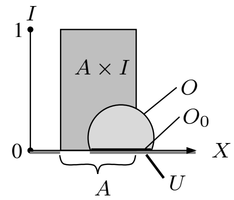
1.19 The homotopy extension property

A pair $(X;A)$ of a space X and its subspace $A \subset X$ has the **homotopy extension property** such that for every homotopy $H_A : A \times I \to Y$ and every given continuous extension $f : X \to Y$ with $f|_A = H_{A;0}$ exists a homotopy $H : X \times I \to Y$ with $f = H_0$ iff $X \times \{0\} \cup A \times I$ is a **retract** of $X \times I$.

Proof: For any set $B \subset X \times I$ we use the abbreviations $X_0 = X \times \{0\}$ and $B_0 = B \cap \{t = 0\} = \{(x;0) \in B\} \subset X_0$.

$\Rightarrow$: The homotopy $H : X \times I \to Y = X_0 \cup A \times I$ extended from $\mathrm{id} : X_0 \cup A \times I \to Y = X_0 \cup A \times I$ is a **retraction**.

$\Leftarrow$: In the case of a **closed** A the function $H_0 : X \times \{0\} \cup A \times I \to Y$ given by $H_0|_{A\times I} = H_A$ and $H_0|_{X_0}(x;0) = f(x)$ is **continuous** since the two closed sets $A \times I$ and X_0 cover their union and as mentioned in [4, p. 4.6] every topology is **coherent** with its **closed** sets. Therefore $H = H_0 \circ r$ with the given **retraction** $r : X \times I \to X_0 \cup A \times I$ provides the desired Homotopy.

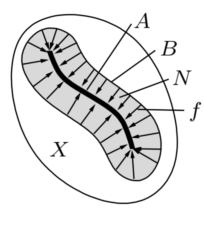


In the general case we start with the demonstration that $O \subset Y = X_0 \cup A \times I$ is open iff $O_0 \cap X_0$ is open in X_0 and $O \cap (A \times I)$ is open in $A \times I$: For $(x;0) \in O \cap X_0 \setminus \bar{A} \times \{0\}$ the hypothesis implies the existence of an neighborhood $V_X \times W$ open in $X \times I$ with $V_X \subset X \setminus \bar{A}$ and $(x;0) \in (V_X \times W) \cap O_0 \cap X_0 = (V_X \times W) \cap O \cap Y$ such that this intersection is also open in Y. Likewise for $t > 0$, i.e. $(x;t) \in O \cap (A \times I) \setminus X_0$ we have the neighborhood $V_A \times W$ open in $X \times I$ with $W \subset]0;1[$ and $(x;t) \in (V_A \times W) \cap O \cap (A \times I) = (V_A \times W) \cap Y$ whence this intersection is also open in Y. Finally for $(x;0) \in \mathring{A}_0$ the neighborhood $V_A \times I$ open in $X \times I$ with $V_A \subset \mathring{A}$ satisfies $(x;0) \in (V_A \times W) \cap O \cap (A \times I) = (V_A \times W) \cap O \cap Y$ and is open in Y.

Thus the only points requiring attention are $(x;0) \in O$ with $x \in \partial A$. For every $n \geq 1$ let U_n be the union of all open sets $U' \subset X$ with $(U' \cap A) \times \left[0; \frac{1}{n}\right[\subset O$ and $U = \bigcup_{n\geq 1} U_n$. For any point $(x;t)$ with $t > 0$ the continuity of the retraction $r = (r_X; r_I) : X \times I \to X_0 \cup A \times I$ and $r|_{A\times I} = \mathrm{id}$ imply $r(x;t) = (r_X(r;t);t)$ and consequently $r_X(x;t) \in A$. Assuming $r_X(x;t) \in U_n$ for some $n \geq 1$ the continuity of r_X implies $r_X[V \times U_\epsilon(t)] \in U_n$ for some neighborhood $x \in V \subset X$ and $\epsilon > 0$. In particular we have $r_X[(V \cap A) \times \{t\}] \in U_n$ and again because of $r|_{A\times I} = \mathrm{id}$ follows $V \cap A \subset U_n$. Since the definition of U_n only concerns parts of U' inside A this implies $V \subset A$ and in particular $x \in U_n$. If $x \notin U$ the preceding argument implies $r_X(x;t) \in A \setminus U$. Since $O \cap (A \times I)$ is open in $A \times I$ we have $O_0 \cap A \subset U$ whence follows $r_X(x;t) \in A \setminus O_0$. As this is true for every $t > 0$ and $r_X : X \times I \to X_0$ is continuous we can infer that $x = r_X(x;0) \in \overline{A \setminus O_0} \subset \bar{A} \setminus O_0$ in contradiction to the choice of $(x;0) \in O$. Hence the assumption $x \notin U$ is false and we have shown that $x \in U$ and in particular $x \in U_n$ for some $n \geq 1$. Then $V \times W = (U_n \cap A) \times \left[0; \frac{1}{n}\right[\subset O \cap Y$ is a neighborhood of x in O which is open in Y. Thus we have demonstrated that $O \subset Y = X_0 \cup A \times I$ is open iff $O_0 \cap X_0$ is open in X and $O \cap (A \times I)$ is open in $A \times I$. But this means that H_0 from above is also continuous in the general case which concludes the proof.

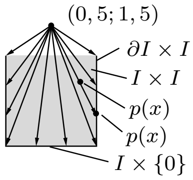
Examples: In the case of $X = [0;1]$ and $A =]0;1[$ the set $O = \{0;0\} \cup \{x < t\}$ is open in $A \times I$ and $O_0 = O \cap X_0 = [0;1[\times \{0\}$ is open in X_0 but due to the point $(0;0)$ the set O is not open in $X \times I$. Since continuous images of **compact** spaces are again compact there is no retraction $r : X \times I \to X_0 \cup A \times I$. For the same reason the homotopy extension fails in the case of the **closed** set $A = \left(\frac{1}{n}\right)_{n\geq 1} \cup \{0\}$.

1.20 Mapping cylinder neighborhoods



A subspace $A \subset X$ has the **homotopy extension property** if it has a **mapping cylinder neighborhood** N in X with a **closed** subset B such that $A \subset B \subset N \subset X$ and a **continuous** map $f : B \to A$ such that its **mapping cylinder** M_f is **homeomorphic** to N via $h : M_f \to N$ with $(h \circ \pi_\Phi)(a;1) = a$ for $a \in A$ and $(h \circ \pi_\Phi)(b;0) = b$ for $b \in B$.

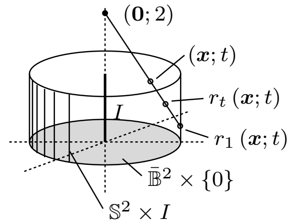
Proof: As before we abbreviate $A_1 = A \times \{1\}$ and $B_0 = B \times \{0\}$. By the simple **projection** p as shown in the drawing $I \times I$ retracts onto $(I \times \{0\}) \cup (\partial I \times I)$, hence by $\mathrm{id}_B \times p$ the product $B \times I \times I$ retracts onto $(B \times I \times \{0\}) \cup (B \times \partial I \times I)$. which provides a retraction $r = (\pi_\Phi \times \mathrm{id}_I) \circ (\mathrm{id}_B \times p) \circ \left(\pi_\Phi^{-1} \times \mathrm{id}_I\right) : M_f \times I \to (M_f \times \{0\}) \cup ((B_0 \cup A_1) \times I)$ given by $r(\pi_\Phi(b; s); t) = (\pi_\Phi(b; p_1(s; t)); p_2(s; t))$ resp. $r(\pi_\Phi(a; 1); t) = (\pi_\Phi(a; p_1(1; t)); p_2(1; t))$ for $(b; s; t) \in B \times I \times I$ and $a \in A$. According to the preceding theorem 1.19 the subspace $B_0 \cup A_1 \subset M_f$ has the **homotopy extension property** which extends to the homeomorphic pair $B \cup A \subset N$. So for any continuous $f : X \to Y$ and any homotopy $H : A \times I \to Y$ coinciding on A with $f|_A = H_0$ in a first step we can apply the homotopy extension property to $f|_N$ and the homotopy $H : (A \cup B) \times I \to Y$ with constant value $H(b; t) = f(b)$ for every $b \in B$ to obtain $H : N \times I \to Y$ with $f|_N = H_0$ which in a second step can be simply extended to $H : X \times I \to Y$ with constant value $H(x; t) = f(x)$ for every $x \in (X \setminus N) \cup B$.



1.21 Homotopy extension on CW complexes

Every **subcomplex** A of a CW complex X has the **homotopy extension property**.

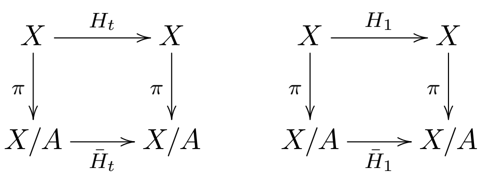
Proof: We use the same notation as before, whence in this proof $X_0 = X \times \{0\}$ is not to be confused with the 0-skeleton. We start with a **projection** $r : \bar{\mathbb{B}}^n \times I \to \bar{\mathbb{B}}^n_0 \cup \left(\mathbb{S}^{n-1} \times I\right)$ given by $r(\boldsymbol{x}; s) = (\boldsymbol{0}; 2) + \tau((\boldsymbol{x}; s) - (\boldsymbol{0}; 2))$ for $\tau = \min\left\{\frac{1}{\|\boldsymbol{x}\|}; \frac{2}{2-t}\right\}$ which due to the **linear homotopy** $r_s = s \cdot r + (1 - s) \cdot \mathrm{id}$ is a **deformation retraction**. Then for every n-cell $e \subset X_n \setminus (X_{n-1} \cup A_n)$ the **characteristic maps** $\varphi_e : \bar{a}_e \to \bar{e} \subset X_n$ with $\varphi_e[\partial \bar{a}_e] \subset X_{n-1}$ and the homeomorphisms $\psi_e : \bar{\mathbb{B}}^n \to \bar{a}_e$ combine with $r_n(\boldsymbol{x}; t) = \begin{cases} \boldsymbol{x} & \text{for } 0 \le t \le \frac{1}{2^{n+1}} \\ r\left(2^{n+1} \cdot t - 1\right) & \text{for } \frac{1}{2^{n+1}} \le t \le \frac{1}{2^n} \\ r(\boldsymbol{x}; 1) & \text{for } \frac{1}{2^n} \le t \le 1 \end{cases}$ to form a deformation retraction $r_e = ((\varphi_e \circ \psi_e) \times \mathrm{id}_I) \circ r_n \circ \left(\left(\psi_e^{-1} \circ \varphi_e^{-1}\right) \times \mathrm{id}_I\right) : \bar{e} \times I \to \bar{e}_0 \cup (X_{n-1} \times I)$ compressing the complete retraction into the interval $I_n = \left[\frac{1}{2^{n+1}}; \frac{1}{2^n}\right]$. Finally by concatenation of these retractions via $r : X \times I \to X_0 \cup (A \times I)$ with $r|_{I_n} = \sum_{e \subset X_n \setminus (X_{n-1} \cup A_n)} r_e + \sum_{e \subset A_n \setminus X_{n-1}} \mathrm{id}$ we obtain a map which is continuous on every $X_n \times I$ and hence continuous on X. The assertion then follows from 1.19.



1.22 Contractible subcomplexes

For every **contractible** subset $A \subset X$ with the **homotopy extension property** the canonical projection $\pi : X \to X/A$ is a **homotopy equivalence**.

Proof: The hypothesis implies the extension of the homotopy $H_A : A \times I \to A$ between the **identity** $H_{A;0} = \mathrm{id}_A$ and the **contraction** $H_{A;1} \equiv p \in A$ to a homotopy $H : X \times I \to X$ with $H_0 = \mathrm{id}_X$ and $H|_A = H_A$. Due to $H_t[A] \subset A$ the continuous map $\bar{H}_t = \pi \circ H_t \circ \pi^{-1} : X/A \to X/A$ with $\bar{H}_t(\bar{p}) = \bar{p}$ yields a **homotopy inverse** $g = H_1 \circ \pi^{-1} = \pi^{-1} \circ \bar{H}_1 : X/A \to X$ with $g(\bar{p}) = p$ for π since $\pi \circ g = \bar{H}_1 = \mathrm{id}_{X/A}$ and $g \circ \pi = H_1 = \mathrm{id}_X$.



1.23 Homotopic attaching maps

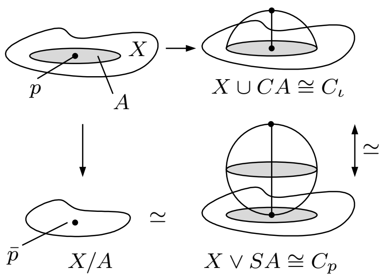
For a subset A of topological space X and a **deformation retraction** $r_0 : X \times I \to X_0 \cup (A \times I)$ with **homotopic attaching maps** $f \simeq g : A \to Y$ into a further space Y the **adjunction spaces** $Y \cup_f X \simeq Y \cup_g X$ rel Y are **homotopy equivalent relative to** Y, i.e. there are continuous maps $\varphi : Y \cup_f X \to Y \cup_g X$ and $\psi : Y \cup_g X \to Y \cup_f X$ with $\varphi|_Y = \psi|_Y = \mathrm{id}_Y$ and $\varphi \circ \psi \simeq \mathrm{id}$ resp. $\psi \circ \varphi \simeq \mathrm{id}$ via homotopies also restricting to the identity on Y.

Corollary: Due to 1.21 every **subcomplex** A of a **CW complex** X satisfies the hypothesis of this lemma.

Proof: The space $Y \sqcup_H (X \times I)$ with regard to the homotopy $H : A \times I \to Y$ with $H_0 = f$ and $H_1 = g$ contains the subspaces $Y \cup_f X \cong Y \cup_f X_0$ and $Y \cup_g X \cong Y \cup_g X_0$. The deformation retraction $r_0 : X \times I \to X_0 \cup (A \times I)$ extends to a deformation retraction $\bar{r}_0 : Y \cup_H (X \times I) \to Y \cup_{H_0} X = Y \cup_f X$ given by $(\bar{r}_0 \circ \pi_H)(a;t) = (\pi_{H_0} \circ H \circ r_0)(a;t)$, $(\bar{r}_0 \circ \pi_H)(x;t) = (\pi_{H_0} \circ \pi_0 \circ r_0)(x;t)$ and $(\bar{r}_0 \circ \pi_H)(y) = \pi_{H_0}(y)$ for $a \in A$, $x \in X \setminus A$ and $\pi_0(x;0) = x$ while the analogous deformation retraction $r_1 : X \times I \to X_1 \cup (A \times I)$ obtained from r_0 by exchanging $0 \leftrightarrow 1$ likewise induces a deformation retraction onto $Y \cup_{H_1} X = Y \cup_g X$. Both maps restrict to the identity on Y whence their composition yields the desired homotopy equivalence $Y \cup_f X \simeq Y \cup_g X$ rel Y.

1.24 Quotient spaces and wedge sums

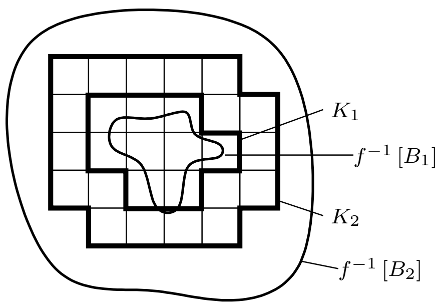
For every **contractible subcomplex** A of a **CW complex** X holds $X/A \simeq X \vee SA$.



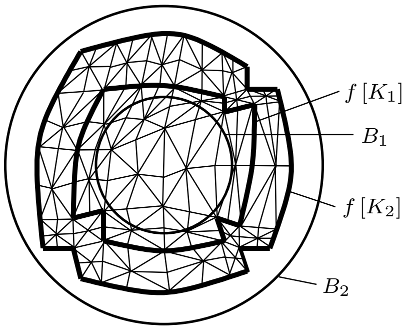
Proof: The **cone** $CA = (A \times I)/A_1$ is a **contractible** subcomplex of $X \cup CA = (X_0 \cup (A \times I))/A_1$ since the **projection** $\pi \circ H : CA \times I \to CA$ of the given contraction $H : A \times I \to A$ with $H_0 = \mathrm{id}_A$ and $H_1 \equiv p \in A$ is again a contraction. By 1.22 then follows, that $X/A \cong (X_0 \cup A_o)/A_0 = (X_0 \cup (A \times I))/(A \times I) = (X \cup CA)/CA \simeq X \cup CA$. According to the hypothesis the **constant function** $p : A \to \{p\}$ is homotopic to the **injection** $\iota : A \to X$ whence the corresponding **mapping cones** $X \cup CA \cong C_\iota \simeq C_p \cong X \vee SA$ are **homotopy equivalent** whence follows the assertion.

1.25 Piecewise linear maps

A map $f : K \to \mathbb{R}^n$ on the union K of finitely many **convex polyhedra** is **piecewise linear** iff its restriction to each of the convex polyhedra is linear. Obviously every polyhedron of a **simplicial complex** as defined in [4, p. 22.1] is such a union and conversely every convex polyhedron can be decomposed into the polyhedron of a simplicial complex formed by the simplices obtained from the **convex hulls** of the centre with n vertices. Note that piecewise linearity depends on the decomposition^of the polyhedron. The following lemma is needed for the cellular approximation theorem:



For every continuous map $f : I^n \to W \cup_\varphi \bar{b}$ from the closed n-cube I^n to a k-cell d attached by φ to a space W exists a homotopy $f_t : I^n \to W \cup_\varphi \bar{b}$ rel $f^{-1}[W]$ from $f = f_0$ to a map f_1 with $f_t|_{f^{-1}[W]} = f|_{f^{-1}[W]}$ which is **piecewise linear** on a **compact** set $K \subset I^n$ whose image includes a nonempty **open** set U such that $U \subset f_1[K] \subset \bar{b}$.



Proof: Since f is **uniformly continuous** on the **compact** sets $(f \circ \psi)^{-1}\left[\bar{\mathbb{B}}_1\right]$, $(f \circ \psi)^{-1}\left[\bar{\mathbb{B}}_2\right] \in I^n$ with regard to the homeomorphism $\psi : d \to \mathbb{R}^k$ there is an $\epsilon > 0$ such that $|\boldsymbol{x} - \boldsymbol{y}| < \epsilon$ implies

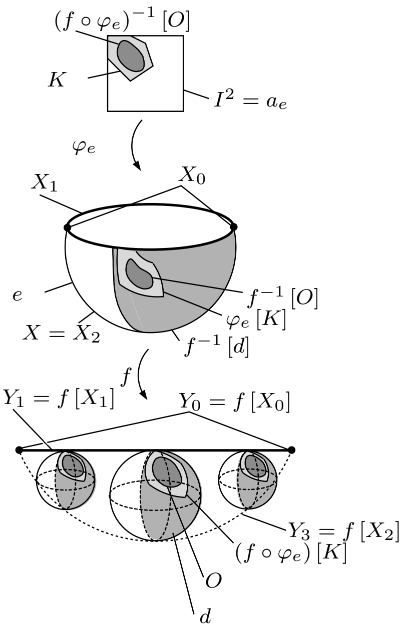
$|f(\boldsymbol{x}) - f(\boldsymbol{y})| < \frac{1}{2}$ for all $\boldsymbol{x}; \boldsymbol{y} \in (f \circ \psi)^{-1}\left[\bar{\mathbb{B}}_2\right]$. We subdivide the interval I so that the induced subdivision of I^n into cubes has each cube lying in a ball of diameter less than ϵ. Let K_1 be the union of all the cubes meeting $(f \circ \psi)^{-1}\left[\bar{\mathbb{B}}_1\right]$ and K_2 be the union of all cubes meeting K_1. We may assume ϵ is chosen smaller than half the distance between the compact sets $(f \circ \psi)^{-1}\left[\bar{\mathbb{B}}_1\right]$ and $I^n \setminus (f \circ \psi)^{-1}[\mathbb{B}_2]$ such that $(f \circ \psi)^{-1}\left[\bar{\mathbb{B}}_1\right] \subset K_2 \subset (f \circ \psi)^{-1}\left[\bar{\mathbb{B}}_2\right]$. Now we subdivide all the cubes of K_2 into simplices by joining the center point of each cube of dimension n to the vertices of the cubes of dimension $n-1$ which inductively may be taken to form its boundary. Then map $g : K_2 \to d$ by $g\left(\sum_{i=0}^n \alpha_i \boldsymbol{a}_i\right) = \sum_{i=0}^n \alpha_i f(\boldsymbol{a}_i)$ for every $\boldsymbol{x} \in \sigma \subset K_2$ in a simplex $\sigma = [\boldsymbol{a}_0; ...; \boldsymbol{a}_n]$ is linear on each simplex and coincides with f on its vertices. Likewise we define the map $\varphi : K_2 \to I$ which is linear on all simplices and defined by its values $\varphi(\boldsymbol{p}) = 1$ on vertices $\boldsymbol{p} \in K_1$ and $\varphi(\boldsymbol{q}) = 0$ on vertices $\boldsymbol{q} \in K_2 \setminus K_1$. The homotopy $f_t = (1 - t \cdot \varphi) \cdot f + t \cdot \varphi \cdot g : K_2 \to d$ then satisfies $f_0 = f$ and $f_1|_{K_1} = f|_{K_1}$. Since f_t is constant and equal to f on simplices in K_2 disjoint from K_1 and in particular on simplices in the closure of $I^n \setminus K_2$ we may extend it to the constant homotopy of f on $I^n \setminus K_2$. Then the extended map $\psi \circ f_1 : I^n \to \mathbb{R}^k$ takes the closure of $I^n \setminus K_1$ to a set C which does not contain the origin $\mathbf{0}$ since

1. on $I^n \setminus K_2$ we have $f_1 = f$ and $\psi \circ f$ takes $I^n \setminus K_2$ outside $\bar{\mathbb{B}}_1$ by construction and
2. for any simplex $\sigma \subset K_2 \setminus K_1$ there is an $\boldsymbol{x} \in (\psi \circ f)[\sigma] \subset \bar{B}_{1/2}(\boldsymbol{x})$ and since $\bar{B}_{1/2}(\boldsymbol{x})$ is convex we conclude $(\psi \circ g)[\sigma] \subset \bar{B}_{1/2}(\boldsymbol{x})$, hence $(\psi \circ f_t)[\sigma] \subset \bar{B}_{1/2}(\boldsymbol{x})$ for $t \in I$ and in particular $(\psi \circ f_1)[\sigma] \subset \bar{B}_{1/2}(\boldsymbol{x})$. Since σ contains points outside K_1, hence outside $(f \circ \psi)^{-1}\left[\bar{\mathbb{B}}_1\right]$ this inclusion implies that $\bar{B}_{1/2}(\boldsymbol{x})$ is not contained in $\bar{\mathbb{B}}_1$, hence does not contain $\mathbf{0}$ and neither does $(\psi \circ f_1)[\sigma]$.

Since C is **compact** there is even a disjoint neighborhood U of the origin and this proves the lemma with $K = K_1$ because $(\psi \circ f_1)^{-1}[U] \subset K_1$ and f_1 is **piecewise linear** on K_1 where it coincides with g.

1.26 The cellular approximation theorem

Every continuous map $f : X \to Y$ between CW complexes is homotopic to a **cellular map** $g : X \to Y$ with $g[X_n] \subset Y_n$ for all $n \in \mathbb{N}$. If f is already cellular on a subcomplex $A \subset X$, the homotopy can be taken to be **stationary** on A.



Proof: By [4, p. 21.8] the image $f[e]$ of an n-cell $e \subset X_n$ meets only finitely many cells of Y. Of these we choose a k-cell $d \subset Y_k$ of maximal dimension k and if f is **not cellular** it follows that $k > n$. Inductively assuming that f is already cellular on X_{n-1} with $f[\partial \bar{e}] \subset f[X_{n-1}] \subset Y_{n-1} \subset Y_k \setminus d$ by the preceding lemma 1.25 exists a homotopy $H_t : \bar{a}_e \to Y_k$ of the composition $H_0 = f \circ \varphi_e : \bar{a}_e \to (Y_k \setminus d) \cup_{\varphi_d} \bar{a}_d$ with the **characteristic functions** $\varphi_e : \bar{a}_e \to X_{n-1} \cup e$ and $\varphi_d : \bar{a}_d \to Y_{k-1} \cup d \subset (Y_k \setminus d) \cup d$ which is stationary on $\partial \bar{a}_e \subset \varphi_e^{-1}[\partial \bar{e}] \subset (f \circ \varphi_e)^{-1}[Y_k \setminus d]$, i.e. coinciding with $f \circ \varphi_e$ on this set, and leads to a continuous map H_1 for which there is a **polyhedron** $K \subset \bar{a}_e$ and a nonempty **open** set $U \subset H_1[K] \subset d$ such that $H_1|_K$ is **piecewise linear** with respect to some decomposition of K by convex polyhedra of dimension at most n. This means that the (nonstationary) action of the homotopy is restricted to the set $(f \circ \varphi_e)^{-1}[d] \subset a_e$ while the existence of U with $H_1^{-1}[U] \cap a_e \neq \emptyset$ guarantees that $H_1[K] \cap d \neq \emptyset$ resp. $K \cap a_e \neq \emptyset$. The induced homotopy $f_t : X_{n-1} \cup e \to (Y_k \setminus d) \cup_{\varphi_d} \bar{a}_d$ given by $f_t(x) = \begin{cases} \left(H_t \circ \varphi_e^{-1}\right)(x) & \text{for } x \in \bar{e} \\ f(x) \subset Y_k \setminus d & \text{for } x \in X_{n-1} \end{cases}$ then leads to a map f_1 with $f_1^{-1}[U] \cap e \neq \emptyset$ whence its image $f_1[e] = H_1[a_e]$ intersects U in a set contained in the union of finitely many polyhedra of dimension at most

$n < k$ which cannot cover any k-dimensional ball contained in U. Hence we have at least a point $p \in U \subset d$ which is not in $f_1[e]$ and since the k-cell d is **contractible** a further composition of f with a **deformation retraction** of $Y_k \setminus \{p\}$ to $Y_k \setminus d$ yields a homotopic $f_2 : X_{n-1} \cup e \to Y_k \setminus d$. Since open cells in a CW complex are disjoint this can be done for **all** cells d in Y meeting $f[e]$ with dimension greater than n and for all n-cells e in X_n staying fixed on all n-cells in A_n where f is already cellular we obtain a homotopy of $f|_{X_n}$ rel $X_{n-1} \cup A_n$ to a cellular map. By the **homotopy extension property** 1.21 this homotopy together with the constant homotopy on A extends to a homotopy on X. Letting n go to ∞ the resulting infinite string of homotopies can be realized as a single homotopy by performing the n-th homotopy during the t-interval $\left[1 - \frac{1}{2^n}; 1 - \frac{1}{2^{n+1}}\right]$ as shown in 1.21. This is possible since each point of X lies in some X_n which is eventually stationary in the infinite chain of homotopies thus defined.

1.27 Cellular approximation on subcomplexes

Every continuous map $f : (X; A) \to (Y; B)$ between **CW pairs** each of a CW complex and a subcomplex with $f[A] = B$ can be deformed through homotopies $f_t : (X; A) \to (Y; B)$ with $f = f_0$ to a **cellular map** f_1.

Proof: By the preceding **cellular approximation theorem** 1.26 we start with deforming the restriction $f|_A : A \to B$ to be cellular, then apply the **homotopy extension property** 1.21 to extend the corresponding homotopy to all of X and finally use 1.26 again to deform the resulting map to be cellular staying fixed on A. As we have seen in the proof of 1.26 the homotopies can be taken to be stationary on any subcomplex where f is already cellular.

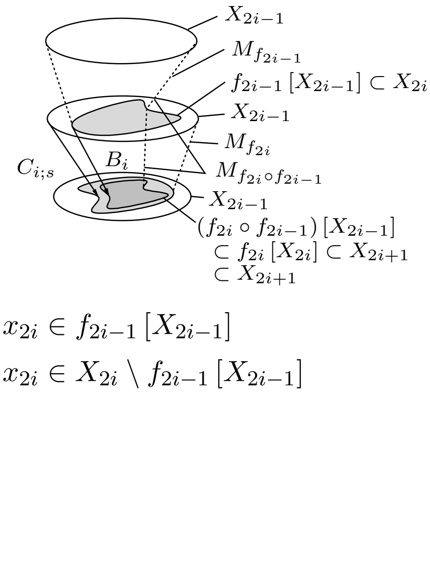
1.28 Spaces dominated by CW complexes

A topological space X **dominated by a CW complex** Y with maps $X \xrightarrow{i} Y \xrightarrow{r} X$ satisfying $r \circ i \simeq \mathrm{id}_X$ is **homotopy equivalent** to some CW complex Z.

Corollary: Every **retract of a CW complex** and due to 1.13 every **compact manifold** is homotopy equivalent to a CW complex.

Proof: The **mapping telescope** $T(f_1; f_2; ...) = \bigsqcup_{i \geq 1} (X_i \times [i; i+1]) / R_\varphi$ of a sequence of **continuous** maps $f_i : X_i \to X_{i+1}$ with regard to $\varphi : \bigsqcup_{i \geq 1} (X_i \times \{i+1\}) \to \bigsqcup_{i \geq 1} (X_{i+1} \times [i+1; i+2])$ defined by $\varphi(x_i; i+1) = (f_i(x_i); i+1)$ for $x_i \in X_i$ and $i \in \mathbb{N}_*$ has the following properties:

1. $T(f_1; f_2; ...) = T(g_1; g_2; ...)$ if $f_i \simeq g_i$ for $i \geq 1$ according to 1.23.
2. $T(f_1; f_2; ...) \simeq T(f_2; f_3; ...)$ since due to the map $A : T(f_1; f_2; ...) \to T(f_2; f_3; ...)$ given by $A(\pi_\Phi(x_1; s)) = \pi_\Phi(f_1(x_1); 2)$ for $(x_1; s) \in X_1 \times [1; 2]$ and $A = \mathrm{id}$ else is a **strong deformation retraction** and in particular a **homotopy equivalence**.
3. $T(f_1; f_2; ...) \simeq T(f_2 \circ f_1; f_4 \circ f_3; ...)$ is a consequence of 1.23: in a first step the **homeomorphism** $B_i : M_{f_{2i}} \cup_{f_{2i-1}} M_{f_{2i-1}} \to M_{f_{2i}} \cup_{f_{2i} \circ f_{2i-1}} M_{f_{2i} \circ f_{2i-1}}$ given by $(B_i \circ \pi_{f_{2i-1}})(f_{2i-1}(x_{2i-1}); 1) = \pi_{f_{2i}}(f_{2i-1}(x_{2i-1}); 1)$ $(B_i \circ \pi_{f_{2i-1}})(f_{2i-1}(x_{2i-1}); 1) = \pi_{f_{2i} \circ f_{2i-1}}((f_{2i} \circ f_{2i-1})(x_{2i-1}); 1)$ and $B_i = \mathrm{id}$ else slides the attaching area $f_{2i-1}[X_{2i-1}]$ from $X_{2i} \subset M_{f_{2i}}$ down to $(f_{2i} \circ f_{2i-1})[X_{2i-1}] \subset f_{2i}[X_{2i-1}] \subset M_{f_{2i} \circ f_{2i-1}}$ with its inverse defined by



$$\left(B_{2i}^{-1} \circ \pi_{f_{2i} \circ f_{2i-1}}\right)(f_{2i}(x_{2i}); 1) = \begin{cases} \pi_{f_{2i-1}}\left(f_{2i-1}^{-1}(x_{2i}); 1\right) & \text{if } x_{2i} \in f_{2i-1}[X_{2i-1}] \\ \pi_{f_{2i}}(f_{2i}(x_{2i}); 1) & \text{if } x_{2i} \in X_{2i} \setminus f_{2i-1}[X_{2i-1}] \end{cases}$$

and $B_i^{-1} = \mathrm{id}$ else.

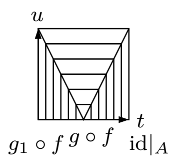
In a second step the **homotopy** $C_{i,s}: M_{f_{2i}} \cup_{f_{2i} \circ f_{2i-1}} M_{f_{2i} \circ f_{2i-1}} \to M_{f_{2i}} \cup_{f_{2i} \circ f_{2i-1}} M_{f_{2i} \circ f_{2i-1}}$ given by $(C_{i,s} \circ \pi_{f_{2i}})(x_{2i}; t) = \pi_{f_{2i} \circ f_{2i-1}}(f_{2i}(x_{2i}); 1)$ for $s < t < 1$ and $(C_{i,s} \circ \pi_{f_{2i}})(x_{2i+1}; 1) = \pi_{f_{2i} \circ f_{2i-1}}(x_{2i+1}; 1)$ for $s < 1$ and $C_{i,s} = \mathrm{id}$ else such that $C_{i,1} = \mathrm{id}$ whence $C_{i,0}: M_{f_{2i}} \cup_{f_{2i} \circ f_{2i-1}} M_{f_{2i} \circ f_{2i-1}} \to M_{f_{2i} \circ f_{2i-1}}$ is a **strong deformation retraction**. Hence we obtain $M_{f_{2i}} \cup_{f_{2i-1}} M_{f_{2i-1}} \simeq M_{f_{2i}} \cup_{f_{2i} \circ f_{2i-1}} M_{f_{2i} \circ f_{2i-1}} \simeq M_{f_{2i} \circ f_{2i-1}}$ for every $i \geq 1$ and hence the assertion.

By 2. and 3. we have $T(i \circ r; i \circ r; ...) \simeq T(r; i; r; i; ...) \simeq T(i; r; i; r; ...) \simeq T(r \circ i; r \circ i; ...) \simeq T(\mathrm{id}_X; \mathrm{id}_X; ...) \simeq X \times [0; \infty[\cong X$. According to the **cellular approximation theorem** 1.26 the composition $i \circ r$ is homotopic to a **cellular map** $f: Y \to Y$ such that $T(i \circ r; i \circ r; ...) \simeq T(f; f; ...)$ which is a CW complex.

1.29 Homotopy extension and homotopy equivalence

A **homotopy equivalence** $f: X \to Y$ with $f|_A = \mathrm{id}$ is also a **homotopy equivalence rel** A if $A \subset X \cap Y$ satisfies the **homotopy extension property** both in X and in Y.

Proof: Due to $f|_A = \mathrm{id}$ the given homotopy $h_t: X \to Y$ from $h_0 = g \circ f$ to $h_1 = \mathrm{id}_X$ is also a homotopy from $g|_A$ to id_A which by the hypothesis can be extended to a homotopy $g_t: Y \to X$ from $g_0 = g$ to a map g_1 with $g_1|_A = \mathrm{id}_A$. A homotopy $k_t: X \to X$ from $k_0 = g_1 \circ f$ to $k_1 = \mathrm{id}_X$ is then given by $k_t = \begin{cases} g_{1-2t} \circ f & \text{for } 0 \leq t \leq \frac{1}{2} \\ h_{2t-1} & \text{for } \frac{1}{2} \leq t \leq 1 \end{cases}$. Since $f|_A = \mathrm{id}_A$ and $g_t|_A = h_t|_A$ we have $k_0|_A = k_1|_A = \mathrm{id}_A$ and $k_t|_A = k_{1-t}|_A$.
Now we define a homotopy $k_{t;u}: A \to X$ from $k_{t;0} = k_t|_A$ to $k_{t;1} = k_0 = k_1 = \mathrm{id}_A$ by $k_{t;u} = \begin{cases} k_t & \text{for } u \leq \max\{1-2t; 2t-1\} \\ k_u & \text{else} \end{cases}$ such that on the left, top and right edges of the parameter square $I \times I$ drawn on the right hand side holds $k_{0;u} = k_{t;1} = k_{1;u} = \mathrm{id}_A$. According to 1.19 the homotopy extension property extends to $(X \times I; A \times I)$ such that we can extend the homotopy $k_{t;u}$ of $k_t|_A$ to all of X starting with $k_{t;0} = k_t: X \to X$ and by restricting this extension to the detour along the three edges where it coincides with the identity on A via $k_s = \begin{cases} k_{0;3s} & \text{for } 0 \leq s \leq \frac{1}{3} \\ k_{3s;1} & \text{for } \frac{1}{3} \leq \mathrm{s} \leq \frac{2}{3} \\ k_{1;3s} & \text{for } \frac{2}{3} \leq s \leq 1 \end{cases}$ we obtain a homotopy $g_1 \circ f \simeq \mathrm{id}_X$ rel A. Since $g_1 \simeq g$, we have $f \circ g_1 \simeq f \circ g \simeq \mathrm{id}$,so that the preceding construction for $g \circ f$ can be applied to the pair $f \circ g_1$. The result is a map $f_1: X \to Y$ with $f_1|_A = \mathrm{id}_A$ and $f_1 \circ g_1 \simeq \mathrm{id}$ rel A. Hence $f_1 \simeq f_1 \circ (g_1 \circ f) = (f_1 \circ g_1) \circ f \simeq f$ rel A. From this we deduce that $f \circ g_1 \simeq f_1 \circ g_1 \simeq \mathrm{id}$rel A.



1.30 Higher homotopy groups

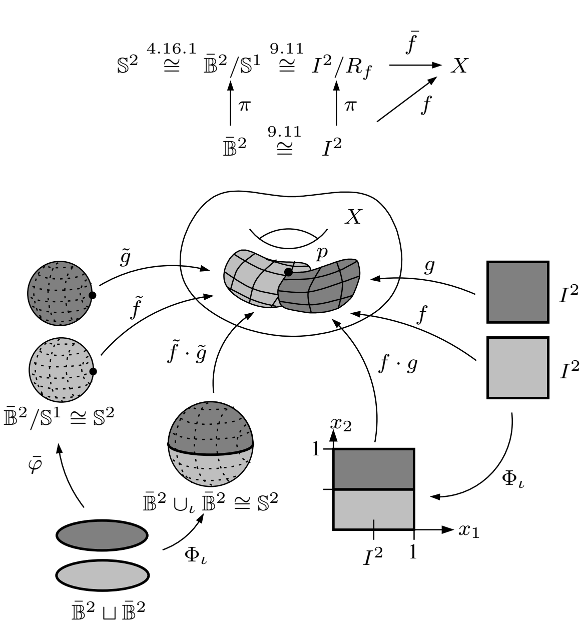
For every $n \geq 1$ an n-dimensional loop based at p in a topological space X is a **continuous** $f: I^n \to X$ with $f[\partial I^n] = \{p\}$ and the nth homotopy group $\pi_n(X; p)$ of X based at p is the set of equivalence classes of loops based at p modulo homotopy. Under the **multiplication** resulting in the **product** $f \cdot g: I^n \to X$ defined by

$$(f \cdot g)(t_1; t_2; ...; t_n) = \begin{cases} f(2t_1; t_2; ...; t_n) & \text{for } 0 \leq t_1 \leq \frac{1}{2} \\ g(2t_1 - 1; t_2; ...; t_n) & \text{for } \frac{1}{2} \leq t_1 \leq 1 \end{cases}$$

it is an **abelian group** and it is also a **topological invariant**, i.e. homeomorphic spaces have isomorphic nth homotopy groups. The converse is not true, i.e. spaces with isomorphic nth homotopy groups need not be homeomorphic.

Due to $\mathbb{S}^{n-1} \overset{9.11}{\cong} \partial I^n$ and $\mathbb{S}^n \overset{4.16.1}{\cong} \bar{\mathbb{B}}^n / \mathbb{S}^{n-1} \overset{9.11}{\cong} I^n / \partial I^n = I^n / R_f$ resp. according to the **universal property** [4, p. 4.7] of the quotient space for every **continuous** $f : I^n \to X$ with $f[\partial I^n] = \{p\}$ exists a **continuous** $\tilde{f} : \mathbb{S}^n \to X$ with $\tilde{f}(\boldsymbol{e}_{n+1}) = p$ and vice versa which allows us to extend the definition to $n \in \mathbb{N}$. Due to [4, p. 20.16.3] the injection $\iota : \mathbb{S}^{n-1} \to \bar{\mathbb{B}}^n$ yields the homeomorphism $\pi_{\mathbb{S}} : \mathbb{S}^n \cong \bar{\mathbb{B}}^n \cup_\iota \bar{\mathbb{B}}^n$ while [4, p. 20.16.1] provides the homeomorphism $\bar{\varphi} : \bar{\mathbb{B}}^n / \mathbb{S}^{n-1} \cong \mathbb{S}^n$. Hence the projection $\pi : \bar{\mathbb{B}}^n \to \bar{\mathbb{B}}^n / \mathbb{S}^{n-1}$ with $\pi[\mathbb{S}^{n-1}] = \{\boldsymbol{e}_{n+1}\}$ together with the **attaching lemma** [4, p. 4.11] for any two $\tilde{f}; \tilde{g} : \mathbb{S}^n \overset{\bar{\varphi}^{-1}}{\cong} \bar{\mathbb{B}}^n / \mathbb{S}^{n-1} \to X$ with $\tilde{f}(\boldsymbol{e}_{n+1}) = \tilde{g}(\boldsymbol{e}_{n+1}) = p$ provides a **product** $\tilde{f} \cdot \tilde{g} : \mathbb{S}^n \overset{\pi_{\mathbb{S}}}{\cong} \bar{\mathbb{B}}^n_{\tilde{f}} \cup_\iota \bar{\mathbb{B}}^n_{\tilde{g}} \to X$ defined by $\left(\tilde{f} \cdot \tilde{g}\right)(x) =$

$$\begin{cases} \left(\tilde{f} \circ \pi_{\mathbb{S}}^{-1} \circ \Phi_\iota \circ \pi^{-1} \circ \bar{\varphi}^{-1}\right)(x) & \text{for } x \in \bar{\varphi}\left[\bar{\mathbb{B}}^n_{\tilde{f}} / \mathbb{S}^{n-1}\right] \\ \left(\tilde{g} \circ \pi_{\mathbb{S}}^{-1} \circ \Phi_\iota \circ \pi^{-1} \circ \bar{\varphi}^{-1}\right)(x) & \text{for } x \in \bar{\varphi}\left[\bar{\mathbb{B}}^n_{\tilde{g}} / \mathbb{S}^{n-1}\right] \end{cases}.$$



The corresponding multiplication is obviously abelian such that we have obtained an **equivalent definition of the *n*th homotopy group**.

In particular $\pi_0(X; p)$ is defined as the set of equivalence classes of continuous maps $\varphi : \mathbb{S}^0 = \{\pm 1\} \to X$ with $\varphi(1) = p$ modulo homotopy. According to [4, p. 5.8] the single element of $\pi_0(X; p)$ coincides with the **path component** $P(p)$ of p in X.

Due to the **cellular approximation theorem** 1.26 we have $\pi_n\left(\mathbb{S}^k\right) = \{\boldsymbol{e}_{k+1}\}$ for $n < k$ since the cellular representations $\varphi : \mathbb{S}^n \to \mathbb{S}^k$ with regard to the **regular CW-decomposition** of $\mathbb{S}^k$ resp. $\mathbb{S}^n$ from [4, p. 21.6] are homotopic to the **inclusion** $\iota : \mathbb{S}^n \to \mathbb{S}^k$ and the complement $\mathbb{S}^k \setminus \{\boldsymbol{e}_{k+1}\} \cong \{\boldsymbol{e}_{k+1}\} \cup_\pi \mathbb{B}^k \setminus \{\boldsymbol{e}_{k+1}\} \cong \mathbb{B}^k$ is **contractible** whence due to 1.1 the continuous map $\iota : \mathbb{S}^n \to \mathbb{S}^k \subset \mathbb{S}^k \setminus \{\boldsymbol{e}_{k+1}\}$ is **nullhomotopic**.

2 Categories

2.1 Objects and morphisms

A **category** $C = \{\mathrm{Ob}(C); \mathrm{Hom}(C)\}$ is the ordered pair of any **class** $\mathrm{Ob}(C)$ of **objects** and a class $\mathrm{Hom}(C)$ of ordered triples $(f; X_f; Y_f)$ where each **morphism** f has an assigned **source** $X_f \in \mathrm{Ob}(C)$ and a **target** $Y_f \in \mathrm{Ob}(C)$. The set $\mathrm{Hom}_C(X; Y) \subset \mathrm{Hom}(C)$ denotes all morphisms with source X and target Y. For each triple $(X; Y; Z) \in \mathrm{Ob}(C)$ there is an **associative composition** $\circ : \mathrm{Hom}_C(X; Y) \times \mathrm{Hom}_C(Y; Z) \to \mathrm{Hom}_C(X; Z)$ with the usual inverted order of notation $(f; g) \mapsto f \circ g$ of compatible morphisms with $(h \circ g) \circ f = h \circ (g \circ f)$ for any triple of compatible morphisms. Also for every pair of objects $X; Y \in \mathrm{Ob}(C)$ we claim **identity morphisms** $\mathrm{id}_X \in \mathrm{Hom}(X; X)$ and $\mathrm{id}_Y \in \mathrm{Hom}(Y; Y)$ with $f \circ \mathrm{id}_X = \mathrm{id}_Y \circ f = f$ for every $f \in \mathrm{Hom}_C(X; Y)$. Hence for every $X \in \mathrm{Ob}(C)$ the ordered pair $(\mathrm{Hom}_C(X; X); \circ)$ is a **monoid**, i.e. a **semigroup** with an **identity element**. Finally $f \in \mathrm{Hom}_C(X; Y)$ is an **isomorphism** iff it has an **inverse** $f^{-1} \in \mathrm{Hom}_C(Y; X)$ with $f \circ f^{-1} = \mathrm{id}_Y$ and $f^{-1} \circ f = \mathrm{id}_X$ whence the set of isomorphisms on any object $X \in \mathrm{Ob}(C)$ together with the composition forms a **group**. Any subset $D = \{\mathrm{Ob}(D); \mathrm{Hom}(D)\}$ with $\mathrm{Ob}(D) \subset \mathrm{Ob}(C)$ and $\mathrm{Hom}(D) \subset \mathrm{Hom}(C)$ is a **subcategory** and it is a **full subcategory** iff $\mathrm{Hom}_D(X; Y) = \mathrm{Hom}_c(X; Y)$ for every pair $X; Y \in \mathrm{Ob}(D)$.

Pointed spaces $(X; p)$ are ordered pairs of a nonempty topological space X and a point $p \in X$. Together with the **pointed maps** of the form $f : (X; P) \to (Y; q)$ with $f(p) = q$ they form a category. Other obvious examples are:

Set: **sets** and **maps**

Grp: **groups** and **group homomorphisms**

Ab: **abelian groups** and **group homomorphisms**.

Rng: **rings** and **ring homomorphisms**

CRng: **commutative rings** and **ring homomorphisms**

$\mathrm{Vec}_{\mathbb{R}}$: **real vector spaces** and **$\mathbb{R}$-linear maps**

$\mathrm{Vec}_{\mathbb{C}}$: **complex vector spaces** and **$\mathbb{C}$-linear maps**

Top: **topological spaces** and **continuous maps**

Top_*: **pointed spaces** and **pointed continuous maps**

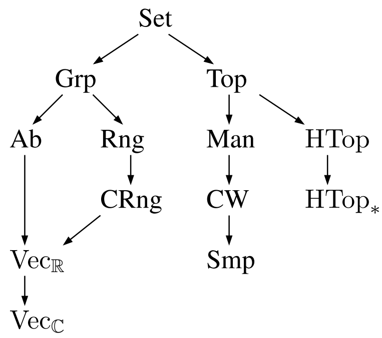
Man: **topological manifolds** and **continuous maps**

CW: **CW complexes** and **continuous maps**

Smp: **simplicial complexes** and **continuous maps**

HTop: **topological spaces** and **homotopy classes** of **continuous maps**. Its **isomorphisms** are the homotopy classes of **homotopy equivalence**.

HTop_* : **pointed topological spaces** and **homotopy classes** of **pointed continuous maps** relative to the **base point**.



2.2 Functors

A **covariant functor** is a map $\mathcal{F} : \mathrm{Ob}(C) \to \mathrm{Ob}(D)$ such that for each pair $X; Y \in \mathrm{Ob}(C)$ exists a map $\mathcal{F}_{X;Y} : \mathrm{Hom}_C(X;Y) \to \mathrm{Hom}_D(\mathcal{F}(X);\mathcal{F}(Y))$ such that $\mathcal{F}_{Y;Z}(g) \circ \mathcal{F}_{X;Y}(f) = \mathcal{F}_{X;Z}(g \circ f)$ for $f \in \mathrm{Hom}_C(X;Y)$ and $g \in \mathrm{Hom}_C(Y;Z)$ as well as $\mathcal{F}(\mathrm{id}_X) = \mathrm{id}_{\mathcal{F}(X)}$. In a **contravariant functor** the associated map has the form $\mathcal{F}_{X;Y} : \mathrm{Hom}_C(X;Y) \to \mathrm{Hom}_D(\mathcal{F}(Y);\mathcal{F}(X))$ with$\mathcal{F}_{Y;Z}(g) \circ \mathcal{F}_{X;Y}(f) = \mathcal{F}_{Z;X}(f \circ g)$. In conformity with the most obvious incarnation as a **dual linear map** as defined in [3] 3.12 and the index notation of the respective **tensors** introduced in [3] 3.13 and 7.2 the images of **covariant** functors are often abbreviated as $f_* = \mathcal{F}(f) \in \mathrm{Hom}_D(\mathcal{F}(X);\mathcal{F}(Y))$ while those referring to **contravariant functors** are denoted $f^* = \mathcal{F}(f) \in \mathrm{Hom}_D(\mathcal{F}(Y);\mathcal{F}(X))$.

Examples for covariant functors:

1. The **functor** $\pi_0 : \mathrm{Top} \to \mathrm{Set}$ assigns to each **topological space** X its set of **path components** $\pi_0(X)$ and to each **continuous map** $f : X \to Y$ the map $f_* : \pi_0(X) \to \pi_0(Y)$ defined by $f_*(P(x)) = P(f(x))$ for every **path component** $P(x) \in \pi_1(X;p)$ according to 2.1 such that $(g_* \circ f_*)(P(x)) = g_*(P(f(x))) = P((g \circ f)(x)) = (g \circ f)_*(P(x))$.
2. The **fundamental group functor** $\pi_1 : \mathrm{Top}_* \to \mathrm{Grp}$ assigns to each **pointed topological space** $(X;p)$ its **fundamental group** $\pi_1(X;p)$ and to each **pointed continuous map** $f : (X;p) \to (Y;q)$ its **induced homomorphism** $f_* : \pi_1(X;p) \to \pi_1(Y;q)$ defined by $f_*[\varphi] = [f \circ \varphi]$ such that $(g_* \circ f_*)[\varphi] = [g \circ f \circ \varphi] = (g \circ f)_*[\varphi]$ for every **loop** $[\varphi] \in \pi_1(X;p)$ and pointge continuous $g : (Y;q) \to (Z;w)$ according to 1.10. If we descend to the **homotopy class** $[f] : (X;p) \to (Y;q)$ between **pointed topological spaces** $(X;p)$ and $(Y;q)$ exactly the same definition as above results in the functor $\pi_1 : \mathrm{HTop}_* \to \mathrm{Grp}$.
3. Given a fixed topological space W the functor $\pi_W : \mathrm{HTop} \to \mathrm{Set}$ assigns to each **topological space** X its set of $\pi(W;X)$ of **homotopy classes** and to each **homotopy class** $[f] \in \pi(X;Y)$ the morphism $f_* : \pi(W;X) \to \pi(W;Y)$ defined by $f_*[\varphi] = [f \circ \varphi]$ with $(g_* \circ f_*)[\varphi] = [g \circ f \circ \varphi] = (g \circ f)_*[\varphi]$ for every functor $g_* : \pi(W;Y) \to \pi(W;Z)$

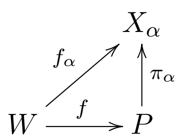
4. Every **group** $\{X;*\}$ forms a category with $\mathrm{Ob}(\{X;*\}) = X$ and $\mathrm{Hom}(\{X;*\}) = \mathrm{End}(X)$. The subcategory $(X;\mathrm{Aut}(X))$ contains the group $\{\mathrm{Aut}(X);\circ\}$ of **automorphisms** with reference to the composition. Conversely for every category C consisting of a **single object** $\mathrm{Ob}(C) = \{X\}$ whose morphisms are **all isomorphisms** the ordered pair $(\mathrm{Hom}(C);\circ) = (\mathrm{Aut}_C(X);\circ)$ forms a **group**. The **group homomorphisms** $f : \mathrm{Aut}_C(X) \to \mathrm{Aut}_D(Y)$ defined by $f(\varphi) = \varphi \circ f$ for every automorphism $\varphi : X \to X$ may be regarded as **functors** between the categories C with $\mathrm{Ob}(C) = \{X\}$ and D with $\mathrm{Ob}(D) = \{Y\}$.

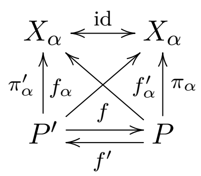
Examples for contravariant functors:

5. The **dual space functor** $\mathcal{D} : \mathrm{Vec}_{\mathbb{R}} \to \mathrm{Vec}_{\mathbb{R}}$ assigns to each **vector space** X its **dual space** X^* and to each **linear map** $f : X \to Y$ its **dual linear map** or **transpose** $f^* : Y^* \to X^*$ defined by $f^*(y^*) = y^* \circ f$ for $y^* \in Y^*$.
6. The **dual group functor** $\mathcal{D}_Z : \mathrm{Ab} \to \mathrm{Ab}$ for any **fixed abelian group** Z assigns to each **abelian group** X its dual abelian group $\mathrm{Hom}(X;Z)$ and to each **homomorphism** $f : X \to Y$ between **abelian groups** $X;Y$ its **dual homomorphism** $f^* : \mathrm{Hom}(Y;Z) \to \mathrm{Hom}(X;Z)$ defined by $f^*(\varphi) = \varphi \circ f$ for $\varphi \in \mathrm{Hom}(Y;Z)$.
7. The functor $\mathcal{C} : \mathrm{Top} \to \mathrm{CRng}$ assigns to each **topological space** X its **ring** $C(X)$ of **real-valued continuous functions** $\varphi : X \to \mathbb{R}$ and to each **continuous** map $f : X \to Y$ the **induced map** $f^* : C(Y) \to C(X)$ defined by $f^*(\varphi) = \varphi \circ f$.
8. The contravariant version of 2.2.3 is the functor $\pi^W : \mathrm{HTop} \to \mathrm{Set}$ assigning to each **topological space** X the set $\pi(X;W)$ and to each $[f] \in \pi(X;Y)$ the morphism $f^* : \pi(X;W) \to \pi(Y;W)$ defined by $f^*[\varphi] = [\varphi \circ f]$ with $(g^* \circ f^*)[\varphi] = [\varphi \circ f \circ g] = (f \circ g)_*[\varphi]$ for every $g_* : \pi(Y;W) \to \pi(Z;W)$.

2.3 Categorical products

An object $P \in \mathrm{Ob}(C)$ in a category C is a **product** of a family $(X_\alpha)_{\alpha\in A} \subset \mathrm{Ob}(C)$ iff there is a family $(\pi_\alpha)_{\alpha\in A} \subset \mathrm{Hom}(C)$ of **projections** $\pi_\alpha \in \mathrm{Hom_C}(\mathrm{P};\mathrm{X}_\alpha)$ such that the following **universal property** holds: For every family $(f_\alpha)_{\alpha\in A} \subset \mathrm{Hom}(C)$ with $f_\alpha \in \mathrm{Hom_C}(\mathrm{W};\mathrm{X}_\alpha)$ for some $W \in \mathrm{Ob}(C)$ exists a **unique morphism** $f \in \mathrm{Hom_C}(\mathrm{W};\mathrm{P})$ with $\pi_\alpha \circ f = f_\alpha$. The product P is unique up to a unique isomorphism that respects the projections, i.e. for products $\left(P;(\pi_\alpha)_{\alpha\in A}\right)$ and $\left(P';(\pi'_\alpha)_{\alpha\in A}\right)$ of the same family $(X_\alpha)_{\alpha\in A} \subset \mathrm{Ob}(C)$ exists a unique isomorphism $f : P' \to P$ with $\pi_\alpha \circ f = \pi'_\alpha$ for every $\alpha \in A$.





Proof: The universal property implies the existence of unique morphisms $f' : P \to P'$ with $f'_\alpha = \pi'_\alpha \circ f' = \mathrm{id}_{X_\alpha} \circ \pi_\alpha \Leftrightarrow f' = \pi_\alpha \circ {\pi'}_\alpha^{-1}$ and $f : P' \to P$ with $f_\alpha = \pi_\alpha \circ f = \mathrm{id}_{X_\alpha} \circ \pi'_\alpha \Leftrightarrow f = \pi'_\alpha \circ \pi_\alpha^{-1}$ which implies $f \circ f' = \mathrm{id}_P$ and $f' \circ f = \mathrm{id}_{P'}$.

Examples:

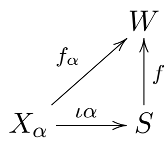
Set : The categorical product of a family of **sets** $(X_\alpha)_{\alpha\in A}$ is their **Cartesian product** $\prod_{\alpha\in A} X_\alpha$.

Top : The categorical product of a family of **topological spaces** $(X_\alpha)_{\alpha\in A}$ is their **Cartesian product** $\prod_{\alpha\in A} X_\alpha$ with the **product topology**. Due to [4, p. 4.2] the **universal property** of the **initial** resp. **product topology** implies the **continuity** of the unique isomorphism.

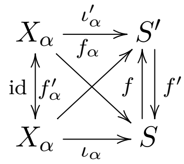
Grp: The categorical product of a family of **groups** $(X_\alpha)_{\alpha\in A}$ is their **direct product group** $\prod_{\alpha\in A} X_\alpha$ with **componentwise multiplication**.

2.4 Categorical sums

An object $S \in \mathrm{Ob}(C)$ in a category C is a **sum** or **coproduct** of a family $(X_\alpha)_{\alpha\in A} \subset \mathrm{Ob}(C)$ iff there is a family $(\iota_\alpha)_{\alpha\in A}$ of **injections** $\iota_\alpha \in \mathrm{Hom_C}(\mathrm{X_\alpha};\mathrm{S})$ such that the following **universal property** holds: For every family $(f_\alpha)_{\alpha\in A}$ with $f_\alpha \in \mathrm{Hom_C}(\mathrm{X_\alpha};\mathrm{W})$ for some $W \in \mathrm{Ob}(C)$ exists a **unique morphism** $f \in \mathrm{Hom_C}(\mathrm{S};\mathrm{W})$ with $f \circ \iota_\alpha = f_\alpha$.



The sum S is unique up to a unique isomorphism that respects the injections, i.e. for sums $\left(S;(\iota_\alpha)_{\alpha\in A}\right)$ and $\left(S';(\iota'_\alpha)_{\alpha\in A}\right)$ of the same family $(X_\alpha)_{\alpha\in A} \subset \mathrm{Ob}(C)$ exists a unique isomorphism $f : S \to S'$ with $f \circ \iota_\alpha = \iota'_\alpha$ for every $\alpha \in A$.



Proof: The universal property implies the existence of unique morphisms $f' : S' \to S$ with $f'_\alpha = f' \circ \iota'_\alpha = \iota_\alpha \circ \mathrm{id}_{X_\alpha} \Leftrightarrow f' = \iota'^{-1}_\alpha \circ \iota^{-1}_\alpha$ and $f : S \to S'$ with $f_\alpha = f \circ \iota_\alpha = \iota'_\alpha \circ \mathrm{id}_{X_\alpha} \Leftrightarrow f = \iota^{-1}_\alpha \circ \iota'_\alpha$ which implies $f \circ f' = \mathrm{id}_{S'}$ and $f' \circ f = \mathrm{id}_S$.

Examples:

Top: The categorical sum of a family of **topological spaces** $(X_\alpha)_{\alpha\in A}$ is their **topological sum** $\bigsqcup_{\alpha\in A} X_\alpha$. The **universal property** of the **final topology** [4, p. 4.5] with regard to the **injections** resp. the **topological sum** [4, p. 4.10] implies the **continuity** of the unique isomorphism.

Top$_*$: The categorical sum of a family of **pointed topological spaces** $(X_\alpha; p_\alpha)_{\alpha\in A}$ is their **wedge sum** $\bigvee_{\alpha\in A} X_\alpha$. The **universal property** of the **final topology** [4, p. 4.5] with regard to the **projection** resp. the **wedge sum** [4, p. 4.14] implies the **continuity** of the unique isomorphism.

Ab: The categorical sum of a family of **abelian groups** $(X_\alpha)_{\alpha\in A}$ is their **direct sum** $\bigoplus_{\alpha\in A} X_\alpha \subset \prod_{\alpha\in A} X_\alpha$, i.e. the subgroup of the Cartesian product with **componentwise multiplication** where only a **finite** number of components deviates from the neutral element. The unique isomorphism is the **finite product of the corresponding components** modulo automorphisms $g : W \to W$ with $\iota'_\alpha = g \circ \iota_\alpha$.

3 Free Groups

3.1 Topological groups

A **topological group** is a **group** $(G;\circ)$ endowed with a topology such that the group operation $\circ : G \times G \to G$ is **continuous**. Obviously every **subgroup** $H \subset G$ of a topological group is a topological group with regard to the **trace topology** and every **product** $(G \times H; \circ \times \diamond)$ of topological groups $(G;\circ)$ and $(H;\diamond)$ is a topological group with regard to the **product topology**. For every **continuous left action** $G \times X \to X$ of a **topological group** G on a **topological space** X defined according to [3] 1.13 by a map $(g;x) \mapsto g \cdot x$ with **associative law** $g_2 \cdot (g_1 \cdot x) = (g_2 g_1) \cdot x$ and conformity with the **neutral element** $e \cdot x = x$ for every $x \in X$ the **left translations** $g \cdot X$ are **homeomorphic** to X since in that case **all** maps $x \mapsto g \cdot x$ by the composition $x \mapsto (g;x) \mapsto g \cdot x$ are **continuous** and so are their **inverses** $g \cdot x \mapsto x = e \cdot x$. In the case of a **discrete topology** on G the converse is also true in that the **homeomorphy** $g \cdot X \approx X$ implies the **continuity of the left action** $(g;x) \mapsto g \cdot x$ since all open sets $g \cdot O$ are homeomorphic to $O \subset X$ whence O itself must be open in X and the preimage $\{g\} \times O$ is then open in $G \times X$. The **orbit**s $G \cdot x$ of all $x \in X$ defined in [3] 1.14 form a **partition** of $X = \mathring{\bigcup}_{x\in X} G \cdot x$ since $g \cdot x = h \cdot y \in (G \cdot x) \cap (G \cdot y) \Rightarrow (h^{-1}g) \cdot x = y \Rightarrow y \in G \cdot x \Rightarrow G \cdot y \subset G \cdot x$ and vice versa. The corresponding **orbit space** X/G is the quotient space with regard to $xGy \Leftrightarrow \exists g \in G : \; y = g \cdot x$. The action is **transitive** iff $G \cdot x = X$ for every $x \in X$ and it is **free** iff $g \cdot x = x$ implies $g = e$. According to [3] 1.7 in the case of a **subgroup** $H \subset G$ acting on a topological group the orbits gH of the **left** action are the **right cosets** and vice versa. The **orbit space** G/H is then called the **left coset space** of G by H and in the case of coinciding cosets $gH =$

Hg according to [3] 1.8 the orbit space inherits the algebraic structure of a **factor group**. Analogous statements hold for the **right actions** defined by $x \cdot g = g^{-1} \cdot x$.

Obvious examples of topological groups are provided by the real numbers $(\mathbb{R}_*^+; \cdot) \subset (\mathbb{R}_*; \cdot) \subset (\mathbb{C}_*; \cdot)$, the **circle** $(\mathbb{S}^1; \cdot) \subset (\mathbb{C}_*; \cdot)$ with the **complex multiplication** and the **torus** $(\mathbb{T}^n; \cdot) = (\mathbb{S}^1 \times ... \times \mathbb{S}^1; \cdot)$with the **direct group structure** defined according to [3] 1.4 as componentwise multiplication $\boldsymbol{xy} = (x_i \cdot y_i)_{1 \le i \le n}$ for $x_i; y_i \in \mathbb{C}$.

3.2 The general linear group

The continuity of the multiplication and addition on $\mathbb{C}^2 \to \mathbb{C}$, the resulting continuity of polynomials on $\mathbb{C}^{2n} \to \mathbb{C}$ resp. the continuity of the components of the matrix multiplication $\mathbb{C}^{2n^2} \to \mathbb{C}$ imply the continuity of the matrix multiplication $\mathbb{C}^{2n^2} \to \mathbb{C}^{n^2}$ with regard to the corresponding product spaces while **Cramer's rule** [3] 4.3 assures the continuity of the inversion whence the **general linear groups** $GL(n; \mathbb{R}) \subset GL(n; \mathbb{C})$ are topological group with regard to the **product topology** on $\mathbb{R}^{n^2} \subset \mathbb{C}^{n^2}$. Among its subgroups we have the **orthogonal group** $O(n; \mathbb{R}) \subset GL(n; \mathbb{R})$ and the **normal subgroup** $U(n) \subset GL(n; \mathbb{C})$ of the **unitary matrices** defined in [3] 6.6.

From the argument above follows that the **general linear group** $GL(n; \mathbb{R})$ by matrix multiplication **continuously** acts on the left on $\mathbb{R}^n$. According to [3] Def. 3.10 for any pair $\boldsymbol{x}; \boldsymbol{y} \in \mathbb{R}^n$ with $x_i \neq 0$ and $y_j \neq 0$ exists a $T_{\mathcal{B}}^{\mathcal{A}} = \left(T_{\mathcal{B}}^{\mathcal{E}}\right)^{-1} * C_{ij} * T_{\mathcal{A}}^{\mathcal{E}} \in GL(n; \mathbb{R})$ with regard to the bases $\mathcal{A} = (\boldsymbol{e}_1; ...; \boldsymbol{e}_{i-1}; \boldsymbol{x}; \boldsymbol{e}_{i+1}; ...; \boldsymbol{e}_n)$ and $\mathcal{B} = (\boldsymbol{e}_1; ...; \boldsymbol{e}_{j-1}; \boldsymbol{y}; \boldsymbol{e}_{j+1}; ...; \boldsymbol{e}_n)$ with the **coordinate systems** $T_{\mathcal{A}}^{\mathcal{E}} * \boldsymbol{x} = \boldsymbol{e}_i$ resp. $T_{\mathcal{B}}^{\mathcal{E}} * \boldsymbol{y} = \boldsymbol{e}_j$ and the **index exchange**

$$C_{ij} = \begin{pmatrix} 1 & & & & & \cdots & 0 \\ & \ddots & & & & & \vdots \\ & & 0 & & 1 & & \\ & & & \ddots & & & \\ & & 1 & & 0 & & \\ \vdots & & & & & \ddots & \\ 0 & \cdots & & & & & 1 \end{pmatrix}$$

$C_{ij} * \boldsymbol{e}_i = \boldsymbol{e}_j$ which implies $T_{\mathcal{B}}^{\mathcal{A}} * \boldsymbol{x} = \boldsymbol{y}$. Hence we have $GL(n; \mathbb{R}) * \mathbb{R}_*^n = \mathbb{R}_*^n$ and consequently **two orbits** $\mathbb{R}_*^n$ and $\{\boldsymbol{0}\}$ resp. the orbit space $\mathbb{R}_*^n / GL(n; \mathbb{R}) = \{\pi(\boldsymbol{e}_1); \pi(\boldsymbol{0})\}$. Its quotient topology comprises the three sets $\{\pi(\boldsymbol{e}_1); \pi(\boldsymbol{0})\}$, $\{\pi(\boldsymbol{e}_1)\}$ and $\emptyset$ whence it is **not a Hausdorff** space. The corresponding orbits of the **orthogonal group** are the **spheres** $r\mathbb{S}^{n-1}$ with the orbit space $\mathbb{R}_*^n / O(n; \mathbb{R}) = \{\pi(r\boldsymbol{e}_1) : r \ge 0\}$. This space is homeomorphic to $\mathbb{R}^+$ and hence a **Hausdorff** space.

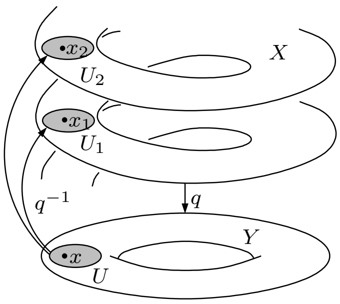
3.3 The torus

The group $(\{\pm 1\}; \cdot)$ endowed with the **discrete topology** by multiplication **freely** and **continuously** acts on the **sphere** $\mathbb{S}^n$ with orbits consisting of pairs of **antipodal points** $\pm\boldsymbol{e}$ for $\boldsymbol{e} \in \mathbb{S}^n$ such that its orbit space is homeomorphic to the **projective space** $\mathbb{P}^n$ defined in [3] 9.1 as the orbit space of $(\mathbb{R}_*; \cdot)$ acting on $\mathbb{R}^{n+1} \setminus \{\boldsymbol{0}\}$ resulting in the orbits $\mathbb{R}_* \cdot \boldsymbol{e}$ for $\boldsymbol{e} \in \mathbb{S}^n$. The action of the additive group $(\mathbb{Z}^n; +)$ endowed with the **direct group structure**, i.e. $\boldsymbol{x} + \boldsymbol{y} = (x_i + y_i)_{1 \le i \le n}$ by $\boldsymbol{g} \cdot \boldsymbol{x} = \boldsymbol{g} + \boldsymbol{x}$ on $\mathbb{R}^n$ results in the orbit space $\mathbb{R}^n / \mathbb{Z}^n$ which due to the **commutativity** of the componentwise addition due to [3] 1.8 inherits the algebraic structure of a **factor group**. The **exponential quotient map** $\epsilon : \mathbb{R}^n \to \mathbb{S}^{n-1}$ defined by $\epsilon(\boldsymbol{r}) = \left(e^{2\pi i r_k}\right)_{1 \le k \le n}$ is **continuous**, **open** and **surjective** so that according to [4, p. 4.8.2] it is a **quotient map** and its canonical bijection $\bar{\epsilon} : \mathbb{R}^n / \mathbb{Z}^n \to \mathbb{S}^{n-1}$ is a **homeomorphism**.

4 Covering maps

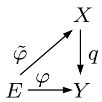
4.1 Elementary properties of covering maps

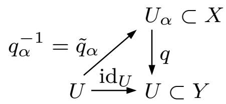
A **continuous right inverse** $q_\alpha^{-1} : Y \to X_\alpha \subset X$ with $q \circ q_\alpha^{-1} = \mathrm{id}_Y$ of a **surjective continuous** map $q : X \to Y$ is called a **section** of q and in the case of $q_\alpha : U_\alpha \to V$ for an **open** subset $V \subset Y$ it is a **local section** of q over U.



An **open** subset $V \subset Y$ is **evenly covered** by a **continuous** map $q : X \to Y$ iff every $y \in V$ has an open neighborhood $y \in U \subset V$ such that $q^{-1}[U] = \bigsqcup_{\alpha \in A} U_\alpha$ is a **disjoint** union of **connected open sheets** $U_\alpha \subset X$ with **homeomorphisms** $q_\alpha : U_\alpha \to U$. Consequently their inverses q_α^{-1} are **local sections**, the sheets are the **components** of $q^{-1}[U]$ and U is a **connected** set. Every **open connected subset** of an evenly covered subset is itself evenly covered. A **covering map** is a **continuous surjective** map $q : X \to Y$ from a **connected** and **locally path connected covering space** X onto the **base of the covering** Y such that every point of Y has an **evenly covered neighborhood**. According to [4, p. 5.7] this implies that the covering space X is (globally) **path connected** whence its continuous image Y is also **path connected**. Also the covering map q is a **local homeomorphism** whence it is an **open quotient map**. The equivalence classes defined on an evenly covered set Y by points $y \in V$ with coinciding cardinalities $|A| = |q^{-1}(y)|$ for $q^{-1}[U] = \bigsqcup_{\alpha \in A} U_\alpha$ and the open neighborhood $y \in U \subset V$ are all open whence the **connectedness** of X implies that there is but a single equivalence class which defines the unique **number of sheets of the covering**. In the case of **injectivity** the covering map is a global homeomorphism. A **finite product** of covering maps as well as a restriction to a **saturated, connected, open** subset are again covering maps.

A **lift** of a **continuous** $\varphi : E \to Y$ to a **covering map** $q : X \to Y$ is a **continuous** $\tilde{\varphi} : E \to X$ such that $q \circ \tilde{\varphi} = \varphi$. In particular the **local sections** $q_\alpha^{-1} : U \to U_\alpha$ of a **covering map** $q : X \to Y$ are lifts of the **local identity** $\mathrm{id}_U = q \circ q_\alpha^{-1}$.



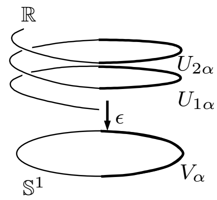


Examples:

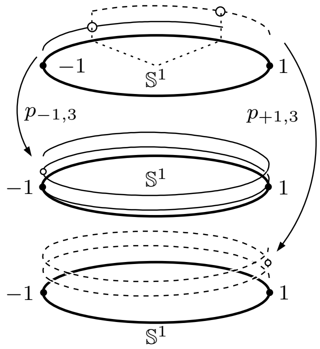
1. The **exponential quotient map** $\epsilon : \mathbb{R} \to \mathbb{S}^1$ with $\epsilon(t) = e^{2\pi i t}$ is a **covering map** since every $z = \mathrm{Re}z + i\mathrm{Im}z \in \mathbb{S}^1$ has a neighborhood included in one of the four halfplanes V_m for $1 \le m \le 4$ with sheets $U_{m,n}$ and local inverses $\epsilon_{m,n}^{-1}$ of the restrictions $\epsilon_{m,n} = \epsilon|_{U_{m,n}} : U_{m,n} \to V_m$ which are also **local sections of the exponential quotient map** ϵ **over the halfplanes** V_m:

m	V_m	$\mathrm{U}_{m,n}$	$\epsilon_{m,n}^{-1}$
1	$\{\mathrm{Re}z > 0\}$	$\left]n - \frac{1}{4}; n + \frac{1}{4}\right[$	$n + \frac{\sin^{-1}(\mathrm{Im}z)}{2\pi}$
2	$\{\mathrm{Im}z > 0\}$	$\left]n; n + \frac{1}{2}\right[$	$n + \frac{\cos^{-1}(\mathrm{Re}z)}{2\pi}$
3	$\{\mathrm{Re}z < 0\}$	$\left]n + \frac{1}{4}; n + \frac{3}{4}\right[$	$n + \frac{1}{2} + \frac{\sin^{-1}(\mathrm{Im}z)}{2\pi}$
4	$\{\mathrm{Im}z < 0\}$	$\left]n + \frac{1}{2}; n + 1\right[$	$n + \frac{1}{2} + \frac{\cos^{-1}(\mathrm{Re}z)}{2\pi}$

2. The **product** $\epsilon^n : \mathbb{R}^n \to \mathbb{T}^n$ defined by $\epsilon^n(x_1; ...; x_n) = (\epsilon(x_1); ...; \epsilon(x_n))$ of n exponential quotient maps is again a covering map.

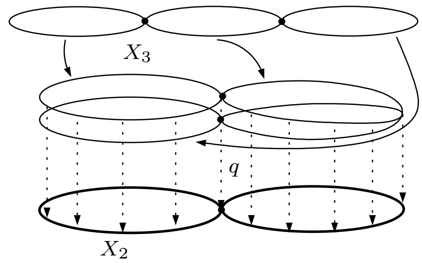


3. The nth **power map** $p_n : \mathbb{S}^1 \to \mathbb{S}^1$ with $p_n(z) = z^n$ is a **covering map** since every $z = e^{2\pi i t} \in \mathbb{S}^1$ has a neighborhood included in one of the two halfplanes $V_\pm = \mathbb{S}^1 \setminus \{\pm 1\}$ with sheets $U_{\pm m,n} = \left\{ e^{2\pi i t} : \frac{2m}{(3\pm 1)n} + \frac{k}{(m+1)n} < t < \frac{2m}{(3\pm 1)n} + \frac{k+1}{(m+1)n} : 0 \leq k \leq m \right\}$ for $0 \leq m < n$ and **local inverses** $p^{-1}_{\pm m,n} : V_\pm \to U_{\pm m,n}$ given by $p^{-1}_{\pm m,n}(z) = z^{\frac{2m}{(3\pm 1)n} + \frac{1}{n}}$.

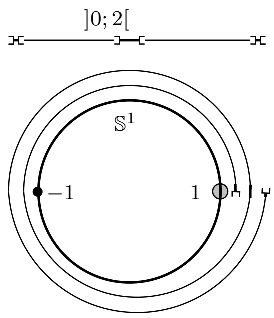


4. The map $q : \mathbb{S}^n \to \mathbb{RP}^n$ defined by $q(x_1; ...; x_{n+1}) \to [x_1 : ... : x_{n+1}]$ is a **covering map** with the open cover given by $V_m = [x_m \neq 0] \subset \mathbb{RP}^n$ with open sheets $U_{+m} = \mathbb{S}^n \cap \{x_m > 0\}$ and $U_{-m} = \mathbb{S}^n \cap \{x_m < 0\}$.

5. The map $q : X_3 \to X_2$ defined by $q(z) = \begin{cases} z & \text{for } z \in S_0 = \partial B^2_1(0) = \mathbb{S}^1 \\ 2 - (z-2)^2 & \text{for } z \in S_2 = \partial B^2_1(2) \\ 4 - z & \text{for } z \in S_4 = \partial B^2_1(4) \end{cases}$ is a **covering map** of the **covering space** $X_3 = S_0 \cup S_2 \cup S_4$ onto the **base** $X_2 = S_0 \cup \delta S_4$. Geometrically it is the identity on S_0, wraps S_2 twice around itself and reflects S_4 onto S_0. The open set $X_3 \setminus \{2; 4\}\}$ is covered by the two **sheets** $S_0 \cup S_2 \cap \{\mathrm{Im} z > 0\}$ and $S_4 \cup S_2 \cap \{\mathrm{Im} z < 0\}$ while the neighborhood of the remaining intersection point $B^2_\epsilon(2)$ is covered by $B^2_\epsilon(2)$ and $B^2_\epsilon(4)$.

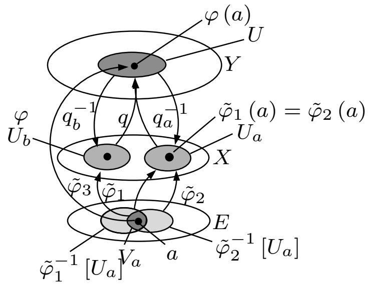


6. The **restriction of the exponential quotient map** $\epsilon|_{]0;2[}$ is a **local homeomorphism** but **not a covering map** since every preimage $\epsilon^{-1}\left[B^2_\delta(1) \cap \mathbb{S}^1\right]$ for $\delta > 0$ contains the open neighborhood $B^1_\delta(1)$ of $t = 1$ but also the intervals $]0; \delta[$ and $]2 - \delta; 2[$ such that there can be no continuous right inverse $\sigma : U_\delta(1) = B^2_\delta(1) \cap \mathbb{S}^1 \to]0; \delta[\cup]0; \delta[$ with $\epsilon \circ \sigma = \mathrm{id}|_{U_\delta(1)}$.



4.2 The unique lifting property

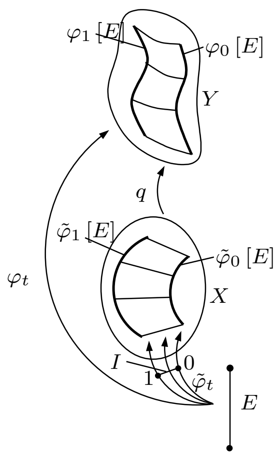
Two **lifts** $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$ of a continuous $\varphi : E \to X$ to a **covering map** $q : X \to Y$ on a **connected** space E are **equal** iff they coincide at a single point $e \in E$ such that $\tilde{\varphi}_1(e) = \tilde{\varphi}_2(e)$.



Proof: For every $a \in A = \{a \in E : \tilde{\varphi}_1(a) = \tilde{\varphi}_2(a)\}$ exists an evenly covered neighborhood U of $\varphi(a) \in U \subset Y$ and a sheet U_a containing $\tilde{\varphi}_1(a) = \tilde{\varphi}_2(a) = x_a \in U_a \subset X$ whence $V_a = \tilde{\varphi}_1^{-1}[U_a] \cap \tilde{\varphi}_2^{-1}[U_a] \subset A \subset E$ due to $\tilde{\varphi}_1|_{V_a} = \tilde{\varphi}_2|_{V_a}$ $= q_a^{-1} \circ \varphi|_{V_a}$ such that V_a is a neighborhood of a in A which implies that A is **open** in E. Conversely for every $a \notin A$ there are sheets $U_{1,a} \ni \tilde{\varphi}_1(a) \neq \tilde{\varphi}_2(a)$ $\in U_{2,a}$ and since the sheets of U must be disjoint we conclude that $U_{1,a} \cap U_{2,a}$ $= \emptyset$. Due to the continuity of the lifts the open set $W_a = \tilde{\varphi}_1^{-1}[U_a] \cap \tilde{\varphi}_2^{-1}[U_a] \subset$ $E \setminus A$ is a neighborhood of a in $E \setminus A$ whence $E \setminus A$ is also **open** in E. Since E is **connected** the assertion follows.

4.3 The homotopy lifting property

For every **continuous** $\varphi_1 : E \to Y$ on a **locally connected** space E with a **homotopy** $\varphi : E \times I \to Y$ to a **continuous** $\varphi_0 : E \to Y$ with a **lift** $\tilde{\varphi}_0 : E \to X$ to a **covering map** $q : X \to Y$ exists a **unique lift** $\tilde{\varphi} : E \times I \to X$ of φ to q which is also a **homotopy** from $\tilde{\varphi}_0$ to $\tilde{\varphi}_1$. If φ_t is stationary on some subset $A \subset E$, then so is $\tilde{\varphi}_t$.



Proof: For any two lifts $\tilde{\varphi}_t; \tilde{\varphi}'_t : W \to X$ with $\tilde{\varphi}_0 = \tilde{\varphi}'_0$ on a subset $W \subset E$ and any $e \in W$ the two trace maps $t \mapsto \tilde{\varphi}_t(e)$ and $t \mapsto \tilde{\varphi}'_t(e)$ are lifts of the path $t \mapsto \varphi_t(e)$ starting at the same point whence by the **unique lifting property** applied to the **connected** set $I = [0;1]$ 4.2 they are identical.

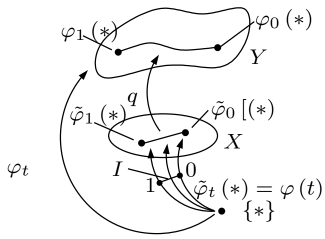
For every $e \in E$ and every $t \in I$ exists an **evenly covered** neighborhood $\varphi_t(e)$ $\in U_t \subset Y$, hence open subsets $V_t \subset E$ and $J_t \subset I$ such that $V_t \times J_t \subset \varphi^{-1}[U_t]$. By **compactness** finitely many of these sets $(V_j \times J_j)_{1 \le j \le m}$ cover $\{e\} \times I$. Due to the **local connectedness** of E exists a **connected** neighborhood of $e \in W \subset \bigcap_{j=1}^{m} V_j$ while according to **Lebesgue's lemma** [4, p. 9.15] the **compactness** of I implies the existence of an $n \in \mathbb{N}$ such that every interval $\left[\frac{k-1}{n}; \frac{k}{n}\right]$ for $1 \le k \le n$ is contained in some J_j whence every column section $W \times \left[\frac{k-1}{n}; \frac{k}{n}\right]$ $\subset V_k \times J_k \subset \varphi^{-1}\left[U_{t_j}\right]$ with $t_j \in J_j$ is mapped by φ into an evenly covered open subset $U_k = U_{t_j} \subset Y$. In particular for $\varphi\left[W \times \left[0; \frac{1}{n}\right]\right] \subset U_1$ exist a **sheet** $V_1 \subset X$ and a **local section** $q_1^{-1} : U_1 \to V_1$ with $\left(q_1^{-1} \circ \varphi_0\right)(e) = \tilde{\varphi}_0(e)$ so that we can define a starting section of the desired homotopy by $\tilde{\varphi} = q_1^{-1} \circ \varphi : W \times \left[0; \frac{1}{n}\right] \to V_1$, which due to the **uniqueness property** 4.2 coincides for $t = 0$ with the given lift $\tilde{\varphi}_0$ on the **connected** domain $W \times \{0\}$. Assuming the existence of a lift $\tilde{\varphi} : W \times \left[0; \frac{k-1}{n}\right] \to X$ coinciding with $\tilde{\varphi}_0$ on $W \times \{0\}$ we proceed by induction with the **evenly covered** U_k containing $\varphi\left[W \times \left[\frac{k-1}{n}; \frac{k}{n}\right]\right]$ and the **local section** $q_k^{-1} : U_k \to V_k$ with $\left(q_k^{-1} \circ \varphi_{(k-1)/n}\right)(e) = \tilde{\varphi}_{(k-1)/n}(e)$. By $\tilde{\varphi} = q_k^{-1} \circ \varphi$ defined on $W \times \left[\frac{k-1}{n}; \frac{k}{n}\right]$ we obtain a lift of φ coinciding with the previously defined section on the single point $\left(e; \frac{k-1}{n}\right)$ whence by 4.2 they must coincide on the common domain $W \times \left\{\frac{k-1}{n}\right\}$ and by the **attaching lemma** [4, p. 4.11] they must agree on the whole connected set $W \times \left[0; \frac{k}{n}\right]$

Two such lifts on neighborhoods $e \in W \subset E$ and $e' \in W' \subset E$ agree on $(W \cap W') \times I$ whence by the attaching lemma the extension $\tilde{\varphi} : E \times I \to X$ is well defined and by construction it is a lift of φ to q coinciding with the given lift $\tilde{\varphi}_0$ on $E \times \{0\}$.

Finally, if φ is **stationary** on $A \subset E$, then for each $e \in A$ the path $t \mapsto \varphi(e;t) = \varphi_0(e)$ is constant whose unique lift starting at $\tilde{\varphi}_0(a)$ is the constant path $t \mapsto \tilde{\varphi}(e;t) = \tilde{\varphi}_0(e)$ whence $\tilde{\varphi}$ is also constant on E.

4.4 The path lifting property

For any **path** $\varphi : I \to Y$ and every $x \in (q^{-1} \circ \varphi)(0)$ exists a **unique lift** $\tilde{\varphi} : I \to X$ such that $\tilde{\varphi}(0) = x$.



Corollary: For every path $\varphi : I \to \mathbb{S}^1$ and any $s_0 \in (\epsilon^{-1} \circ \varphi)(0)$ exists a unique lift $\tilde{\varphi} : I \to \mathbb{R}$ to the **exponential quotient map** ϵ given by $\epsilon(s) = e^{2\pi i s}$ with $\tilde{\varphi}(0) = s_0$ and any other lift of φ to ϵ has the form $\tilde{\varphi} + z$ with $z \in \mathbb{Z}$.

Proof: Directly follows from 4.3 since for the homotopy $\Phi : \{*\} \times I \to Y$ defined by $\Phi_t(*) = \varphi(t)$ from $\Phi_0 : * \mapsto \varphi(0)$ to $\Phi_1 : * \mapsto \varphi(1)$ exists a unique lift $\tilde{\Phi} : \{*\} \times I \to X$ with $\tilde{\Phi}_0(*) = x$ which results in the desired lift defined by $\tilde{\varphi}(t) = \tilde{\Phi}_t(*)$. The special claim of the corollary follows from $(\epsilon^{-1} \circ \varphi)(0) = s_0 + \mathbb{Z}$ whence every other lift $\tilde{\varphi}'$ has an initial point $\tilde{\varphi}'(0) = s_0 + z$ with $z \in \mathbb{Z}$ and the uniquely determined lift from $\varphi' - \varphi = \varphi - \varphi \equiv 0$ to ϵ is given by $\tilde{\varphi}' - \tilde{\varphi} \equiv \tilde{\varphi}'(0) - \tilde{\varphi}(0) = z$.

4.5 The Monodromy theorem

The lifts $\tilde{\varphi}_0; \tilde{\varphi}_1 : I \to X$ with coinciding initial point $\tilde{\varphi}_0(0) = \tilde{\varphi}_1(0)$ of any two **path homotopic** paths $\varphi_0; \varphi_1 : I \to Y$ with **coinciding initial and terminal points** $\varphi_0(0) = \varphi_1(0)$ resp. $\varphi_0(1) = \varphi_1(1)$ to a **covering map** $q : X \to Y$ are **path homotopic** such that they have the same terminal point $\tilde{\varphi}_0(1) = \tilde{\varphi}_1(1)$.

Proof: Since **path homotopy** implies a **stationary** behaviour of the homotopy φ_t on the initial and terminal points the assertion immediately follows from the uniqueness of the **homotopy lifting property** 4.3 with the lift homotopy $\tilde{\varphi}_t : I \to X$ being stationary on $A = \{0; 1\}$.

4.6 Path homotopy criterion for the circle

Two paths $\varphi_0; \varphi_1 : I \to \mathbb{S}^1$ with **coinciding initial and terminal points** $\varphi_0(0) = \varphi_1(0)$ resp. $\varphi_0(1) = \varphi_1(1)$ are **path-homotopic** iff any lifts $\tilde{\varphi}_0; \tilde{\varphi}_1 : I \to \mathbb{R}$ to the **exponential quotient map** $\epsilon : \mathbb{R} \to \mathbb{S}^1$ given by $\epsilon(s) = e^{2\pi i s}$ with coinciding initial point $\tilde{\varphi}_0(0) = \tilde{\varphi}_1(0)$ have the same terminal point $\tilde{\varphi}_0(1) = \tilde{\varphi}_1(1)$.

Proof:

$\Rightarrow$ is a direct consequence of the **monodromy theorem** 4.5.

$\Leftarrow$ follows from the **simple connectivity** of $\mathbb{R}$ due to 1.4.3 and the composition $\varphi_t = \epsilon \circ \tilde{\varphi}_t$.

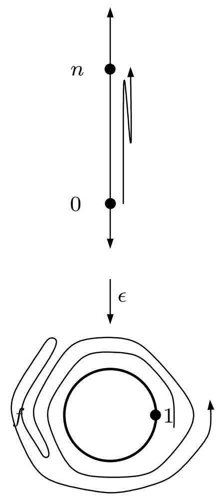
4.7 The injectivity theorem

For every **covering map** $q : X \to Y$ and every $x \in X$ the **induced homomorphism** $q_* : \pi_1(X; x) \to \pi_1(Y; q(x))$ is **injective**.

Proof: For every $\varphi \in q_*^{-1}\left(c_{q(x)}\right)$ we have $q_*(\varphi) = q \circ \varphi = c_{q(x)}$ whence by the **unique lifting property** 4.2 the path $\varphi : I \to X$ is also the uniquely determined lift of the constant function $t \mapsto q(x)$ to q whence it must coincide with the stationary lift $\varphi \equiv \mathrm{c}(x)$.

4.8 The winding number

For a loop $\varphi : I \to \mathbb{S}^1$ based at $\varphi(0) = \varphi(1) \in \mathbb{S}^1$and a lift $\tilde{\varphi} : I \to \mathbb{R}$ of φ according to 4.6 the points $\tilde{\varphi}(0); \tilde{\varphi}(1) \in (\epsilon^{-1}\varphi)(0)$ differ by the **winding number** $N(\varphi) \in \mathbb{N}$ depending only on φ. According to the he **path-homotopy criterion** 4.6 two loops in $\mathbb{S}^1$ based on the same point are **path-homotopic** iff they have the same winding number. The lift of the product $\varphi \cdot \psi$ of two loops based at the same point is given by $(\varphi \cdot \psi)^{\sim}(s) = \begin{cases} \tilde{\varphi}(2s) & \text{for } 0 \le s \le \frac{1}{2} \\ \tilde{\psi}(2s-1) & \text{for } \frac{1}{2} \le s \le 1 \end{cases}$ with $\tilde{\varphi}(1) = \tilde{\psi}(0)$ such that the **winding number of the product** is $N(\varphi \cdot \psi)$ $= \tilde{\psi}(1) - \tilde{\varphi}(0) = \tilde{\psi}(1) - \tilde{\psi}(0) + \tilde{\varphi}(1) - \tilde{\varphi}(0) = N(\varphi) + N(\psi)$.



4.9 The fundamental group of the circle

The fundamental group $\pi_1(\mathbb{S}^1; 1) = \langle \epsilon \rangle$ is an **infinite cyclic group** generated by the **exponential quotient map**.

Proof: The loop $\epsilon_z : I \to \mathbb{S}^1$ given by $\epsilon_z(s) = e^{2\pi i z s}$ for $z \in \mathbb{Z}$ satisfies $\epsilon_1 = \epsilon$, $\epsilon_{-1} = \bar{\epsilon}$ and by the reparametrization $\begin{cases} \epsilon_{z-1}(2s) = \epsilon_z\left(\frac{2(z-1)s}{z}\right) & \text{for } 0 \le s \le \frac{1}{2} \\ \epsilon(2s-1) = \epsilon_z\left(\frac{2s-1}{z}\right) = \epsilon_z\left(\frac{2s-1}{z} + \frac{z-1}{z}\right) & \text{for } \frac{1}{2} \le s \le 1 \end{cases}$ we get $[\epsilon_{z-1}] \cdot [\epsilon] = [\epsilon_z]$ whence by induction follows $[\epsilon_z] = [\epsilon]^z$. Hence the map $J : \mathbb{Z} \to \pi_1(\mathbb{S}^1; 1)$ defined by $J(z) = [\epsilon_z]$ is a **homomorphism**. The **lift** from ϵ_z to ϵ is given by $\tilde{\epsilon}_z : I \to \mathbb{R}$ with $\tilde{\epsilon}_z(s) = z \cdot s$ such that its **winding number** is $N(\epsilon_z) = z$. According to 4.8 every loop $\varphi : I \to \mathbb{S}^1$ based on 1 with the same winding number $N(\varphi) = z$ is path-homotopic to ϵ_z whence the map $K : \pi_1(\mathbb{S}^1; 1) \to \mathbb{Z}$ given by $K(\varphi) = N(\varphi)$ is well defined. Due to 4.8 it is also a **homomorphism** with $K \circ J = \mathrm{id}_{\mathbb{Z}}$ and also $J \circ K = \mathrm{id}_{\pi_1(\mathbb{S}^1;1)}$ since $[\varphi] = [\epsilon_z]$ for every $[\varphi] \in \pi_1(\mathbb{S}^1; 1)$ with $N(\varphi) = z$. Hence K and J are **isomorphisms** which proves the assertion.

4.10 The fundamental group of the torus

The map $\varphi : \mathbb{Z}^n \to \pi_1(\mathbb{T}^n; p)$ defined by $\varphi(k_1; ...; k_n) = [\omega_1]^{k_1} \cdot ... \cdot [\omega_n]^{k_n}$ with the base point $p = (1; ...; 1)$and the loops $\omega_j : \mathbb{R} \to \mathbb{T}^n$ given by $\omega_j(s) = (1; ...; 1; e^{2\pi i s}; 1; ...; 1)$ is an isomorphism.

Proof: Directly follows from 4.9 and 1.9.

5 Homology

5.1 Homology groups of CW complexes

Due to 1.13 the **homology groups** and the **fundamental group** of a **compact** ENR are **finitely generated**.

Proof: [1] p. 140 application (ii) of theorem 2.35 on p. 140

5.2 Invariance of dimension

The dimension n of a finite-dimensional manifold is **uniquely** determined.

Proof: [1] p. 379 problem 13-3

5.3 Invariance of the boundary

Every **manifold with a boundary** is the **disjoint** union of its **interior** and its **boundary**..

Proof: [1] p. 379 problem 13-4

5.4 Brouwer's fixed point theorem

Every continuous map $f : \bar{\mathbb{B}}^n \to \bar{\mathbb{B}}^n$ has a fixed point $\boldsymbol{x}$ with $f(\boldsymbol{x}) = \boldsymbol{x}$.

Proof: [1] p. 379 problem 13-7

5.5 The Jordan-Brouwer separation theorem

Every embedding $\mathbb{S}^n \cong X \subset \mathbb{R}^{n+1}$ separates $\mathbb{R}^{n+1}$ into a bounded interior and an unbounded exterior.

Proof: [2] ch. 3 p. 31 - 41

5.6 The Jordan-Schoenflies theorem

Every embedding $\varphi : \mathbb{S}^n \to X \subset \mathbb{R}^{n+1}$ extends to a homeomorphism $\Phi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$.

Proof: [2] ch. 4 & ch. 9

5.7 Topological invariance of the Euler characteristic

For every **finite CW complex** X the **Euler characteristic** computes by $\chi(X) = \sum_p (-1)^p \operatorname{rank} H_p(X)$.
Therefore, the Euler characteristic is a **homotopy invariant**.

Proof: [1] p. 373 theorem 13.36

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[1] John M. Lee. *Introduction to topological manifolds*. 2nd ed. Springer, 2011.

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