Functional Analysis

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1.1 Topological vector spaces

A topology $\mathcal{O}$ on a complex vector space X is a **vector topology** and the pair $(X, \mathcal{O})$ is a **topological vector space** iff it satisfies T_1 and the **multiplication with scalars** $\cdot : \mathbb{C} \times X \to X$ as well as the **vector addition** $+ : X \times X \to X$ are **continuous**. Hence for $\alpha; \beta \in \mathbb{C}$ resp. $\mathbf{x}; \mathbf{x}_1; \mathbf{x}_2 \in X$ and every neighbourhood $U \in \mathcal{U}(\alpha\mathbf{x})$ there are neighbourhoods $B_\delta(\alpha)$ resp. $V \in \mathcal{U}(\mathbf{x})$ with $\beta V \subset U \, \forall \beta \in B_\delta(\alpha)$ and for each neighbourhood $U \in \mathcal{U}(\mathbf{x}_1 + \mathbf{x}_2)$ there are neighbourhoods $V_1 \in \mathcal{U}(\mathbf{x}_1)$ resp. $V_2 \in \mathcal{U}(\mathbf{x}_2)$ with $V_1 + V_2 \subset U$.

1.2 Invariance

The **translation operator** $T_\mathbf{a} : X \to X$ with $T_\mathbf{a}(\mathbf{x}) = \mathbf{a} + \mathbf{x}$ and the two **multiplication operators** $M_\alpha : X \to X$ with $M_\alpha(\mathbf{x}) = \alpha\mathbf{x}$ for $\alpha \neq 0$ resp. $M_\mathbf{x} : \mathbb{C} \to \langle\{\mathbf{x}\}\rangle$ with $M_\mathbf{x}(\alpha) = \alpha\mathbf{x}$ for $\mathbf{x} \neq \mathbf{0}$ are **homeomorphisms**, i.e. **continuous, bijective** and **open** on account of the continuity of the inverse mappings $T_\mathbf{a}^{-1} = T_{-\mathbf{a}}$ resp. $M_\alpha^{-1} = M_{\frac{1}{\alpha}}$ resp. $M_\mathbf{x}^{-1}(\alpha\mathbf{x}) = \alpha$. Note that these operators are **not linear** since $\alpha T_\mathbf{a}(\mathbf{x}) \neq T_\mathbf{a}(\alpha\mathbf{x})$ resp. $\mathbf{a} + M_\alpha(\mathbf{x}) \neq M_\alpha(\mathbf{a} + \mathbf{x})$ and hence they are not **homomorphisms**. On account of $\mathbf{a} + X \cong X$ the vector topology $\mathcal{O}$ is uniquely determined by the **local basis** $\mathcal{B}(\mathbf{0})$ of the **origin**: $\mathbf{x} + U = T_\mathbf{x}[U] \in \mathcal{U}(\mathbf{x}) \Leftrightarrow U \in \mathcal{U}(\mathbf{0})$. **Hence every study of convergence can be restricted to null sequences** $(\mathbf{x}_n)_{n\in\mathbb{N}}$ with $\lim\limits_{n\to\infty} \mathbf{x}_n = \mathbf{0}$. Every neighbourhood $U \in \mathcal{U}(\mathbf{x})$ can be **symmetrized** by taking $V = U \cap -U$ with $V = -V$. On account of the continuity of the vector addition at the origin with $\mathbf{0} + \mathbf{0} = \mathbf{0}$ for every neighbourhood $U \in \mathcal{U}(\mathbf{0})$ there is a symmetric neighbourhood $V \in \mathcal{U}(\mathbf{0})$ with $V + V \subset U$ resp. a $W \in \mathcal{U}(\mathbf{0})$ with $W + W + W \subset U$ (cf. [11, th. 12.1]). Due to the homeomorphic character of the translation we have $T_\mathbf{x}[U] \in T_\mathbf{x}(\mathcal{U}(\mathbf{0})) = \mathcal{U}(\mathbf{x})$ and $T_\mathbf{x}[V + V] = \mathbf{x} + V + V \subset T_\mathbf{x}[U]$.

1.3 Separation axioms and uniformization

1. In a topological vector space X the **compact** set $K \subset X$ and the **closed** set $A \subset X$ with $A \cap K = \emptyset$ can be separated by an **open neighbourhood** $U \in \mathcal{U}(\mathbf{0})$ of the origin: $(K + U) \cap (A + U) = \emptyset$: For every $\mathbf{x} \in K$ there is a symmetric neighbourhood $U_\mathbf{x} \in \mathcal{U}(\mathbf{0})$ with $\mathbf{x} + U_\mathbf{x} + U_\mathbf{x} + U_\mathbf{x} \subset X \setminus A$, i.e. $\mathbf{x} + U_\mathbf{x} + U_\mathbf{x} \cap A + U_\mathbf{x} = \emptyset$. The finite subcover $K \subset \bigcup_{\mathbf{x}\in E}(\mathbf{x} + U_\mathbf{x})$ with finite $E \subset K$ yields the desired neighbourhood $U = \bigcap_{\mathbf{x}\in E} U_\mathbf{x}$(cf. [11, p. 9.5]).
2. **Topological vector spaces are regular** and can be **uniformized** due to [11, p. 13.5]. The **local basis** $\mathcal{B}(\mathbf{0})$ induces the basis $\mathcal{B}$ of a **translation invariant neighbourhood filter** $\mathcal{U}$ with $(\mathbf{x}; \mathbf{y}) \in B \in \mathcal{B} \Leftrightarrow \mathbf{x} - \mathbf{y} \in B(\mathbf{0}) \in \mathcal{B}(\mathbf{0})$.

1.4 Boundedness and metrization

A subset $A \subset X$ of a topological vector space X is **bounded** iff for every neighbourhood $U \in \mathcal{U}(\mathbf{0})$ there is a $\tau > 0$ with $A \subset \tau U$.

1. Since the multiplication operator$M_\mathbf{x} : K \to \langle\{\mathbf{x}\}\rangle$ is **continuous** for every $\mathbf{x} \in X$ at $\alpha = 0$ there is an $n_\mathbf{x} \in \mathbb{N}$ such that $\frac{\mathbf{x}}{n_\mathbf{x}} \in U$ resp. $\mathbf{x} \in n_\mathbf{x} U$ for every given neighbourhood $U \in \mathcal{U}(\mathbf{0})$. Thus we have $X = \bigcup_{n\geq 1} nU$ and in particular every **compact** subset of X is **bounded**.
2. For a **bounded** neighbourhood $V \in \mathcal{U}(\mathbf{0})$ and every $U \in \mathcal{U}(\mathbf{0})$ there is an $n \in \mathbb{N}$ such that $V \subset nU$ resp. $\frac{1}{n}V \subset U$. Hence the sets $\left(\frac{1}{n}V\right)_{n\geq 1}$ constitute a **countable local basis** and on account of [11, th. 13.3] the space X is **metrizable**.

3. A **translation invariant metric** with $d(\mathbf{x}+\mathbf{y};\mathbf{x}+\mathbf{z}) = d(\mathbf{y};\mathbf{z})$ induces a translation invariant neighbourhood filter by means of $(\mathbf{y};\mathbf{z}) \in U \Leftrightarrow \mathbf{y}-\mathbf{z} \in U(\mathbf{0}) \Leftrightarrow (\mathbf{x}+\mathbf{y};\mathbf{x}+\mathbf{z}) \in U$. The translation invariance is by no means obligatory since the continuity of the vector addition according to $\forall\epsilon > 0 \exists\delta > 0 : (d(\mathbf{y};\mathbf{z}) < \delta \Rightarrow d(\mathbf{x}+\mathbf{y};\mathbf{x}+\mathbf{z}) < \epsilon)$ can also be satisfied **by non translation invariant metrics** allowing corresponding **dilations of space** with $\delta \neq \epsilon$. (cf. [11, th. 14.10])

4. For every **translation invariant metric** on a topological vector space X the **triangle inequality** and induction over $n \geq 1$ yields the estimate $d(n\mathbf{x};\mathbf{0}) \leq d(n\mathbf{x};(n-1)\mathbf{x}) + d((n-1)\mathbf{x};\mathbf{0}) = d(\mathbf{x};\mathbf{0}) + d((n-1)\mathbf{x};\mathbf{0}) \leq n \cdot d(\mathbf{x};\mathbf{0})$. In comparison to this strong condition the **continuity of the scalar multiplication** is much less demanding and asks only for $\forall\epsilon > 0\exists\delta > 0 : d(\delta\mathbf{x};\mathbf{0}) < \epsilon d(\mathbf{x};\mathbf{0})$. The construction of the metric on regular uniform spaces according to [11, th. 13.2] yields the comparable relation $2^{-k-1}d(\mathbf{x};\mathbf{0}) \leq d\left(2^{-k}\mathbf{x};\mathbf{0}\right) \leq 2^{-k}d(\mathbf{x};\mathbf{0})$ but this inequality only holds in the local neighbourhood $B_1(\mathbf{0})$ whereas for $\mathbf{x} \notin B_1(\mathbf{0})$ the construction defines $d(\mathbf{x};\mathbf{0}) := 1$ and hence provides no global structure since far reaching properties like compactness or algebraic operations are not presumed. For **translation invariant** metrics and every $\mathbf{x} \in X$ the ϵ-balls $B_\epsilon(\mathbf{x})$ are of the same size such that the **Cauchy property** of a sequence does not depend on the individual metric (cf. [11, th. 14.10]). In the case of a **norm** we have $d(\alpha\mathbf{x};\mathbf{0}) = \|\alpha\mathbf{x}\| = \alpha\|\mathbf{x}\| = \alpha d(\mathbf{x};\mathbf{0})$.

1.5 Closed subsets

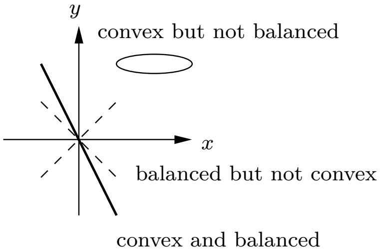
For $\alpha \in K$ and the closures $\overline{A}, \overline{B}$ of subsets $A, B \subset X$ of a topological vector space X the following statements hold:

1. $\overline{A} = \bigcap\{A + U : U \in \mathcal{U}(\mathbf{0})\}$ on account of 1.2. (cf. [11, th. 7.7])
2. $\overline{A} + \overline{B} \subset \overline{A+B}$ since for $\mathbf{a} \in \overline{A}$, $\mathbf{b} \in \overline{B}$ and every neighbourhood $U \in \mathcal{U}(\mathbf{0})$ with symmetric $W \in \mathcal{U}(\mathbf{0})$ such that $W + W \subset U$ we have $(\mathbf{a}+W) \cap A \neq \emptyset$, $(\mathbf{b}+W) \cap B \neq \emptyset$ and hence $(\mathbf{a}+\mathbf{b}+W+W) \cap (A+B) \neq \emptyset$. (cf. [11, th. 2.7])
3. $\alpha\overline{A} = \overline{\alpha A}$ since due to M_α being a homeomorphism we have $\mathcal{U}(\mathbf{0}) = \alpha\mathcal{U}(\mathbf{0}) = \frac{1}{\alpha}\mathcal{U}(\mathbf{0})$.
4. For **bounded** $A \subset X$ the **closure** $\overline{A}$ is still **bounded:** On account of [11, th. 7.7] and 1.3.2 the **closed** $\overline{U} \in \mathcal{U}(\mathbf{0})$ already form a local basis and for each of them there is a $\tau > 0$ with $A \subset \tau\overline{U} = \overline{\tau U} \Rightarrow \overline{A} \subset \overline{\tau U} = \tau\overline{U}$.
5. On account of 1.4.1 the set X is the only **open subspace** of X. For every **vector subspace** A the **topological closure** $\overline{A}$ is also **algebraically closed** since owing to 2. and 3. as well as $\alpha A + \beta A = A$ we have $\alpha\overline{A} + \beta\overline{A} = \overline{\alpha A} + \overline{\beta A} \subset \overline{\alpha A + \beta A} = \overline{A}$.
6. A **proper** vector subspace $A = \langle A_0\rangle$ of $X = \langle X_0\rangle$ with $A_0 \subsetneq X_0$ **is closed** and of **first category** since on account of the continuity of the multiplication with scalars $M_\mathbf{e} : K \to X$ with $M_e(\alpha) = \alpha\mathbf{e}$ for every **basis vector** $\mathbf{e} \in X_0$ at $\alpha = 0$ for every **basis set** $B \in \mathcal{B}(\mathbf{0})$ there exists an $\epsilon > 0$ such that $\tau\mathbf{e} \in B \,\forall\tau \in B_\epsilon(\mathbf{0})$ (cf. 1.7.1). For $\mathbf{e} \notin B_A$ it follows that $B \subsetneq \langle A_0\rangle = A$.
7. The sum $A + B$ of a **closed** subspace A and a **finite dimensional** subspace B is **closed** since according to 1.12 the set $\pi[B]$ is a **closed** subspace of X/A and due to π being **continuous** its inverse image $\pi^{-1}[\pi[B]] = A + B$ is closed in X.

1.6 Convex and balanced sets

A subset $A \subset X$ of a topological vector space X is

- **convex** iff for $\mathbf{x}, \mathbf{y} \in A$ the **line** $\{\tau\mathbf{x} + (1-\tau)\mathbf{y} : 0 \le \tau \le 1\}$ is contained in A such that $\forall \tau \in \mathbb{R} : 0 \le \tau \le 1 \Rightarrow \tau A + (1-\tau)\, A = A$.
- **balanced** iff for $\mathbf{x} \in A$ the **disc** $\{\tau\mathbf{x} : |\tau| \le 1\}$ is contained in A such that $\forall \tau \in \mathbb{C} : |\tau| \le 1 \Rightarrow \tau A \subset A$ resp. $\mathbf{0} \in A \wedge \forall \tau \in \mathbb{C} : 0 < |\tau| \le 1 \Rightarrow A \subset \tau^{-1} A := \sigma A$ resp. $A = \bigcup_{|\tau| \le 1} \tau A = \{\mathbf{0}\} \cup \bigcap_{1 \le |\sigma|} \sigma A$.



Geometrically a **convex** set can be imagined as a an obviously **connected** shape **without holes** or **recesses**. The **convex hull** co $A = \left\{ \sum\limits_{k=1}^{n} \alpha_k \mathbf{x}_k : 0 \le \alpha_k ; \sum\limits_{k=1}^{n} \alpha_k = 1; \mathbf{x}_k \in A; n \in \mathbb{N} \right\}$ of a set A is the **intersection** of all convex sets containing A. A **balanced set** is **symmetrical to the origin** and can be imagined as a **radial symmetric** and **connected** set **without holes** centered around the **origin**. Note that **convexity** relates to **real** $\tau \in \mathbb{R}$ whereas **balance** refers to **complex** $\tau \in \mathbb{C}$. The following properties hold in general:

1. **Convexity** and **balance** obviously transfer to **intersections** and **linear images**.
2. A **convex** $A \subset X$ has a **convex closure** $\overline{A}$ since on account of 1.5.3 and 1.5.5 for every $\tau \in \mathbb{R}$ with $0 \le \tau \le 1$ we have $\tau\overline{A} + (1-\tau)\,\overline{A} = \overline{\tau A + (1-\tau)\, A} = \overline{A}$. The **interior** $\mathring{A}$ is **convex** since from $\mathring{A} \subset A$ and the convexity of A follows that $\tau\mathring{A} + (1-\tau)\,\mathring{A} \subset A$. On account of 1.2 the set $\tau\mathring{A} + (1-\tau)\,\mathring{A}$ is open and hence must be included in $\mathring{A}$.
3. A **balanced** $A \subset X$ has a **balanced closure** $\overline{A}$ and a **balanced interior** $\mathring{A}$ following the same arguments as in 2.

1.7 Properties of the local neighbourhood basis

A topological vector space is **locally convex** iff it has a **convex local basis**.

1. Every topological vector space has a **balanced** local basis.
2. Every **locally convex** topological vector space has a **balanced convex** local basis.
3. In the case of a **balanced** neighbourhood $V \in \mathcal{U}(\mathbf{0})$ the topology can be induced by a metric with **balanced** ϵ-balls $B_\epsilon(\mathbf{0})$ being **convex** if V is **convex**.

Proof:

1. On account of the multiplication with scalars being continuous for every neighbourhood $U \in \mathcal{U}(\mathbf{0})$ there is an open $V \in \mathcal{U}(\mathbf{0})$ and a $\delta > 0$ such that $\tau V \subset U \, \forall \tau \in B_\delta(0)$. Then $W = \bigcup_{|\tau| < \delta} \tau V \subset U$ is a **balanced** and **open** neighbourhood in U.
2. For a **convex** $U \in \mathcal{U}(\mathbf{0})$ let $W \subset U$ be a **balanced** and **open** neighbourhood in U according to 1. and $A = \bigcap_{|\tau|=1} \tau U = \{\mathbf{x} \in X : \tau\mathbf{x} \in U \, \forall \tau \in \mathbb{C} : |\tau| = 1\}$. Since W is **balanced** we have $\bigcap_{|\tau|=1} \tau W = W$ and hence $W \subset A \subset U$. Because W is an open neighbourhood it follows that $\mathring{A} \in \mathcal{U}(x)$. Due to 1.6.1 the set A is **convex** and owing to 1.6.2 this is also true for $\mathring{A}$. For $\sigma \in \mathbb{C}$ we have $\sigma A = \bigcap_{|\tau|=1} |\sigma| \frac{\sigma}{|\sigma|} \tau U = \bigcap_{|\tau|=1} |\sigma| \tau U$. Since τU is **convex** and contains $\mathbf{x}_2 = \mathbf{0}$ for $|\sigma| \le 1$ we conclude that $|\sigma|\tau U \subset \tau U$ and hence $\sigma A \subset A$, i.e. A is **balanced** and so is $\mathring{A}$ due to 1.6.3.
3. Following the construction in [11, th. 13.2] for

$$d(\mathbf{x};\mathbf{y}) := \inf \left\{ \sum_{0 \le i \le n-1} g\left(\mathbf{x}_i; \mathbf{x}_{i+1}\right) : \left(\mathbf{x}_i\right)_{i \in K} \in M_{\mathbf{xy}} \right\} < \epsilon$$

and $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1$ implies $d(\alpha\mathbf{x};\mathbf{0}) < \epsilon$ since with $B_1 = V$ all other $B_k = 2^{-k}V$ are **balanced** too such that with every pair $(\mathbf{x}_i;\mathbf{x}_{i+1}) \in B_k \Leftrightarrow (\mathbf{x}_i - \mathbf{x}_{i+1};\mathbf{0}) \in B_k$ we have $(\alpha(\mathbf{x}_i - \mathbf{x}_{i+1});\mathbf{0}) \in B_k \Leftrightarrow (\alpha\mathbf{x}_i;\alpha\mathbf{x}_{i+1}) \in B_k$ such that $(\alpha\mathbf{x}_i)_{i\in K} \in M_{\mathbf{xy}}$ and $g(\alpha\mathbf{x}_i;\alpha\mathbf{x}_{i+1}) \leq g(\mathbf{x}_i;\mathbf{x}_{i+1})$. The case for **convexity** is proved in the same way.

1.8 Bounded sets

$A \subset X$ is **bounded** iff for every **sequence** $(\mathbf{x}_n)_{n\in\mathbb{N}} \subset A$ and every **null sequence** $(\alpha_n)_{n\in\mathbb{N}} \subset \mathbb{C}$ the sequence $(\alpha_n\mathbf{x}_n)_{n\in\mathbb{N}} \subset X$ is a **null sequence**.

Proof:

$\Rightarrow$: Due to the hypothesis for every $U \in \mathcal{U}(\mathbf{0})$ there is a $\tau \geq 1$ with $\mathbf{x}_n \in \tau U \Leftrightarrow \frac{1}{\tau}\mathbf{x}_n \in U \, \forall n \in \mathbb{N}$ and an $m \in \mathbb{N}$ with $|\alpha_n| < \frac{1}{\tau} \, \forall n \geq m$ such that $|\alpha_n|\,\mathbf{x}_n \in U$. On account of 1.7.1 we can choose a **balanced** U and hence obtain $\alpha_n\mathbf{x}_n \in U$.

$\Leftarrow$: Assuming there is a $U \in \mathcal{U}(\mathbf{0})$ and for every $n \in \mathbb{N}$ an $\mathbf{x}_n \in A \setminus nU \neq \emptyset \Rightarrow \frac{1}{n} \cdot \mathbf{x}_n \notin U$ we obtain a sequence $(\mathbf{x}_n)_{n\in\mathbb{N}} \subset A$ and $\left(\frac{1}{n}\right)_{n\in\mathbb{N}} \subset \mathbb{C}$ with $\lim\limits_{n\to\infty} \frac{1}{n} = 0$ and $\lim\limits_{n\to\infty} \frac{1}{n}\mathbf{x}_n \neq \mathbf{0}$.

1.9 d-bounded sets

A subset $A \subset X$ of a topological vector space X is **d-bounded with reference to the inducing metric** d iff there is a $\tau < \infty$ such that the **diameter** $\delta(A) < \tau$. Every bounded set is d-bounded for every inducing metric d but the converse is not true since the metric itself may be bounded. E.g. for a translation invariant metric d the bounded metric $d' = \frac{d}{1+d}$ (cf. [11, th. 1.7] and 1.4.4) a translation invariant metric inducing the same topology with every single subset being d'-bounded on account of $\delta'(X) = 1$.

1.10 Bounded and continuous linear mappings

A linear mapping $\Lambda : X \to Y$ between complex topological vector spaces X and Y is **bounded** iff the images $\Lambda A \subset Y$ of bounded sets $A \subset X$ are bounded again in Y. For **metrizable** X the following statements are **equivalent**. Without this hypothesis we only have 1. ⇒ 2. ⇒ 3. ⇒ 4.

1. Λ is **continuous** at the **origin**.
2. Λ is **uniformly continuous**.
3. Λ is **bounded**.
4. For every null sequence $(\mathbf{x}_n)_{n\in\mathbb{N}} \subset X$ the image sequence $(\Lambda\mathbf{x}_n)_{n\in\mathbb{N}} \subset Y$ is bounded.
5. For every null sequence $(\mathbf{x}_n)_{n\in\mathbb{N}} \subset X$ the image sequence $(\Lambda\mathbf{x}_n)_{n\in\mathbb{N}} \subset Y$ is again a null sequence.

Proof:

1. ⇒ 2. : For a $\mathbf{x} \in X$ and a neighbourhood $U \in \mathcal{U}(\Lambda\mathbf{x})$ there is a $V \in \mathcal{U}(\mathbf{0})$ with $\Lambda[V] \subset U(\Lambda\mathbf{0})$ and hence $\Lambda[\mathbf{x} + V] = \mathbf{x} + \Lambda[V] \subset \mathbf{x} + U(\Lambda\mathbf{0})$.

2. ⇒ 3. : Follows from 1.8 since for $(\Lambda\mathbf{x}_n)_{n\in\mathbb{N}} \subset \Lambda A$ with **bounded** $A \subset X$ and every null sequence $(\alpha_n)_{n\in\mathbb{N}} \subset \mathbb{C}$ we have $(\mathbf{x}_n)_{n\in\mathbb{N}} \subset A$ and hence $\lim\limits_{n\to\infty} \alpha_n\mathbf{x}_n = 0$ so that on account of Λ being **linear** and **continuous** we conclude $\lim\limits_{n\to\infty} \alpha_n\Lambda\mathbf{x}_n = \lim\limits_{n\to\infty} \Lambda(\alpha_n\mathbf{x}_n) = 0$.

3. ⇒ 4. : Every null sequence is **bounded**.

4. ⇒ 5. : Let d be the inducing metric for the topology on X and $(\mathbf{x}_n)_{n\in\mathbb{N}} \subset X$ a null sequence such that $d(\mathbf{x}_n;0) < 2^{-k}$ for $n \geq n_k$. Choose $\alpha_1 = 1$ and $\alpha_n = \frac{1}{k}$ for $n_k \leq n < n_{k+1}$ such that $\left(\frac{1}{\alpha_n} \cdot \mathbf{x}_n\right)_{n\in\mathbb{N}}$ is still a null sequence. Owing to the hypothesis $\left(\Lambda\left(\frac{1}{\alpha_n} \cdot \mathbf{x}_n\right)\right)_{n\in\mathbb{N}} \subset Y$ is bounded and with 1.8 we infer that $\left(\alpha_n\Lambda\left(\frac{1}{\alpha_n} \cdot \mathbf{x}_n\right)\right)_{n\in\mathbb{N}} = (\Lambda\mathbf{x}_n)_{n\in\mathbb{N}} \subset Y$ is a null sequence.

5. $\Rightarrow$ 1. : Let d be the inducing metric for the topology on X and $U \in \mathcal{U}(0)$ an arbitrary neighbourhood of the origin in Y. Assuming that for every $n \in \mathbb{N}$ there is an $\mathbf{x}_n \in B_{\frac{1}{n}}(0)$ with $\Lambda\mathbf{x}_n \notin U$ we had a null sequence $(\mathbf{x}_n)_{n\in\mathbb{N}} \subset X$ whose image sequence $(\Lambda\mathbf{x}_n)_{n\in\mathbb{N}} \subset Y$ would not be a null sequence.

1.11 Closed and continuous linear functionals

Every **linear functional** $\Lambda : X \to \mathbb{C}$ is **open** since for every $\mathbf{x} \in X$ and every neighbourhood $U \in \mathcal{U}(\mathbf{x})$ due to 1.2 there is an $\epsilon > 0$ with $(1+\lambda)\mathbf{x} \in U$ for all $\lambda \in \mathbb{C}$ and $|\lambda| < \epsilon$ and hence $(1+\lambda)\Lambda\mathbf{x} = \Lambda(1+\lambda)\mathbf{x} \in \Lambda[U]$ resp. $B_{\epsilon\cdot|\Lambda x|}(\Lambda\mathbf{x}) \subset \Lambda[U]$. The following statements concerning **continuity** are equivalent:

1. Λ is **continuous** in the origin.
2. $\ker\Lambda$ is **closed** in X.
3. $\ker\Lambda$ is **not dense** in X if $\Lambda \neq 0$.
4. There is a $V \in \mathcal{U}(\mathbf{0})$ with a **bounded** image $\Lambda V \subset \mathbb{C}$.

Proof:

1. $\Rightarrow$ 2. : $\ker\Lambda = \Lambda^{-1}[\{0\}]$ is **closed** since due to 1.10.2 the linear functional Λ is **uniformly continuous** and $\{0\}$ is **closed** in $\mathbb{C}$.

2. $\Rightarrow$ 3. : $\ker\Lambda$ being **dense** and **closed** would imply $\ker\Lambda = X$.

3. $\Rightarrow$ 4. : Owing to the hypothesis there is an $\mathbf{x} \in X \setminus \ker\Lambda$ and a balanced neighbourhood $V \in \mathcal{U}(\mathbf{0})$ with $\mathbf{x} + V \cap \ker\Lambda = \emptyset$ resp. $-\Lambda\mathbf{x} \notin \Lambda V$. Due to 1.6.1 the image ΛV is **balanced** in $\mathbb{C}$ and hence includes the line $\{\lambda\Lambda\mathbf{y} : |\lambda| \leq 1\}$ for every $\mathbf{y} \in V$. Assuming $\Lambda\mathbf{y} \in \mathbb{C}$ with $|\Lambda\mathbf{y}| \geq |\Lambda\mathbf{x}|$ implies $-\Lambda\mathbf{x} \in \{\lambda\Lambda\mathbf{y} : |\lambda| \leq 1\} \subset \Lambda V$ in contradiction to $-\Lambda\mathbf{x} \notin \Lambda V$ whence follows $\Lambda V \subset B_{|\Lambda\mathbf{x}|}(\mathbf{0})$.

4. $\Rightarrow$ 1. : For $B_\epsilon(0) \subset \mathbb{C}$ and $s = \sup\{|\mathbf{y}| : \mathbf{y} \in \Lambda V\}$ follows $\Lambda\frac{\epsilon}{s}V = \frac{\epsilon}{s}\Lambda V \subset B_\epsilon(\mathbf{0})$, i.e. Λ is **continuous** in the origin.

1.12 Finite dimensional topological vector spaces

On a **finite dimensional** vector subspace $Y \subset X$ with the basis $(\mathbf{b}_j)_{1\leq j\leq n}$ the **isomorphism** $i : \mathbb{C}^n \to Y$ with $i(\mathbf{e}_i) = \mathbf{b}_i$ and $i\left(\sum\limits_{1=1}^{n} x_i\mathbf{e}_i\right) = \sum\limits_{1=1}^{n} x_i\mathbf{b}_i$ on account of 1.11.3 is **continuous** as well as the inverse $i^{-1}Y \to \mathbb{C}^n$ with $i^{-1}(\mathbf{b}_i) = \mathbf{e}_i$ for the basis $(\mathbf{e}_i)_{1\leq i\leq n}$ of $\mathbb{C}^n$. Hence Y is homeomorphic to $\mathbb{C}^n$, **locally compact** with a **countable basis**, **metrizable** und **complete**. Due to [11, th. 14.2.3] and 1.3.2 Y is **closed**.

1.13 Locally compact vector spaces

A topological vector space X is **locally compact** iff it is of **finite dimension**.

Proof: Owing to 1.4.1 the **compact** neighbourhood $V \in \mathcal{U}(\mathbf{0})$ is **bounded** and hence $\left(\frac{1}{m}V\right)_{m\geq 1}$ is a **local basis**. Let $Y = \langle\{\mathbf{b}_1; ..., \mathbf{b}_n\}\rangle$ with $(\mathbf{b}_i)_{1\leq i\leq n} \subset V$ taken from the **finite subcover** $V \subset \bigcup\limits_{i=1}^{n}\left(\mathbf{b}_i + \frac{1}{2}V\right)$. On account of $V \subset Y + \frac{1}{2}V \subset Y + Y + \frac{1}{4}V = Y + \frac{1}{4}V$, etc. we have $V \subset Y + \frac{1}{n}V$ resp. $V \subset \bigcap_{m\geq 1}\left(Y + \frac{1}{m}V\right) = \overline{Y} = Y$ since Y is **closed** in X due to 1.5.6. Because Y is a vector space we conclude that $mV \subset Y\ \forall m \geq 1$ and hence $Y = \bigcup_{m\geq 1} mV = X = \langle\{\mathbf{b}_1; ..., \mathbf{b}_n\}\rangle$.

2 Normalization

2.1 Seminorms

A **seminorm** (cf. [11, th. 1.3]) resp. a **real seminorm** on a topological vector space X is a function $p : X \to [0;\infty[$ with

1. $p(\lambda \cdot \mathbf{x}) = |\lambda| \cdot p(\mathbf{x})$ for all $\lambda \in \mathbb{C}$. (**Absolute homogeneity**)
2. $p(\mathbf{x}+\mathbf{y}) \le p(\mathbf{x}) + p(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in X$. (**sub-additivity** resp. **triangle inequality**)

2.2 Properties

1. $p(\mathbf{0}) = 0$ due to 2.1.2 with $\lambda = 0$.
2. $|p(\mathbf{x}) - p(\mathbf{y})| \le p(\mathbf{x}-\mathbf{y})$ follows from 2.1.2.
3. $p(\mathbf{x}) \ge 0$ follows from 2. with $\mathbf{y} = \mathbf{0}$.
4. $\ker p \subset X$ is a **vector subspace** since for $\alpha, \beta \in \mathbb{C}$ and $\mathbf{x}, \mathbf{y} \in p^{-1}(\{0\})$ we have $0 \le p(\alpha\mathbf{x} + \beta\mathbf{y}) \le |\alpha| p(\mathbf{x}) + |\beta| p(\mathbf{x}) = 0$ and hence $\alpha\mathbf{x} + \beta\mathbf{y} \in p^{-1}(\{0\})$.
5. $B_\epsilon = \{p < \epsilon\}$ is **convex** since for $0 \le t \le 1$ and $\mathbf{x}, \mathbf{y} \in B_\epsilon$ we have $p(t\mathbf{x} + (1-t)\mathbf{y}) \le tp(\mathbf{x}) + (1-t)p(\mathbf{y}) < \epsilon$. B_ϵ is **balanced** since for $0 \le |\tau| \le 1$ and $\mathbf{x}, \mathbf{y} \in B_\epsilon$ we have $p(\tau\mathbf{x}) = |\tau| p(\mathbf{x}) < \epsilon$

2.3 Examples

1. The space $X = L^2(Y;\mathbb{C})$ of **square integrable** complex valued functions on a **measure space** $(Y;\mathcal{A};\mu)$ is provided with a **seminorm** by means of $p(f) := \sqrt{\int \left(f \cdot \overline{f}\right) d\mu}$ with $p(f) = 0 \Leftrightarrow f = 0$ μ-**almost everywhere**.
2. The space $X = C(Y;\mathbb{C})$ of **continuous** complex-valued functions on a **topological space** $(Y;\mathcal{O})$ for any **compact** set $K \subset Y$ is provided with a **seminorm** by means of $p_K(f) := \sup\{|f(\mathbf{x})| : \mathbf{x} \in K\}$ with $p(f) = 0 \Leftrightarrow f(\mathbf{x}) = 0 \, \forall \mathbf{x} \in K$.
3. Every **seminorm** $\|\|$ generates a **pseudometric** by means of $d(\mathbf{x};\mathbf{y}) := \|\mathbf{x}-\mathbf{y}\|$ (cf. 2.8)

2.4 Absorbing sets and Minkowski functionals

1. A **convex** set $A \subset X$ is **absorbing** iff for every $\mathbf{x} \in X$ there is a $\tau > 0$ with $\mathbf{x} \in \tau A$. Every absorbing set contains the **origin** and every **local neighbourhood** is absorbing due to 1.4.1. Hence absorbing sets which are not neighbourhoods of the origin must contain the origin as a **boundary point**.
2. For every **absorbing** set $A \subset X$ the **Minkowski functional** $\mu_A : X \to [0;\infty[$ with $\mu_A(\mathbf{x}) = \inf\{\tau > 0 : \mathbf{x} \in \tau A\}$ is a **real seminorm** and even a (**complex**) **seminorm** if A is **balanced** since for $\mathbf{x}, \mathbf{y} \in X$ and $\lambda \in \mathbb{C}$ we have
    - $\mu_A(\lambda\mathbf{x}) = \lambda \cdot \mu_A(\mathbf{x})$ for $\lambda \ge 0$ and $\mu_A(\mathbf{x}) < \infty$ since A is **absorbing.**
    - $\mu_A(\mathbf{x}+\mathbf{y}) \le \mu_A(\mathbf{x}) + \mu_A(\mathbf{y})$ since $\mathbf{x} \in \tau A \wedge \mathbf{y} \in \sigma A \Leftrightarrow \frac{1}{\tau}\mathbf{x}, \frac{1}{\sigma}\mathbf{y} \in A \Rightarrow \frac{\tau}{\tau+\sigma} \cdot \frac{1}{\tau}\mathbf{x} + \frac{\sigma}{\tau+\sigma} \cdot \frac{1}{\sigma}\mathbf{y} = \frac{1}{\tau+\sigma} \cdot (\mathbf{x}+\mathbf{y}) \in A$ since A is **convex.**
    - $\mu_A(\lambda\mathbf{x}) = |\lambda| \cdot \mu_A(\mathbf{x})$ for $\lambda \in \mathbb{C}$ since $\mu_A(\lambda\mathbf{x}) \le \tau \Leftrightarrow \lambda\mathbf{x} \in \tau A = \frac{\lambda}{|\lambda|}\tau A \Leftrightarrow \mathbf{x} \in \frac{\tau}{|\lambda|}A \Leftrightarrow \mu_A(\mathbf{x}) \le \frac{\tau}{|\lambda|}$ in the case of a **balanced** A.
3. $B_\epsilon = \{p < \epsilon\}$ is **absorbing** with $p = \mu_{B_1}$ since for an arbitrary $\mathbf{x} \in X$ we have $p(\mathbf{x}) < \infty$ and thus $\frac{\epsilon}{p(\mathbf{x})}\mathbf{x} \in B_\epsilon \Leftrightarrow \mathbf{x} \in \frac{p(\mathbf{x})}{\epsilon}B_\epsilon = p(\mathbf{x}) \cdot B_1$ as well as $\mu_1(\mathbf{x}) \le p(\mathbf{x})$. Since on the other hand $\frac{1}{\tau}\mathbf{x} \notin B_1$ for all $0 < \tau < p(\mathbf{x})$ we obtain $\mu_1(\mathbf{x}) \ge p(\mathbf{x})$ and hence the equality.

4. For a **bounded, balanced** and **convex** neighbourhood $U \in \mathcal{U}(\mathbf{0})$ the seminorm μ_U becomes a **norm** since for every $\mathbf{x} \neq \mathbf{0}$ there is an $n \geq 1$ with $\mathbf{x} \notin \frac{1}{n}U$ and hence $\mu_U(\mathbf{x}) \geq \frac{1}{n}$ since U is balanced.

2.5 Construction of a family of separating seminorms from a convex local basis

For a **locally convex** vector space X with the **convex** local basis $\mathcal{B}$ the Minkowski functionals $\{\mu_V : V \in \mathcal{B}\}$ form a **separating** family of **continuous and real** seminorms with $V = \{\mu_V < 1\}$ for every **open** $V \in \mathcal{B}$. A family $\mathcal{P}$ of (real) seminorms is **separating** iff for every $\mathbf{x} \neq \mathbf{0}$ there is a $p \in \mathcal{P}$ with $p(\mathbf{x}) \neq 0$. In the case of a **balanced** $\mathcal{B}$ the Minkowski functionals are even **continuous** and **complex** seminorms.

Proof: According to 2.4.2 every μ_V is a (real) **seminorm**. It is **continuous** since for $\epsilon > 0$ and $\mathbf{x} - \mathbf{y} \in \epsilon V$ we infer from 2.4.2 the relation $|\mu_V(\mathbf{x}) - \mu_V(\mathbf{y})| \leq \mu_V(\mathbf{x} - \mathbf{y}) < \epsilon$. Finally on account of 1.3.1 for every $\mathbf{0} \neq \mathbf{x} \in X$ there is a $V \in \mathcal{B}$ with $\mathbf{x} \notin V$ and hence $\mu_V(\mathbf{x}) \geq 1$, i.e. the μ_V are **separating**. For an **open** $V \in \mathcal{B}$ and $\mathbf{x} \in V$ due to the **continuity** of M_x there is a $t < 1$ with $\frac{1}{t}\mathbf{x} \in V \Leftrightarrow \mathbf{x} \in tV$ and hence $\mu_V(\mathbf{x}) \leq t < 1$. On the other hand if $\mathbf{x} \notin V$ and $\mathbf{x} \in tV \Leftrightarrow \frac{1}{t}\mathbf{x} \in V$ implies $t \geq 1$ and hence $\mu_V(\mathbf{x}) \geq 1$ since V is **convex** and contains the origin $\mathbf{0} \in V$. Thus we have shown $V = \{\mu_V < 1\}$.

2.6 Construction of a bounded, balanced and absorbing local basis from a family of separating seminorms

On a topological vector space X with a separating family $\mathcal{P}$ of seminorms the family $\mathcal{B}$ of all finite intersections of sets $B_{p,n} = \left\{p < \frac{1}{n}\right\}$ for $n \in \mathbb{N}^*$ and $p \in \mathcal{P}$ forms a **bounded**, **balanced and absorbing** local basis for a topology $\mathcal{O}$ such that every $p \in \mathcal{P}$ is **continuous** and an arbitrary set $A \subset X$ is **bounded** iff every $p \in \mathcal{P}$ is **bounded** on A.

Proof: The family $\mathcal{O}$ of all unions of arbitrarily shifted basis sets $\mathbf{x} + B$ with $\mathbf{x} \in X$ and $B \in \mathcal{B}$ obviously forms a **translation invariant topology** on X. The $B_{p,n}$ are **balanced** due to 2.1.1, **convex** on account of 2.1.2 and **absorbing** since $\tau B_{p,n} = \left\{p < \frac{\tau}{n}\right\}$ resp. the boundedness of p. According to 1.6.1 these properties extend to the set $\mathcal{B}$ of its finite intersections.

According to the hypothesis for every $\mathbf{0} \neq \mathbf{x} \in X$ there is a $p \in \mathcal{P}$ and an $n \in \mathbb{N}$ with $p(\mathbf{x}) > \frac{1}{n} > 0$ $\Leftrightarrow np(\mathbf{x}) > 1 \Leftrightarrow \mathbf{x} \notin B_{p,n} \in \mathcal{B}(\mathbf{0}) \Leftrightarrow \mathbf{0} \notin \mathbf{x} + B_{p,n} \in \mathcal{B}(\mathbf{x})$. Hence $\{0\}$ is **closed** in X and on account of the translation invariance of the neighbourhood basis this extends to every other $\mathbf{x} \in X$, i.e. X satisfies T_1.

Since $\mathcal{B}$ is a local basis for every neighbourhood $U \in \mathcal{U}(0)$ there are finitely many seminorms $p_i \in \mathcal{P}$ and $n_i \in \mathbb{N}$ with $1 \leq i \leq m$ for an $m \in \mathbb{N}$ such that $\bigcap\limits_{i=1}^{m} B_{p_i,n_i} \subset U$. With $V = \bigcap\limits_{i=1}^{m} B_{p_i,2n_i}$ due to 2.1.2 we have $V + V \subset U$ and hence the **continuity of the addition**. According to 2.4.3 the above chosen set V is **absorbing** such that for $\mathbf{x} \in X$ there is an $s > 0$ with $\mathbf{x} \in sV$. For $\alpha \in \mathbb{C}$ let $\beta \in B_{\frac{1}{s}}(\alpha) \subset \mathbb{C}$ and $\mathbf{y} \in \frac{s}{1+|\alpha|s}V(\mathbf{x})$. Then we have $\beta\mathbf{y} - \alpha\mathbf{x} = \beta(\mathbf{y} - \mathbf{x}) + (\beta - \alpha)\mathbf{x} \in \frac{|\beta|s}{1+|\alpha|s}V + |\beta - \alpha|sV \subset V + V \subset U$ since V is **balanced** on account of 2.2.5 resp. 1.6.1. Hence we have shown the **continuity of the multiplication**. The **continuity** of the $p \in \mathcal{P}$ follows from the definition of the $B_{p,n}$ and 2.2.2. Since $\mathcal{B}$ is a local basis for a **bounded** $A \subset X$ and $p \in \mathcal{P}$ there is a $k < \infty$ with $A \subset kB_{p,1}$ and therefore $p(\mathbf{x}) < k \forall \mathbf{x} \in A$. Hence p is **bounded** on A. Conversely let p be **bounded** on A and $U \in \mathcal{U}(\mathbf{0})$ an arbitrary local neighbourhood. As above we find $p_i \in \mathcal{P}$ resp. $n_i \in \mathbb{N}$ such that $\bigcap\limits_{i=1}^{m} B_{p_i,n_i} \subset U$ and hence for every $1 \leq i \leq m$ a $M_i < \infty$ with $p_i(\mathbf{x}) < M_i \forall \mathbf{x} \in A$. Then for $n = \max\{M_i n_i : 1 \leq i \leq m\}$ we have $A \subset nU$, hence A is **bounded**. In particular we have found that all $B_{p,n}$ are **bounded**.

2.7 Normalization

A topological vector space $(X, \mathcal{O})$ is **normalizable** iff it has a **bounded** and **convex local basis**.

Proof:

$\Rightarrow$: According to 2.6 the given norm $\|\| : X \to [0;\infty[$ generates a locally convex topology $\mathcal{O}'$ with a **bounded and convex local basis** $\mathcal{B} = \left\{B_{\frac{1}{n}}(\mathbf{0}) : n \geq 1\right\}$. Since the given norm also generates $\mathcal{O}$ the two topologies $\mathcal{O}$ resp. $\mathcal{O}'$ must be identical.

$\Leftarrow$: Due to 1.7.1 there is a **bounded, convex, balanced** and **open** local neighbourhood $U \in \mathcal{U}(\mathbf{0})$ which according to 2.5 resp. 2.4.3 induces a **continuous** norm $\|\| = \mu_U : X \to [0;\infty[$ with $U = \{\|\mathbf{x}\| < 1\} = B_1(\mathbf{0})$ resp. $\frac{1}{n}U = B_{\frac{1}{n}}(\mathbf{0})$. The basis $\mathcal{B} = \left\{B_{\frac{1}{n}}(\mathbf{0}) : n \geq 1\right\}$ generates a topology $\mathcal{O}'$ on X coincidong with the topology $\mathcal{O}$ being induced by the same basis $\left\{\frac{1}{n}U : n \geq 1\right\}$.

2.8 Metrizability

Every **locally convex** vector space $(X, \mathcal{O})$ with a **countable basis** can be generated by a **translation invariant metric.**

Proof:

Owing to 2.5 and 2.4.3 the **open**, **convex** and **countable** basis $\mathcal{U}(\mathbf{0}) = (U_n)_{n\geq 1}$ generates a **separating** family of **continuous** seminorms $(p_n)_{n\geq 1}$ with $p_n = \mu_{U_n}$ such that $U_n = \{p_n < 1\}$ for $n \geq 1$. By means of $d(\mathbf{x};\mathbf{y}) := \max\limits_{n\geq 1} \frac{p_n(\mathbf{x}-\mathbf{y})}{n(1+p_n(\mathbf{x}-\mathbf{y}))}$ we obtain a translation invariant metric: The **positive definiteness** [11, th. 1.2.1] results from the separation property 1.3.2; the **translation invariance** as well as the **symmetry** [11, th. 1.2.2] are obvious and the **triangle inequality** [11, th. 1.2.3] follows from 2.1.2 resp.2.3.3 (cf. [11, th. 1.5]). For $\epsilon > 0$ and $\frac{1}{m+1} < \epsilon < \frac{1}{m}$ the ball $B_\epsilon(\mathbf{0}) = \bigcap\limits_{n=1}^{m} \left\{\frac{p_n(\mathbf{x})}{n(1+p_n(\mathbf{x}))} < \epsilon\right\}$ $= \bigcap\limits_{n=1}^{m} \left\{p_n < \frac{\epsilon n}{1-\epsilon n}\right\}$ is **open** with reference to $\mathcal{O}$ on account of the p_n being continuous such that $\mathcal{O}_d \subset \mathcal{O}$. On the other hand for $\epsilon < \frac{1}{2n}$ we have $B_\epsilon(\mathbf{0}) \subset \{p_n < 1\} = U_n$ and hence $\mathcal{O} \subset \mathcal{O}_d$. Hence the metric d induces the given topology $\mathcal{O}$. The balls $B_\epsilon(\mathbf{0})$ are **balanced** due to 2.1.1, **convex** owing to 2.1.2, **absorbing** because of 2.4.3 and due to 2.6 all of these properties extend to finite intersections.

2.9 Fréchet and Banach spaces

Resuming the preceding two theorems we have obtained the following results for a complex topological vector space $(X;\mathcal{O})$:

1. the following three properties are **equivalent**:
   a) $\mathcal{O}$ has a **bounded** and **convex basis.**
   b) $\mathcal{O}$ is generated by a **norm** $\|\ \|$.
   c) $\mathcal{O}$ is generated by a **translation invariant metric** d with $d(\lambda\mathbf{x};\mathbf{0}) = |\lambda| d(\mathbf{x};\mathbf{0})$.
   A **complete and normed** topological vector space is a **Banach** space.
2. **If** $\mathcal{O}$ has a **countable** and **convex basis** it is generated by a **translation invariant metric**.
   A **complete** topological vector space with a **translation invariant metric** is a **Fréchet** space.

2.10 Quotient spaces

For a topological vector space $(X, \mathcal{O})$ and a subspace $Y \subset X$ the **quotient space** X/Y with reference to the **equivalence relation** $\mathbf{x}R\mathbf{y} \Leftrightarrow \mathbf{x} - \mathbf{y} \in Y$ with the **canonical projection** $\pi : X \to X/Y$ resp. $\pi^{-1}(\pi(\mathbf{x})) = \mathbf{x} + Y$, **addition** $\pi(\mathbf{x}) + \pi(\mathbf{y}) := \pi(\mathbf{x}+\mathbf{y})$ **and multiplication with scalars** $\alpha \cdot \pi(\mathbf{x}) := \pi(\alpha\mathbf{x})$ for $\mathbf{x}, \mathbf{y} \in X$ resp. $\alpha \in \mathbb{C}$ is again a **vector space** over $\mathbb{C}$.

1. The **canonical projection** $\pi : X \to X/Y$ is **linear, continuous** and **open**.
2. For a **local basis** $\mathcal{B}$ in X the **canonical image** $\pi(\mathcal{B})$ is a **local basi**s for the **quotient topology** $\mathcal{O}_Y = \{\pi[O] \subset X \setminus Y : \pi^{-1}[\pi[O]] = O + Y \in \mathcal{O}\}$.
3. For a **closed** Y the vector space X/Y with the **quotient topology** $\mathcal{O}_Y$ is a **topological vector space.**
4. For a **bounded** resp. **convex** local basis $\mathcal{B}$ the same is true for its canonical image $\pi(\mathcal{B})$ on X/Y.
5. For a **Fréchet** resp. **Banach space** X the same is true for X/Y.
6. For a **seminorm** p on X the map $\|\| : X/\ker p \to [0;\infty[$ with $\|\pi(\mathbf{x})\| = p(\mathbf{x})$ is a **norm** on $X/\ker p$.

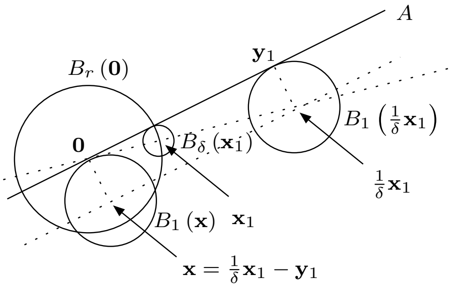
Proof:

1. π is obviously **linear, continuous** (cf. [11, th. 4.5]) and **open** since for open $O \in \mathcal{O}$ we have $\pi^{-1}[\pi[O]] = O + Y \in \mathcal{O}$.
2. Since π is **continuous** for every neighbourhood U of a point $\pi(\mathbf{x}) \in X/Y$ there is an **open** $B \in \mathcal{B}(\mathbf{0})$ with $\mathbf{x} + B \in \mathcal{B}(\mathbf{x})$ and $\pi[\mathbf{x} + B] \subset U$. Since π is linear we have $\pi[\mathbf{x} + B] = \pi(\mathbf{x}) + \pi[B]$ with **open** $\pi[B]$ since π is open. Hence $\pi(\mathbf{x}) + \pi(\mathcal{B})$ is a neighbourhood basis for $\pi(\mathbf{x})$and $\pi(\mathcal{B})$ is a local basis for $(X/Y; \mathcal{O}_Y)$.
3. Due to 2. for every local neighbourhood $U \in \mathcal{U}(\pi(\mathbf{0}))$ there is a local neighbourhood $V \in \mathcal{U}(\mathbf{0})$ with $V + V \subset \pi^{-1}[U]$ and hence $\pi[V] + \pi[V] = \pi[V + V] \subset U$. Thus the **addition** on $X/Y \times X/Y$ is **continuous**. As above and due to 2. for $U \in \mathcal{U}(\pi(\mathbf{0}))$ there is a local neighbourhood $V \in \mathcal{U}(\mathbf{0})$ with $V \subset \pi^{-1}[U]$. Since the **multiplication is continuous** on $\mathbb{C} \times X$ for $\alpha \in \mathbb{C}$ and $\mathbf{x} \in X$ there is an $\epsilon > 0$ and a $W \in \mathcal{U}(\mathbf{0})$ with $\beta\mathbf{y} \in \mathbf{x} + V \; \forall \mathbf{y} \in \mathbf{x} + W; \beta \in B_\epsilon(\alpha)$ such that $\beta \cdot \pi(\mathbf{y}) = \pi(\beta\mathbf{y}) \in \pi(\mathbf{x}) + \pi[V] \subset \pi(\mathbf{x}) + U$. Thus the **multiplication** on $\mathbb{C} \times X/Y$ is **continuous**. Since Y is closed for $\pi(\mathbf{x}) \in X/Y$ the inverse image $\pi^{-1}(\pi(\mathbf{x})) = \mathbf{x} + Y$ is closed in X. Due to 1. $\pi(\mathbf{x})$ is closed in X/Y and hence $\mathcal{O}_Y$ satisfies T_1.
4. This is a simple consequence of the linear character of π: For convex $W \subset X$ its image $\pi[W]$ is convex too since for $\pi(\mathbf{x}), \pi(\mathbf{y}) \in \pi[W]$ and $0 \leq t \leq 1$ the linear combination $t \cdot \pi(\mathbf{x}) + (1 - t) \cdot \pi(\mathbf{y}) = \pi(t\mathbf{x} + (1 - t)\mathbf{y}) \in \pi[W]$. The bounded character of W analogously extends to $\pi[W]$.
5. For a translation invariant metric d on X the mapping $\delta : X/Y \times X/Y \to [0;\infty[$ given by $\delta(\pi(\mathbf{x}); \pi(\mathbf{y})) = \inf\{d(\mathbf{x} + \mathbf{u}; \mathbf{y} + \mathbf{v}) : \mathbf{u}, \mathbf{v} \in Y\} = \inf\{d(\mathbf{x} - \mathbf{y}; \mathbf{z}) : \mathbf{z} \in Y\}$ defines a **translation invariant metric** on X/Y inducing $\mathcal{O}_Y$ since $\pi(\mathbf{x}) \in B_\epsilon(\pi(\mathbf{0})) \Leftrightarrow \delta(\pi(\mathbf{x}); \pi(\mathbf{0})) < \epsilon \Leftrightarrow \inf\{d(\mathbf{x}; \mathbf{z}) : \mathbf{z} \in Y\} < \epsilon \Leftrightarrow \pi(\mathbf{x}) \in \pi(B_\epsilon(\mathbf{0}))$. According to the construction of δ for a **Cauchy sequence** $(\pi(\mathbf{x}_n))_{n \in \mathbb{N}}$ there is a sequence $(\mathbf{u}_n)_{n \in \mathbb{N}} \subset Y$ such that $(\mathbf{x}_n + \mathbf{u}_n)_{n \in \mathbb{N}} \subset X$ converges to an $\mathbf{x} \in X$. Hence the **continuous image** $\pi(\mathbf{x}_n + \mathbf{u}_n)_{n \in \mathbb{N}} = \pi(\mathbf{x}_n)_{n \in \mathbb{N}}$ converges to $\pi(\mathbf{x})$. For a **norm** $\|\|$ on X we obtain a **norm** by means of $\|\pi(\mathbf{x})\| := \inf\{\|\mathbf{x} - \mathbf{u}\| : \mathbf{u} \in Y\}$ on X/Y inducing the given **metric** $\delta(\pi(\mathbf{x}); \pi(\mathbf{y})) := \|\pi(\mathbf{x}) - \pi(\mathbf{y})\|$ and hence the **quotient topology** $\mathcal{O}_Y$.
6. On account of $\pi(\mathbf{x}) = \pi(\mathbf{y}) \Rightarrow 0 \leq |p(\mathbf{x}) - p(\mathbf{y})| \leq p(\mathbf{x} - \mathbf{y}) = 0 \Rightarrow \|\pi(\mathbf{x})\| = \|\pi(\mathbf{y})\|$ the map $\|\|$ is well defined and obviously a norm.

2.11 Closed subspaces

For every **closed** subspace $A \subsetneq X$ of a **normed** space X and every $r > 1$ there exists an $\mathbf{x} \in X \setminus A$ with $\|\mathbf{x}\| < r$ and $d(\mathbf{x}; A) = \inf\limits_{y \in A} \|\mathbf{x} - \mathbf{y}\| \geq 1$.

Proof: Due to the hypothesis there is an $\mathbf{x}_1 \in B_r(\mathbf{0}) \setminus A$ with $\delta = d(\mathbf{x}_1; A) > 0$ resp. $d\left(\frac{1}{\delta}\mathbf{x}_1; A\right) = 1$. Hence there exists a $\mathbf{y}_1 \in A$ with $1 \leq \|\mathbf{x}_1 - \mathbf{y}_1\| < r$ and the assertion holds with $\mathbf{x} = \mathbf{x}_1 - \mathbf{y}_1$.



2.12 The Banach spaces $\mathrm{L^p}(\mu)$

According to [8, th. 7.5] the terms

$$\|f\|_p = \left(\int |f|^p d\mu\right)^{\frac{1}{p}}$$

for $p < \infty$ resp.

$$\|f\|_\infty = \inf\{0 < \alpha < \infty : \mu(\{|f| > \alpha\}) = 0\}$$

define a seminorm on the vector space $\mathcal{L}^p(\mu) := \left\{f : (X;\mathcal{A};\mu) \to \left(\overline{\mathbb{C}};\overline{\mathcal{B}};\lambda\right) : \|f\|_p :< \infty\right\}$ of the **complex valued Borel measurable** functions on the **measure space** $(X;\mathcal{A};\mu)$. Special cases are the **integrable** functions $\mathcal{L}^1(\mu)$ and the μ **almost everywhere bounded** functions $\mathcal{L}^\infty(\mu)$ with the **supremum norm** $\|\|_\infty$. According to 2.10.6 we obtain a **norm** by moving on to the **quotient space** $\mathrm{L^p}(\mu) := \mathcal{L}^p/\ker\|\|$. For $1 \le p \le \infty$ all $f \in \mathcal{L}^p$ are μ almost everywhere **finite** (but except for $p = \infty$ not necessarily **bounded**). Since in every equivalence class we can find representants with only finite values the study of integrable functions w.l.o.g. can be restricted to the range $\mathbb{C}$. Due to [8, th. 7.7] all $\mathrm{L^p}(\mu)$ are **complete** and hence **Banach** spaces.

2.13 The Fréchet space $(\mathcal{C}(\Omega;\mathbb{C});\mathcal{O}_c)$

Due to [11, th. 18.7.5 and th. 18.8] for an open set $\Omega \subset \mathbb{C}^n$ on the vector space $\mathcal{C}(\Omega;\mathbb{C})$ of the **continuous complex-valued** functions the **compact-open topology** $\mathcal{O}_c$ is **metrizable** and coincides with the **topology of compact convergence**. For compact $K_m \subset \mathbb{C}^n$ with $m \ge 1$, w.l.o.g. $K_m \subset K_{m+1}$ and $\bigcup_{m\ge1} K_m = \mathbb{C}^n$ the **separating** family of **seminorms** $\|\,\|_{K_m}$ defined by $\|f\|_{K_m} = \sup\{|f(\mathbf{x})| : \mathbf{x} \in K_m\}$ resp. the **pseudometrics** d_m with $d_m(f;g) = \|f-g\|_{K_m}$ induces a **metric** D with $D(f;g) := \max\limits_{m\ge1}\frac{d_m(f;g)}{m(1+d_m(f;g))}$ for $f;g \in \mathcal{C}(\Omega;\mathbb{C})$. According to 2.6 the neighbourhoods $B_m = \left\{\|f\|_{K_m} < \frac{1}{m} : f \in \mathcal{C}(\Omega;\mathbb{C})\right\}$ form a **convex local basis** for $\mathcal{O}_c$ and due to [11, th. 18.4] the space $(\mathcal{C}(\Omega;\mathbb{C});\mathcal{O}_c)$ is **complete**. It is **not normalizable** since every basis set $W(K;U)$ and especially all B_m contain functions with unbounded arbitrary values outside of K resp. K_m.

2.14 The Fréchet space $(H(\Omega;\mathbb{C});\mathcal{O}_c) \subset (\mathcal{C}(\Omega;\mathbb{C});\mathcal{O}_c)$

Due to [6, th. 2.7] for an **open** set $\Omega \subset \mathbb{C}$ the vector subspace $(H(\Omega;\mathbb{C});\mathcal{O}_c)$ of the **holomorphic functions** is **closed** in $(\mathcal{C}(\Omega;\mathbb{C});\mathcal{O}_c)$ relative to **compact convergence** and therefore **complete** on account of [11, th.14.2.2]. Furthermore $(H(\Omega;\mathbb{C});\mathcal{O}_c)$ has the **Heine-Borel property**, i.e. every **bounded** and **closed** subset is **compact**: According to **Montel's theorem** [6, th. 2.11] every **bounded** subset $\mathcal{F} \subset (H(\Omega;\mathbb{C});\mathcal{O}_c)$ is **normal**, i.e. every sequence in $\mathcal{F}$ has a subsequence which converges on **compact** sets to a limit function which is also an element of the **closed** set $\mathcal{F}$ such that $\mathcal{F}$ is **sequentially compact** and hence **compact** due to [11, th. 10.12].

3 Complete spaces

3.1 The Banach-Steinhaus theorem

A family Γ of **continuous** linear mappings $\Lambda : X \to Y$ from a **Banach** space X to a **normed** vector space Y is **equicontinuous** and **uniformly bounded** if the set B of all points $\mathbf{x} \in X$ with **bounded ranges** $\Gamma(\mathbf{x}) = \{\Lambda\mathbf{x} : \Lambda \in \Gamma\}$ is of **second category** in X.

Proof: For an arbitrary neighbourhood U of the origin 0_Y in Y and a balanced neighbourhood $W \in \mathcal{U}(0_Y)$ with $\overline{W} + \overline{W} \subset U$ let $E = \bigcap_{\Lambda\in\Gamma}\Lambda^{-1}\left(\overline{W}\right)$. For every $\mathbf{x} \in B$ there is a $n \ge 1$ with $\Gamma(\mathbf{x}) \subset nW$ resp. $\mathbf{x} \in nE$ and hence $B \subset \bigcup_{n\ge1} nE$. Due to Baire's theorem [11, th. 16.4.1] the Banach space X is **Baire** and hence according to the **category theorem** [11, th. 16.3.1] at least one nE is of second category. Since the multiplication operator $M_n : X \to X$ is a homeomorphism with $M_n \left[\bigcup_{n\in\mathbb{N}} \overline{\mathring{A_n}}\right] = \left[\bigcup_{n\in\mathbb{N}} \overline{M\left(\mathring{A_n}\right)}\right]$ the set E itself must be of second category. Since Λ is continuous E is **closed** and has an interior point $\mathbf{x} \in \mathbf{x} + V \subset \mathring{E}$ with a neighbourhood $V \in \mathcal{U}(\mathbf{0})$ and $\Lambda V \subset \Lambda E - \Lambda \mathbf{x} \subset \overline{W} + \overline{W} \subset U$ for every $\Lambda \in \Gamma$. Hence we have shown that Γ is **equicontinuous**.

For an arbitrary bounded set $A \subset X$ let U be an arbitrary neighbourhood of $\mathbf{0}_Y$ in Y with $V \in \mathcal{U}(\mathbf{0}_Y)$ such that $\Lambda V \subset U$ for all $\Lambda \in \Gamma$. Then there is a $n \geq 1$ with $A \subset nV$ and hence $\Lambda A \subset \Lambda nV = n\Lambda V \subset nU$ for all $\Lambda \in \Gamma$, i.e. $\bigcup_{\Lambda\in\Gamma} \Lambda A \subset nU$. Thus we have shown that Γ is **uniformly bounded** on Y. In particular with $A = \{\mathbf{x}\}$ every $\Gamma(\mathbf{x})$ is bounded in Y and hence $B = X$.

3.2 Convergence of continuous linear mappings

For a sequence $(\Lambda_n)_{n\in\mathbb{N}}$ of **continuous** linear mappings $\Lambda_n : X \to Y$ on a **Banach space** X to a **normed space** Y we have:

1. If the set C of all points $\mathbf{x} \in X$ with $(\Lambda_n \mathbf{x})_{n\in\mathbb{N}}$ being a **Cauchy sequence** is of **second category** in X we have $C = X$.
2. If the set L of all points $\mathbf{x} \in X$ having a **limit** $\Lambda\mathbf{x} := \lim\limits_{n\to\infty} \Lambda_n \mathbf{x}$ is of **second category** in X we have $L = X$ and Λ is **continuous**.

Proof:

1. On account of the addition and the multiplication with scalars being continuous the set C is a vector subspace of X and owing to 1.5.5 this property extends to $\overline{C}$. Since C and hence $\overline{C}$ are of **second category** it follows from 1.5.6 that $\overline{C} = X$. For an $\mathbf{x} \in X$ and an arbitrary neighbourhood $U \in \mathcal{U}(\mathbf{0}_Y)$ in Y let $V \in \mathcal{U}(\mathbf{0}_Y)$ with $V + V + V \subset U$. Since in a **metric** space every **Cauchy sequence** is **bounded** the set $B = \{\mathbf{x} \in X : \exists N \in \mathbb{N} : \|\Lambda_n \mathbf{x}\| \leq N \, \forall n \in \mathbb{N}\} \subset X$ with $C \subset B \subset \overline{C} = X$ also is of **second category** such that due to **Banach-Steinhaus** 3.1 the family $(\Lambda_n)_{n\in\mathbb{N}}$ is **equicontinuous** and there is a local neighbourhood $W \in \mathcal{U}(\mathbf{0}_X)$ in X with $\mathbf{x} - \mathbf{y} \in W \Rightarrow \Lambda_n \mathbf{x} - \Lambda_n \mathbf{y} \subset V \; \forall n \in \mathbb{N}$. Since C is **dense** in X there actually exists such an $\mathbf{y} \in C$ with $\mathbf{x} - \mathbf{y} \in W$ and due to the hypothesis an $n_0 \in \mathbb{N}$ with $\Lambda_n \mathbf{y} - \Lambda_m \mathbf{y} \in V \; \forall n, m \geq n_0$ we conclude that $\Lambda_n \mathbf{x} - \Lambda_m \mathbf{x} = \Lambda_n \mathbf{x} - \Lambda_n \mathbf{y} + \Lambda_n \mathbf{y} - \Lambda_m \mathbf{y} + \Lambda_m \mathbf{y} - \Lambda_m \mathbf{x} \in W + W + W \subset U$. Hence $(\Lambda_n \mathbf{x})_{n\in\mathbb{N}}$ is a **Cauchy sequence**.
2. As in 1. we prove that $L = X$. For $U \in \mathcal{U}(\mathbf{0}_Y)$ and $W \in \mathcal{U}(\mathbf{0}_X)$ as in 1. we have $\Lambda_n [W] \subset U \, \forall n \in \mathbb{N}$ and due to the hypothesis $\Lambda [W] \subset \overline{U}$.

3.3 Uniform boundedness on compact and convex subsets

Let Γ be a family of continuous linear mappings $\Lambda : K \to Y$ on a **compact** and **convex** subset $K \subset X$ of a topological vector space X into a topological vector space Y. Then if the set of all values $\Gamma(x) = \{\Lambda\mathbf{x} : \Lambda \in \Gamma\}$ is **pointwise bounded**, i.e. for every $\mathbf{x} \in K$ there is a $B_{\mathbf{x}} \subset Y$ with $\Lambda\mathbf{x} \subset B_{\mathbf{x}}$ for every $\Lambda \in \Gamma$ it is already **uniformly bounded** on K, i.e. there is a bounded $B \subset Y$ with $\Lambda[K] \subset B$ for every $\Lambda \in \Gamma$.

Proof: For an arbitrary **balanced** neighbourhood $U \in \mathcal{U}(\mathbf{0}_Y)$ choose a balanced $V \in \mathcal{U}(\mathbf{0}_Y)$ with $\overline{V} + \overline{V} \subset U$ and $E := \bigcap_{\Lambda\in\Gamma} \Lambda^{-1}\left(\overline{V}\right)$. Since for every $\mathbf{x} \in K$ there is an $n \geq 1$ with $\Gamma(\mathbf{x}) \subset nV$ we have $K \subset \bigcup_{n\geq 1} nE$. On account of **Baire's theorem** (cf. [11, th. 16.4.2]) the **compact** set K is **Baire** and since E is **closed** ist due to the **category theorem** (cf. [11, th. 16.3.1]) there is a $n \geq 1$, a $\mathbf{x}_0 \in K \cap nE$ and a **balanced** neighbourhood $W \in \mathcal{U}(0_X)$ with $K \cap (\mathbf{x}_0 + W) \subset nE$. Owing to 1.4.1 there is an $m \geq 1$ with $K \subset \mathbf{x}_0 + mW$. Since K is **convex** for any $\mathbf{x} \in K$ the point $z = \left(1 - \frac{1}{m}\right)\mathbf{x}_0 + \frac{1}{m}\mathbf{x} \in K$ and hence $\mathbf{z} - \mathbf{x}_0 = \frac{1}{m}(\mathbf{x}_0 - \mathbf{x}) \in W$ resp. $\mathbf{z} \in nE$. Since $\Lambda nE \subset \overline{V}$ and $\mathbf{x} = mz + (1-m)\mathbf{x}_0$ we have $\Lambda\mathbf{x} \subset mn\overline{V} + (1-m)n\overline{V} \subset mn\left(\overline{V} + \overline{V}\right) mnU$ and hence the **uniform boundedness** of $\Lambda[K]$.

3.4 The open mapping theorem

A **continuous** and **linear** mapping $\Lambda : X \to Y$ on a **Fréchet space** X into a topological vector space Y is **surjective** and **open** if the **image** ΛX is of **second category** in Y. Moreover in that case Y is a **Fréchet space**.

Proof: For any local basis set $B_n := B_{2^{-n}}\left(\mathbf{0}_X\right)$ with $n \geq 2$ we have $\Lambda X = \bigcup_{m \geq 1} m\Lambda B_n$. Owing to **Baire's theorem** [11, th. 16.4.1] resp. the **category theorem** [11, th.16.3.1] there is a $m \geq 1$ with $\overset{\circ}{\overline{m\Lambda B_n}} \neq \emptyset$ and since $x \to m\mathbf{x}$ is a **homeomorphism** we also have $\overset{\circ}{\overline{\Lambda B_n}} \neq \emptyset$. For an arbitrary $\mathbf{y}_0 \in \overline{\Lambda B_n}$ follows $\left(\mathbf{y}_0 + \overline{\Lambda B_{n+1}}\right) \cap \Lambda B_n \neq \emptyset$, i.e. there is an $\mathbf{x}_0 \in B_n$ with $\mathbf{y}_1 := \mathbf{y}_0 - \Lambda \mathbf{x}_0 \in \overline{\Lambda B_{n+1}}$. Subsequently we can find $\mathbf{x}_k \in B_{n+k}$ with $\mathbf{y}_{k+1} := \mathbf{y}_k - \Lambda \mathbf{x}_k \in \overline{\Lambda B_{n+k+1}}$ such that $\lim\limits_{k\to\infty} \mathbf{y}_k \in \bigcap_{k\in\mathbb{N}} \overline{\Lambda B_{n+k}} = \mathbf{0}_Y$ on account of Λ being **continuous** and Y being **regular** (cf. 1.3.2). Due to the definition of the B_n the partial sums $\left(\sum\limits_{k=0}^{u} \mathbf{x}_k\right)_{u\geq 0}$ form a **Cauchy sequence** converging to na $\mathbf{x} := \sum\limits_{k=0}^{\infty} \mathbf{x}_k \in B_{n-1}$ since X is **complete**. Again due to Λ being **continuous** the partial sums $\left(\sum\limits_{k=0}^{u} \Lambda\mathbf{x}_k\right)_{u\geq 0} = \left(\sum\limits_{k=0}^{u} \left(\mathbf{y}_k - \mathbf{y}_{k+1}\right)\right)_{u\geq 0} = \mathbf{y}_0 - \left(\mathbf{y}_{u+1}\right)_{u\geq 0}$ converge to $\Lambda\mathbf{x} = \mathbf{y}_0 - \lim\limits_{u\to\infty} \mathbf{y}_{u+1} = \mathbf{y}_0 \in \Lambda B_{n-1}$. Thus we have shown that $\overline{\Lambda B_n} \subset \Lambda B_{n-1}$ and from $\overset{\circ}{\overline{\Lambda B_n}} \neq \emptyset$ we conclude that $\Lambda \overset{\circ}{B}_{n-1} \neq \emptyset$. Hence for every open $O \subset X$ and $\Lambda\mathbf{x} \in \Lambda O$ we find an $n \in \mathbb{N}$ with $\mathbf{x} \in \mathbf{x} + B_n \subset O$ and $\Lambda\mathbf{x} \in \Lambda\mathbf{x} + \Lambda \overset{\circ}{B}_n \subset \Lambda O$ and hence ΛO is **open**. Due to 1.5.5 the **open** mapping Λ must be **surjective** but not necessarily **injective**.

In order to prove the **Fréchet properties** of Y we use 2.10.5 and show that $f : X \setminus \ker\Lambda \to Y$ with $f \circ \pi = \Lambda$ is a **homeomorphism**: Due to [11, p. 4.5] the **continuity** extends from Λ to f. On the other hand for open $O \subset X \setminus \ker\Lambda$ the inverse iumage $\pi^{-1}[O]$ is open in X and hence $f[0] = f\left[\pi\left[\pi^{-1}[O]\right]\right] = \Lambda\left[\pi^{-1}[O]\right]$ is open in Y.

3.5 Surjective and bounded linear maps

1. A **continuous** and **linear map** between **Fréchet spaces** is **surjective** iff it is **open**.
2. A **continuous** and **linear bijection** Λ between **Banach spaces** is **bounded** above and below, i.e. there are bounds $0 < a < b < \infty$ with $a\|\mathbf{x}\| \leq \|\Lambda\mathbf{x}\| \leq b\|\mathbf{x}\|\ \forall \mathbf{x} \in X$.

Proof:

1. $\Rightarrow$: According to **Baire's theorem** [11, th. 16.4.1] and [11, th. 16.2.4] Fréchet spaces are of **second category**. Hence in the case of $\Lambda X = Y$ the hypothesis for the **open mapping theorem** 3.4 is satisfied. $\Leftarrow$: Due to 1.5.5 in the case of ΛX being open it follows that $\Lambda X = Y$.
2. Owing to 1.10.3 and 3.4 the maps Λ and $\Lambda^{-1}$are bounded with $\|\Lambda\mathbf{x}\| = \|\mathbf{x}\| \cdot \left\|\Lambda\frac{\mathbf{x}}{\|\mathbf{x}\|}\right\| \leq \|\mathbf{x}\| \cdot b$ and $\|\mathbf{x}\| = \left\|\Lambda^{-1}\Lambda\mathbf{x}\right\| = \|\Lambda\mathbf{x}\| \cdot \left\|\Lambda^{-1}\frac{\Lambda\mathbf{x}}{\|\Lambda\mathbf{x}\|}\right\| \geq \|\Lambda\mathbf{x}\| \cdot \frac{1}{a}$

3.6 The closed graph theorem

A linear mapping $\Lambda : X \to Y$ between **Fréchet spaces** X and Y is **continuous** iff its **graph** $G = \{(\mathbf{x}; \Lambda\mathbf{x}) : \mathbf{x} \in X\}$ is **closed** in $X \times Y$.

Proof:

$\Rightarrow$: follows from [11, th. 7.12] for **any** mapping $\Lambda : X \to Y$ between **topological spaces** X and Y.

$\Leftarrow$: The **product** $X \times Y$ is again a vector space with the **componentwise** addition resp. multiplication according to $\alpha\left(\mathbf{x}_1; \mathbf{y}_1\right) + \beta\left(\mathbf{x}_2; \mathbf{x}_2\right) := \left(\alpha\mathbf{x}_1 + \beta\mathbf{x}_2; \alpha\mathbf{y}_1 + \beta\mathbf{x}_2\right)$. The complete and translation invariant metrics d_X and d_Y generate the **metric** $d\left(\left(\mathbf{x}_1; \mathbf{y}_1\right); \left(\mathbf{x}_2; \mathbf{y}_2\right)\right) := d_X\left(\mathbf{x}_1; \mathbf{x}_2\right) + d_Y\left(\mathbf{y}_1; \mathbf{y}_2\right)$ on $X \times Y$

with corresponding properties inducing the **product topology**. Since Λ is **linear** the graph G is a **vector subspace** $X \times Y$ and hence a **Fréchet space** due to [11, th. 14.2.2]. The **projection** $\pi_1 : G \to X$ with $\pi_1 ((\mathbf{x}; \Lambda\mathbf{x})) = \mathbf{x}$ is a **continuous** and **linear bijection** between **Fréchet** spaces; due to 3.4 its inverse $\pi_1^{-1} : G \to X$ is **continuous** too. Likewise the **projection** $\pi_2 : X \times Y \to Y$ with $\pi_2 (\mathbf{x}; \mathbf{y}) = \mathbf{y}$ is **continuous** and so is the **composition** $\Lambda = \pi_2 \circ \pi_1^{-1}$.

3.7 Continuous bilinear maps

For a **Fréchet space** X and arbitrary topological spaces Y resp. Z the **bilinear map** $B : X \times Y \to Z$ is **sequentially continuous** on the product $X \times Y$ iff it is **separately sequentially continuous** on the components X and Y, i.e. the **cuts** $B_x : Y \to Z$ resp. $B_y : X \to Z$ are sequentially continuous for every fixed $x \in X$ resp. $y \in Y$.

Proof:

$\Rightarrow$: directly follows from the definition of the product topology.

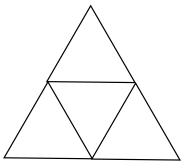
$\Leftarrow$: For sequences $(x_n)_{n \geq 1} \subset X$ resp. $(y_n)_{n \geq 1} \subset Y$ converging to $x_0 = \lim_{n \to \infty} x_n \in X$ resp. $y_0 = \lim_{n \to \infty} y_n \in Y$ for each $x \in X$ the set $(B(x; y_n))_{n \geq 1} \subset Z$ is **bounded** and every $B_{y_n} : X \to Z$ is **continuous** the **Banach-Steinhaus theorem** 3.2.2 implies that the family $(B_{y_n})_{n \geq 1} \subset C(X; Z)$ is **equicontinuous**. Hence for every neighbourhood $W \in \mathcal{U}(0)$ in Z and $U \in \mathcal{U}(0)$ with $U + U \subset W$ there is a further neighbourhood $V \in \mathcal{U}(0)$ such that $B_{y_n}[V] \subset U$ for every $n \geq 1$. Then for sufficiently large n we have $x_n \in x_0 + V$ such that $B_{y_n}(x_n - x_0) \in U$ and due to the continuity of $B_{x_0} : Y \to Z$ holds $B(x_0; y_n - y_0) \in U$. Hence we obtain $B(x_n; y_n) - B(x_0; y_0) = B(x_n - x_0; y_n) + B(x_0; y_n - y_0) \in U + U \subset W$.

3.8 The Sperner decomosition

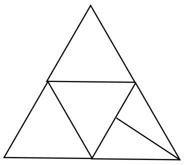
An n-**simplex** $S = \mathrm{co}(A)$ is the **convex hull** of $n+1$ **affinely independent** elements $A = \{\mathbf{x}_0; ...; \mathbf{x}_n\}$ of a **vector space** X with $\dim(X) \geq n$, i.e. the vectors $\mathbf{x}_k - \mathbf{x}_0$ are **linearly independent**.

A **regular decomposition** of S is a finite family $\{S_1; ...; S_m\}$ of n-simplices whose **union** is $\bigcup_{i=1}^{m} S_i = S$ and whose **pairwise intersections** are either empty $S_i \cap S_j = \emptyset$ or there is an $\mathbf{x}_{ik}$ with $S_i \cap S_j = A_i \cap A_j = \{\mathbf{x}_{ik}\}$, i.e. a **vertex** or there are two $\mathbf{x}_{ik}, \mathbf{x}_{jl}$ with $A_i \Delta A_j = \{\mathbf{x}_{ik}, \mathbf{x}_{jl}\}$ and $S_i \cap S_j = \mathrm{co}(A_i \cap A_j)$, i.e. a **boundary segment** consisting of an $n-1$-simplex.

A **Sperner labeling** of a regular decomposition $\{S_1; ...; S_m\}$ with $S_i = \mathrm{co}(A_i)$ of an n-simplex $S = \mathrm{co}(A)$ with $A = \{\mathbf{x}_0; ...; \mathbf{x}_n\}$ is a function $s_i : A_i \to \{0; ...; n\}$ such that for a vertex $\mathbf{x}_{ik} \in \mathrm{co}(A \setminus \{\mathbf{x}_j\})$ only labels from that boundary are used, i.e. $s_i(\mathbf{x}_{ik}) \in \{0; ...; n\} \setminus \{j\}$. The labeling is **complete**, iff s_i is **injective**.



regular



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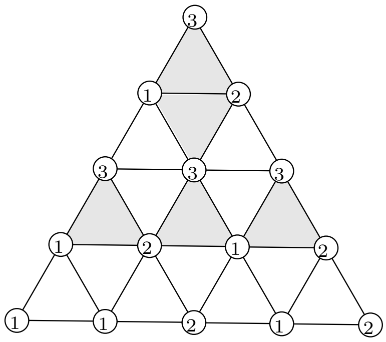


A Sperner-labeled regular decomposition

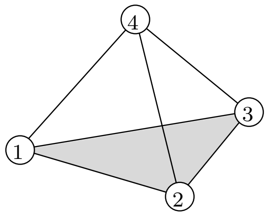
3.9 Sperner's lemma

In a **vector space** X with $\dim(X) \geq n$ every **Sperner-labeled regular decomposition** of an n-simplex contains an **odd** number of **completely labeled elements**.

Proof: By induction over the dimension n:

Induction start $n = 2$: We examine the decomposition of a closed interval $S = [x_1; x_2]$ with $x_1 < x_2$ into closed subintervals $S_i = [x_{i1}; x_{i2}]$ with $1 \leq i \leq m$

starting with $x_1 = x_{11}$, closing with $x_{m2} = x_2$ and any two subintervals sharing at most one boundary point. Since we start with $s_1(x_{11}) = 1$ and close with $s_1(x_{m2}) = 2$ there must be at least one subintervall with $s_i(x_{i1}) = 1$ and $s_i(x_{i2}) = 2$. Any further subintervals with different labeling must be complemented by a second interval with inverse labeling such that we arrive at an odd number of differently, i.e. completely labeled intervals.



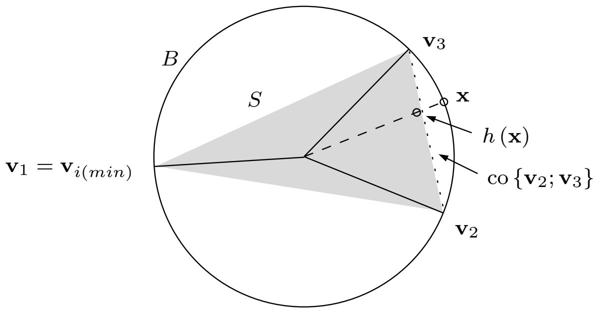
Completely labeled 3-simplex

Induction step $n-1 \Rightarrow n$: Each completely labeled n-simplex S_i includes exactly one boundary segment consisting of a completely labeled $n-1$-simplex with $s_i(\mathbf{x}_{ik}) \in \{0; ...; n-1\}$ for $\mathbf{x}_{ik} \in A_i \setminus \{\mathbf{x}_{il}\}$ with $s_i(\mathbf{x}_{il}) = n$. If such a completely labeled $n-1$-simplex is included in the boundary of S, i.e., if it lies on the outside it belongs to exactly one S_i; if it is not included in the boundary and lies inside, it is shared by exactly two S_i. According to the base case applied to the single Sperner-labeled decomposition boundary $n-1$-simplex containing the desired labels $0; ...; n-1$ this boundary includes an odd number of completely labeled $n-1$-simplices with these labels . The other boundarys include at most completely labeled $n-1$-simplices with other subsets of $\{0; ...; n\}$, i.e. none of the desired sort. Hence we we arrive at an odd number of completely labeled n-simplices on one boundary and an even number lying inside including those looking with the "wrong" face at the other boundaries.

3.10 Every simplex is homeomorphic to a closed unit ball

In a **topological vector space** X with $\dim(X) \geq n$ every **simplex** $\overline{S} = \operatorname{co}\{\mathbf{x}_0; ...; \mathbf{x}_n\}$ with $\mathbf{0} \in S$ is **homeomorphic** to the **closed unit ball** $\overline{B} = \{\mathbf{x} : |\mathbf{x}| \leq 1\}$.

Proof: The condition $\mathbf{0} \in S$ entails that each $\mathbf{x}_j$ can be expressed as a **linear combination** $\mathbf{x}_j = \sum_{i \in N_j} \beta_i \mathbf{x}_i$ of the remaining $\mathbf{x}_i$ with **nonpositive** coefficients $\beta_i \leq 0$ for $i \in N_j$ and $i \in N_j = \{0; ...; n\} \setminus \{j\}$ since S must include a neighbourhood $B_\epsilon(0)$, hence it must contain the vector $-\epsilon \mathbf{x}_j = \sum_{i \in N_{i(min)}} (-\epsilon \beta_i) \mathbf{x}_i$ and due to the linear independence of the remaining $\mathbf{x}_i$ this expression is uniquely determined whence follows $-\epsilon \beta_i > 0$ according to the definition of the convex hull. In particular any subset of n vectors $\{\mathbf{x}_i : i \in N_j\}$ must be linearly independent.



W.l.o.g. we assume that all $|\mathbf{x}_i| = 1$. Then for every $\mathbf{x} \in \overline{S} = \left\{\mathbf{x} = \sum_{i=0}^{n} \alpha_i \mathbf{x}_i : \alpha_i \geq 0; 0 \leq i \leq n; \sum_{i=0}^{n} \alpha_i = 1\right\}$ we have $|\mathbf{x}| = \left|\sum_{i=0}^{n} \alpha_i \mathbf{x}_i\right| \leq \sum_{i=0}^{n} \alpha_i |\mathbf{x}_i| = 1$ with equality only iff $\mathbf{x} \in \{\mathbf{x}_0; ...; \mathbf{x}_n\}$ and hence $\overline{S} \subset \overline{B}$. We start by expressing the component $\mathbf{x}_{i(min)}$ with the **smallest** coefficient $\alpha_{i(min)} = \min\{\alpha_0; ...; \alpha_n\} > 0$ and $N_{i(min)} = \{0; ...; n\} \setminus \{i(min)\}$ as a linear combination $\mathbf{x}_{i(min)} = \sum_{i \in N_{i(min)}} \beta_i \mathbf{x}_i$ of the remaining linearly independent $\mathbf{x}_i$ with **nonpositive** coefficients $\beta_i \leq 0$ for $i \in N_{i(min)}$.

Now we **dilat**e $\mathbf{x} = \sum_{i \in N_{i(min)}} \delta_i \mathbf{x}_i$ with $\delta_i = \alpha_i + \alpha_{i(min)} \beta_i$ until reaches the boundary section of S formed by $H_{min} = \left\{\sum_{i \in N_{i(min)}} \gamma_i \mathbf{x}_i : \gamma_i \geq 0; i \in N_{i(min)}; \sum_{i \in N_{i(min)}} \gamma_i = 1\right\}$ by defining $\mathbf{h}(\mathbf{x}) = \frac{\mathbf{x}}{h(\mathbf{x})}$ with $h(\mathbf{x}) = \sum_{i \in N_{min}} \delta_i < \sum_{i \in N_{min}} \alpha_i < 1$ such that we have $\sum_{i \in N_{i(min)}} \frac{\delta_i}{h(\mathbf{x})} = 1$, i.e. $\mathbf{h}(\mathbf{x}) \in \delta S$ lies on the convex hull $\operatorname{co}\left\{\mathbf{x}_i : i \in N_{i(min)}\right\}$ defining the face **opposite** $\mathbf{x}_{i(min)}$. Because of the minimal character of $\alpha_{i(min)}$ and $\beta_i < 0$ we have $0 < \delta_i < 1$ and hence $|\mathbf{h}(\mathbf{x})| = \frac{|\mathbf{x}|}{h(\mathbf{x})} \leq \frac{\sqrt{\sum_{i \in N_{i(min)}} \delta_i^2}}{\sum_{i \in N_{i(min)}} \delta_i} < 1$ such that $\mathbf{g} : S \to B$ given by $\mathbf{g}(\mathbf{x}) = \frac{\mathbf{x}}{|\mathbf{h}(\mathbf{x})|}$ with $|\mathbf{g}(\mathbf{x})| = h(\mathbf{x}) < 1$ is again a **dilation** and thus **injective** with $\mathbf{g}[S] \subset B$. For $\mathbf{x} \in B$ and $\epsilon > 0$ with $B_\epsilon(0) \subset S$ there are uniquely determined and **continuous** $\alpha_i(\mathbf{x}) \geq 0; 0 \leq i \leq n$ with $\epsilon \mathbf{x} = \sum_{i=0}^{n} \alpha_i(\mathbf{x}) \mathbf{x}_i$ and $\sum_{i=0}^{n} \alpha_i(\mathbf{x}) = 1$. Furthermore we can determine an equally **continuous** $\alpha_{i(min)}(\mathbf{x}) = \min\{\alpha_0; ...; \alpha_n\} > 0$ and hence $0 < h(\epsilon\mathbf{x}) < 1$. The **inverse** $\mathbf{g}^{-1} : B \to S$ is then given by $\mathbf{g}^{-1}(\mathbf{x}) = |\mathbf{h}(\epsilon\mathbf{x})| \cdot \mathbf{x} = \frac{|\mathbf{h}(\epsilon\mathbf{x})|}{\epsilon} \cdot \epsilon\mathbf{x} = \frac{|\mathbf{h}(\epsilon\mathbf{x})| \cdot h(\epsilon\mathbf{x})}{\epsilon} \cdot \mathbf{h}(\epsilon\mathbf{x}) = \frac{|\epsilon\mathbf{x}|}{\epsilon} \cdot \mathbf{h}(\epsilon\mathbf{x}) \in S$ since S is **convex** with $\mathbf{0} \in S$, $\mathbf{h}(\epsilon\mathbf{x}) \in S$ and $\frac{|\epsilon\mathbf{x}|}{\epsilon} \leq 1$.

Thus g is **surjective** and obviously **continuous**. It can be **extended** to a **continuous bijection** $\overline{g} : \overline{S} \to \overline{B}$ by $\overline{g}(\mathbf{x}) = \mathbf{x}$ for $\mathbf{x} \in \{\mathbf{x}_0; ...; \mathbf{x}_n\} = \delta S \cap \delta B$.

3.11 Approximate fixed-point lemma

A **continuous** function $f : X \to X$ on a **compact metric space** $(X; d)$ has a **fixed point** iff for every $\epsilon > 0$ there is an $x_\epsilon \in X$ such that $d(x_\epsilon; f(x_\epsilon)) < \epsilon$.

Proof: Due to (cf. [11, th. 10.12]) and since X is **compact** the sequence $(x_n)_{n \geq 1}$ with $d(x_n; f(x_n)) < \frac{1}{n}$ converges to an $x \in X$. Due to f being **continuous** $(f(x_n))_{n \geq 1} \to f(x)$ and hence $(x_n; f(x_n))_{n \geq 1} \to (x; f(x))$. Since d is continuous we have $d(x_n; f(x_n))_{n \geq 1} \to d(x; f(x))$ such that the assertion follows from $\lim\limits_{n \to \infty} d(x_n; f(x_n)) = 0$.

3.12 Brouwer's fixed point theorem

Every continuous map$\mathbf{f} : \overline{B_1(0)} \to \overline{B_1(0)}$ on a **finite dimensional Banach space** X has a **fixed point** $\mathbf{x} = \mathbf{f}(\mathbf{x}) \in \overline{B_1(0)}$.

Proof: W.l.o.g we assume $X = \operatorname{span}\{\mathbf{e}_0; ...; \mathbf{e}_n\}$ with $\dim(X) = n + 1$ and $\overline{S} = \operatorname{co}\{\mathbf{e}_0; ...; \mathbf{e}_n\}$ homeomorphic and homomorphic to $\overline{T} = \operatorname{co}\{\mathbf{v}_0; ...; \mathbf{v}_n\}$ with $\mathbf{0} \in \overline{T}$ by the linear transformation $t : \overline{S} \to \overline{T}$ with $t(\mathbf{e}_0) = \mathbf{v}_0$ resp. $t(\mathbf{e}_i - \mathbf{e}_0) = \mathbf{v}_i - \mathbf{v}_0$ for $1 \leq i \leq n$. Note that the condition $\mathbf{0} \in \overline{T}$ entails the **linear independence** of the direction vectors $\mathbf{v}_i - \mathbf{v}_0$. Hence lemma 3.10 justifies the assumption.

Any $\mathbf{f} = \sum\limits_{i=0}^{n} f_i \mathbf{e}_i : \overline{S} \to \overline{S}$ must satisfy $\sum\limits_{i=0}^{n} f_i(\mathbf{x}) = 1$ for every $\mathbf{x} \in S$. Hence if we assume that it does **not** fix any **subvertex** of a **regular decomposition** $\left\{\overline{S_1}; ...; \overline{S_m}\right\}$ of $\overline{S}$ it determines a **Sperner labeling** of subervertices:

Since for any subvertex $\mathbf{p}$ holds $\mathbf{f}(\mathbf{p}) \neq \mathbf{p}$ and $\sum\limits_{i=1}^{n} f_i(\mathbf{p}) = 1$ we can choose a $j \in \{0; ...; n\}$ with $f_j(\mathbf{p}) < p_j$ and define $s(\mathbf{p}) = j$. Since we have chosen **basis vectors** $\mathbf{e}_i$ as vertices of the original simplex $\overline{S}$ the k-th coordinate of a new vertex $p \in \operatorname{co}(\{\mathbf{e}_0; ...; \mathbf{e}_n\} \setminus \{\mathbf{e}_k\})$ lying on a boundary face is $p_k = 0$ and hence $j \neq k$, i.e. we have the **Sperner condition** $s(\mathbf{p}) \in \{0; ...; n\} \setminus \{k\}$.

Due to **Heine's theorem** (cf. [11, th. 12.9]) the **continuous** function $\mathbf{f}$ is **uniformly continuous** on the **compact** space X such that for every ϵ there is a δ with $\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})\|_1 < \epsilon$ for every $x; y \in X$ with $\|\mathbf{x} - \mathbf{y}\|_1 < \delta < \epsilon$. In this computation we use the **norm** $\|(x_0; ...x_n)\|_1 = \sum\limits_{i=0}^{n} |x_i|$ for $\mathbf{x} = \sum\limits_{i=0}^{n} x_i \mathbf{e}_i \in X$ being equivalent to the **euclidean norm** $|(x_0; ...x_n)| = \sum\limits_{i=0}^{n} x_i^2$ according to (cf. [11, th. 1.8]).

We now prove that every **completely labeled element** $\overline{S_\epsilon}$ of a **regular decomposition** $\left\{\overline{S_1}; ...; \overline{S_m}\right\}$ of $\overline{S}$ with **diameter** $\delta\left(\overline{S_\epsilon}\right) < \delta < \epsilon$ contains an an **approximate fixed point** $\mathbf{x}_\epsilon \in \overline{S_\epsilon}$ with $\|\mathbf{x}_\epsilon - \mathbf{f}(\mathbf{x}_\epsilon)\|_1 < \epsilon$: Let $(\mathbf{x}_j)_{0 \leq j \leq n} \subset \overline{S_\epsilon}$ be the vertices of $\overline{S_\epsilon}$ labeled by $s(i) = j$ for $0 \leq i \leq n$ such that $f_j(\mathbf{x}_j) < x_{jj}$ for every $\mathbf{x}_j = (x_{j0}; ...; x_{jn})$. Thus we have

$$\begin{aligned}
\left\|\mathbf{x}_j - \mathbf{f}\left(\mathbf{x}_j\right)\right\|_1 =& \sum_{i=0}^{n} \left|x_{ji} - f_i\left(\mathbf{x}_j\right)\right| \\
=& \sum_{i=0}^{n} \left|x_{ji} - x_{ii} + x_{ii} - f_i\left(\mathbf{x}_i\right) + f_i\left(\mathbf{x}_i\right) - f_i\left(\mathbf{x}_j\right)\right| \\
\leq& \sum_{i=0}^{n} \left(\delta + x_{ii} - f_i\left(\mathbf{x}_i\right) + (n+1)\,\epsilon\right) \\
\leq& \sum_{i=0}^{n} x_{ii} - \sum_{i=0}^{n} f_i\left(\mathbf{x}_i\right) + 2\,(n+1)\,\epsilon \\
=& \sum_{i=0}^{n} x_{0i} + \sum_{i=1}^{n} \left(x_{ii} - x_{0i}\right) - \sum_{i=0}^{n} f_i\left(\mathbf{x}_0\right) + \sum_{i=1}^{n} \left(f_i\left(\mathbf{x}_0\right) - f_i\left(\mathbf{x}_i\right)\right) + 2\,(n+1)\,\epsilon \\
\leq& \; 1 + n\epsilon - 1 + n\epsilon + 2\,(n+1)\,\epsilon \\
\leq& \; 4\,(n+1)\,\epsilon.
\end{aligned}$$

Hence **every completely labeled vertex is an approximate fixed point** and the assertion follows from lemma 3.11.

4 Extension of continuous functionals

4.1 Dual spaces

For a complex topological vector space X the **dual space** X^* is the vector space of all complex liner functionals $\Lambda : X \to \mathbb{C}$. We have **real linearity** iff $\Lambda\alpha\mathbf{x} = \alpha\Lambda\mathbf{x}$ holds for **real** $\alpha \in \mathbb{R}$ as opposed to (complex) linearity for $\alpha \in \mathbb{C}$. For **linear** Λ with **(complex) linear real** resp. **imaginary** parts we have $\Lambda i\mathbf{x} = \mathrm{Re}\Lambda i\mathbf{x} + i\mathrm{Im}\Lambda i\mathbf{x} = i\mathrm{Re}\Lambda\mathbf{x} - \mathrm{Im}\Lambda\mathbf{x} = i\Lambda\mathbf{x}$ whence the imaginary part $\mathrm{Im}\Lambda\mathbf{x} = -\mathrm{Re}\Lambda i\mathbf{x}$is already determined by the real part. Conversely every $\Lambda : X \to \mathbb{C}$ defined by $\Lambda\mathbf{x} = \mathrm{Re}\Lambda\mathbf{x} - i\mathrm{Re}\Lambda i\mathbf{x}$ with a **real linear** and **real valued** $\mathrm{Re}\Lambda : X \to \mathbb{R}$ is **complex linear** since for $\alpha + i\beta \in \mathbb{C}$ we have $\Lambda\left(\alpha + i\beta\right)\mathbf{x} = \alpha\mathrm{Re}\Lambda\mathbf{x} + \beta\mathrm{Re}\Lambda i\mathbf{x} - i\alpha\mathrm{Re}\Lambda i\mathbf{x} + i\beta\mathrm{Re}\Lambda\mathbf{x} = \left(\alpha + i\beta\right)\mathrm{Re}\Lambda\mathbf{x} - i\left(\alpha + i\beta\right)\mathrm{Im}\Lambda\mathbf{x} = \left(\alpha + i\beta\right)\Lambda\mathbf{x}$. Moreover Λ is **continuous** iff its **real part** is **continuous**.

4.2 The Hahn-Banach theorem

Every **continuous** and **linear** functional $\Lambda_Y : Y \to \mathbb{C}$ on a vector subspace $Y \subset X$ of a **locally convex** vector space X can be extended to a **continuous** and **linear** $\Lambda : X \to \mathbb{C}$. In the case of a **bounded** Λ_Y with reference to a **real seminorm** p with $|\Lambda_Y\mathbf{y}| \leq p(\mathbf{y})\,\forall\mathbf{y} \in Y$ this property transfers to the extension $|\Lambda\mathbf{x}| \leq p(\mathbf{x})\,\forall\mathbf{x} \in X$.

Note: According to 1.4 for every **continuous** and **linear** functional $\Lambda : X \to \mathbb{C}$ on a **locally convex** vector space X there is a **real seminorm** p with $|\Lambda\mathbf{x}| \leq p(\mathbf{x})\,\forall\mathbf{x} \in X$: On account of Λ being **continuous** for every **convex** local neighbourhood $U \in \mathcal{U}(0)$ with $|\Lambda\mathbf{x}| < 1\,\forall\mathbf{x} \in U$ for $p = \mu_U$ we have $U = \{p < 1\}$ and hence $p(x) \leq t \Rightarrow \mathbf{x} \in tU \Rightarrow |\Lambda\mathbf{x}| < t$.

Proof: Let $\Lambda_Y : Y \to \mathbb{R}$ be **real valued** and **real linear**. The family $\mathcal{M}$ of all **continuous** and **real linear** extensions $\Lambda_Z : Z \to \mathbb{R}$ on vector subspaces $Y \subset Z \subset X$ with $|\Lambda_Z\mathbf{z}| \leq p(\mathbf{z})\,\forall\mathbf{z} \in Z$ is **not empty** since for every vector subspace $Y \subset Z \subsetneq X$ and $\mathbf{x}' \in X \setminus Z$ any Λ_Y can be extended to a $\Lambda_{Z'} : Z \to \mathbb{R}$ on $Z = Y \oplus \langle\mathbf{x}'\rangle$ with $\Lambda_Z\left(\mathbf{y} + \alpha\mathbf{x}'\right) := \Lambda_Y\mathbf{y} - \alpha p\left(\mathbf{x}'\right)$. Since Λ_Z is obviously real linear and continuous we only have to prove an upper bound for $\left|\Lambda_Y\mathbf{y} + \alpha p\left(\mathbf{x}'\right)\right|$: In the case of $\Lambda_Y\mathbf{y} \geq \alpha p\left(\mathbf{x}'\right)$ the absolute value is identical to the argument such that owing to the boundedness of Λ_Y we obtain $\left|\Lambda_Y\mathbf{y} - \alpha p\left(\mathbf{x}'\right)\right| = \Lambda_Y\mathbf{y} - \alpha p\left(\mathbf{x}'\right) \leq p\left(\mathbf{y}\right) - p\left(\alpha\mathbf{x}'\right) \leq p\left(\mathbf{y} + \alpha\mathbf{x}'\right)$. In the opposite case $\Lambda_Y\mathbf{y} < \alpha p\left(\mathbf{x}'\right)$ we compute the absolute value by exchanging minuend and subtrahend to arrive again at the upper bound $\left|\Lambda_Y\mathbf{y} - \alpha p\left(\mathbf{x}'\right)\right| = \alpha p\left(\mathbf{x}'\right) - \Lambda_Y\mathbf{y} \leq p\left(\mathbf{y} + \alpha\mathbf{x}'\right)$ since due to the triangle equation resp. the hypothesis we have $\alpha p\left(\mathbf{x}'\right) - p\left(\mathbf{y} + \alpha\mathbf{x}'\right) \leq -p\left(\mathbf{y}\right) \leq \Lambda_Y\mathbf{y}$. We provide the family $\mathcal{M}$ with an **order** by means of $\Lambda_{Z_1} \preceq \Lambda_{Z_2} \Leftrightarrow Z_1 \subset Z_2 \wedge \Lambda_{Z_1} = \Lambda_{Z_2}|_{Z_1}$ such that for any **linearly ordered** subfamily $\mathcal{N} \subset \mathcal{M}$ we obtain an **upper bound** $\Lambda_{\mathcal{N}} := \bigcup \mathcal{N} = \{(\mathbf{z}; \Lambda_Z \mathbf{z}) : \exists \Lambda_Z \in \mathcal{N} : \mathbf{z} \in Z\} \in \mathcal{M}$ since $\bigcup_{\Lambda_Z \in \mathcal{N}} Z \subset X$ is a **vector subspace** and $\Lambda_N : \bigcup_{\Lambda_Z \in \mathcal{N}} Z \to \mathbb{R}$ is **continuous** and **linear** with $|\Lambda_{\mathcal{N}} \mathbf{z}| = |\Lambda_Z \mathbf{z}| \leq p(\mathbf{z})$ as for every $\mathbf{z} \in \operatorname{dom}(\Lambda_{\mathcal{N}})$ there is a $\Lambda_Z \in \mathcal{N}$ with $\mathbf{z} \in Z$. Due to **Zorn's lemma** [9, th. 14.2.4] there is a **maximal** $\Lambda : Z \to \mathbb{R}$ in $\mathcal{M}$ which has to be the desired extension on the whole set $Z = X$ since otherwise for a $\mathbf{x}' \in X \setminus Z$ we could construct an extension $\Lambda' : Z \oplus \langle \mathbf{x}' \rangle \to \mathbb{R}$ with $\Lambda \preceq \Lambda'$ as described above and contrary to the maximal character of Λ.

For a **complex valued**, bounded, continuous and linear functional $\Lambda_Y : Y \to \mathbb{C}$ according to the preceding part of the proof the **real part** $u := \operatorname{Re}\Lambda_Y$ can be extended to a real valued $U : X \to \mathbb{R}$ which owing to 4.1 uniquely determines a **complex valued**, bounded, continuous and linear functional $\Lambda : X \to \mathbb{C}$ with $\Lambda(\mathbf{x}) = U(\mathbf{x}) - iU(i\mathbf{x})$ coinciding on Y with $\Lambda_Y(\mathbf{x}) = u(\mathbf{x}) - iu(i\mathbf{x})$. The functional Λ is **bounded** since with $\alpha = \frac{|\Lambda \mathbf{x}|}{\Lambda \mathbf{x}}$ we have $|\Lambda \mathbf{x}| = \alpha \Lambda \mathbf{x} = \Lambda(\alpha \mathbf{x}) = U(\alpha \mathbf{x}) \leq p(\alpha \mathbf{x}) = \alpha |p(\mathbf{x})| \leq p(\mathbf{x})$.

4.3 Existence of bounded functionals

On a **Banach** space X for every $\mathbf{x}_0 \in X$ there is a $\Lambda \in X^*$ with $\Lambda \mathbf{x}_0 = \|\mathbf{x}_0\|$ and $|\Lambda \mathbf{x}| \leq \|\mathbf{x}\| \, \forall \mathbf{x} \in X$.

Proof: Apply 4.2 to $Y = \langle \mathbf{x}_0 \rangle$ and $\Lambda_Y(\alpha \mathbf{x}_0) = \alpha \|\mathbf{x}_0\|$.

4.4 Separation properties of linear functionals

For **disjoint**, **convex** subsets $A, B \subset X$ of a topological vector space X the following statements hold:

1. For **open** A there is a $\Lambda \in X^*$ and $\gamma \in \mathbb{R}$ such that for every $\mathbf{a} \in A, \mathbf{b} \in B$ we have $\operatorname{Re}\Lambda \mathbf{a} < \gamma \leq \operatorname{Re}\Lambda \mathbf{b}$.
2. For **locally convex** X, **compact** A and **closed** B there is a $\Lambda \in X^*$ and $\gamma_1, \gamma_2 \in \mathbb{R}$ such that for every $\mathbf{a} \in A, \mathbf{b} \in B$ we have $\operatorname{Re}\Lambda \mathbf{a} < \gamma_1 < \gamma_2 < \operatorname{Re}\Lambda \mathbf{b}$.

Proof: Since only the real part of Λ is concerned it suffices to construct a real valued $U : X \to \mathbb{R}$ which owing to 4.1 is the real part $U = \operatorname{Re}\Lambda$ of a uniquely determined $\Lambda(\mathbf{x}) = U(\mathbf{x}) - iU(i\mathbf{x})$ with the desired properties.

1. Choose an $\mathbf{a}_0 \in A$ resp. $\mathbf{b}_0 \in B$ and define $\mathbf{x}_0 = \mathbf{b}_0 - \mathbf{a}_0$ such that $C = A - B + \mathbf{x}_0$ is a **convex** local neighbourhood. According to 2.4.1 and 2.4.2 the **Minkowski functional** μ_C is a **real seminorm** with $\mu_C(\mathbf{x}_0) \geq 1$ since $\mathbf{x}_0 \notin C$. Due to the **Hahn-Banach theorem** 4.2 the linear functional $\Lambda_0 \langle \mathbf{x}_0 \rangle \to \mathbb{R}$ with $\Lambda(t\mathbf{x}_0) = t \leq t\mu_C(\mathbf{x}_0) = \mu_C(t\mathbf{x}_0)$ can be **extended** to a linear $\Lambda : X \to \mathbb{R}$ with $\Lambda \leq \mu_C$ and particularly $\Lambda \leq 1$ on C resp. $\Lambda \geq -1$ on $-C$ such that Λ is **bounded** on $C \cap (-C)$ and **continuous** owing to 1.11.4. For $\mathbf{a} \in A$ resp. $\mathbf{b} \in B$ we have $\mathbf{a} - \mathbf{b} + \mathbf{x}_0 \in C$ such that $\Lambda \mathbf{a} - \Lambda \mathbf{b} + 1 = \Lambda(\mathbf{b} - \mathbf{a} + \mathbf{x}_0) \leq \mu_C(\mathbf{b} - \mathbf{a} + \mathbf{x}_0) < 1$ since C is **open** and the multiplication $M_{\mathbf{a}-\mathbf{b}+\mathbf{x}_0}$ is **continuous**. Hence we obtain $\Lambda \mathbf{a} < \Lambda \mathbf{b}$ for every $\mathbf{a} \in A$ resp. $\mathbf{b} \in B$ and since $\Lambda[A]$ is **open** due to 1.11 we obtain the proposition by choosing $\gamma := \inf\limits_{b \in B} \Lambda \mathbf{b}$
2. Owing to 1.3.1 there is an **open** and **convex** neighbourhood $U \in \mathcal{U}(\mathbf{0})$ with $(A + U) \cap B = \emptyset$ and with 1. we obtain $\operatorname{Re}\Lambda(\mathbf{a} + \mathbf{x}) = \operatorname{Re}\Lambda \mathbf{a} + \operatorname{Re}\Lambda \mathbf{x} < \gamma \leq \operatorname{Re}\Lambda \mathbf{b}$ for every $\mathbf{a} \in A, \mathbf{b} \in B$ and $\mathbf{x} \in U$. Since Λ is **open** there is an $\epsilon > 0$ with $B_\epsilon(0) \subset \Lambda[U]$ such that we obtain the proposition with e.g. $\gamma_1 = \gamma - \epsilon$ and $\gamma_2 = \gamma - \frac{\epsilon}{2}$.

4.5 Separating functionals on locally convex spaces

For a **locally convex** space X the following propositions hold:

1. For every **vector subspace** $Y \subset X$ and $\mathbf{x}_0 \in X \setminus \overline{Y}$ there is a $\Lambda \in X^*$with $\Lambda \mathbf{x}_0 = 1$ and $\Lambda[Y] \subset \{0\}$.

2. For every **convex** and **balanced** subset $Y \subset X$ and $\mathbf{x}_0 \in X \setminus \overline{Y}$ there is a $\Lambda \in X^*$with $\Lambda \mathbf{x}_0 > 1$ and $\Lambda\left[Y\right] \subset [-1;1]$.

Proof:

1. For $\Lambda \in X^*$ according to 4.4.1 the vector subspace $\Lambda\left[Y\right] \subset \mathbb{R}$ does **not contain** the point $\Lambda \mathbf{x}_0$ so that we conclude $\Lambda\left[Y\right] = \{0\}$and hence $\Lambda \mathbf{x}_0 \neq 0$.
2. On account of 1.6.2 and 1.6.3 the **closure** $\overline{Y}$ also is **convex** and **balanced** such that with 4.4.2 and $A = \{\mathbf{x}_0\}$ we find an $\Lambda_0 \in X^*$ with $\Lambda_0 \mathbf{x}_0 r e^{i\varphi} \notin \overline{\Lambda\left[\overline{Y}\right]}$. Since the balanced character of $\overline{Y}$ implies a balanced $\overline{\Lambda\left[\overline{Y}\right]}$ there is a $0 < s < r$ with $|\mathbf{z}| \leq s$ for every $\mathbf{z} \in \overline{\Lambda\left[\overline{Y}\right]}$. The functional $\Lambda = s^{-1} e^{-i\varphi} \Lambda_0$ then shows the desired properties.

5 Weak convergence

5.1 Linear combinations of functionals

For linear functionals Λ_i, $1 \leq i \leq n$ and Λ on a vector space X with $N = \bigcap\limits_{i=1}^{n} \ker \Lambda_i$ the following three statements are equivalent:

1. $\Lambda = \sum\limits_{i=1}^{n} \alpha_i \Lambda_i$ with $\alpha_i \in \mathbb{C}$ for $1 \leq i \leq n$.
2. There is a $\gamma < \infty$ with $|\Lambda \mathbf{x}| \leq \gamma \max\limits_{1 \leq i \leq n} |\Lambda_i \mathbf{x}|$ for every $\mathbf{x} \in X$.
3. $\Lambda \mathbf{x} = 0$ for every $\mathbf{x} \in N$.

Proof: We only have to show 3. $\Rightarrow$ 1.: We define a linear map $\pi : X \to \mathbb{C}^n$ by $\pi\left(\mathbf{x}\right) = (\Lambda_1 \mathbf{x}; ...; \Lambda_n \mathbf{x})$. Since $\mathbf{0} = \pi\left(\mathbf{x}\right) - \pi\left(\mathbf{y}\right) = \pi\left(\mathbf{x} - \mathbf{y}\right)$ implies $\mathbf{x} - \mathbf{y} \in N$ whence $\Lambda \mathbf{x} - \Lambda \mathbf{y} = \Lambda\left(\mathbf{x} - \mathbf{y}\right) = 0$ the map $f : \pi\left[X\right] \to \mathbb{C}$ defined by $f\left(\pi\left(\mathbf{x}\right)\right) = \Lambda \mathbf{x}$ is a **linear functiona**l on the **vector subspace** $\pi\left[X\right] \subset \mathbb{C}^n$. Due to the **Hahn-Banach-theorem** 4.2 it can be extended to a linear $F : \mathbb{C}^n \to \mathbb{C}$ with $F\left(\mathbf{z}_1; ...; \mathbf{z}_n\right) = \sum\limits_{i=1}^{n} \alpha_i \mathbf{z}_i$ for some $\left(\alpha_i\right)_{1 \leq i \leq n} \subset \mathbb{C}$. This means $\Lambda \mathbf{x} = F\left(\pi\left(\mathbf{x}\right)\right) = \sum\limits_{i=1}^{n} \alpha_i \Lambda_i \mathbf{x}$ for $\mathbf{x} \in X$ which is the assertion.

5.2 The initial topology of a family of linear functionals

Let Γ be a **separating family** of linear functionals on a vector space X, i.e. for every pair $\mathbf{x}_1 \neq \mathbf{x}_2 \in X$ there is a $\Lambda \in \Gamma$ with $\Lambda \mathbf{x}_1 \neq \Lambda \mathbf{x}_2$. Then the **initial topology** $\mathcal{O} = \tau\left(\Gamma\right)$ on X with reference to Γ is **locally convex** and its **dual space** is again $X^* = \Gamma$.

Proof: Since $\mathbb{C}$ is **Hausdorff** and $\Lambda \in \Gamma$ is **continuous** as well as **separating** the topology $\mathcal{O}$ is **Hausdorff** too. Due to [11, th. 12.10] the local neighbourhoods $U\left(0\right) = \{|\Lambda \mathbf{x}| < \epsilon\}$ for $\epsilon > 0$ and $\Lambda \in \Gamma$ form a **subbasis** for the **initial neighbourhood filter** resp. the resulting topology $\mathcal{O}$. The **translation invariance** follows from $|\Lambda \mathbf{x} - \Lambda \mathbf{y}| = |\Lambda\left(\mathbf{x} + \mathbf{a}\right) - \Lambda\left(\mathbf{y} + \mathbf{a}\right)|$. The neighbourhoods $U(\mathbf{0})$ are **convex** and **balanced** on account of $|\Lambda\left(t\mathbf{x} + \left(1 - t\right)\mathbf{y}\right)| \leq t\left|\Lambda \mathbf{x}\right| + \left(1 - t\right)\left|\Lambda \mathbf{y}\right|$. The **addition is continuous** due to $\frac{1}{2}V + \frac{1}{2}V = V$. For $\alpha \in \mathbb{C}$ and $\mathbf{x} \in X$ let $|\alpha - \beta| < \min\left\{\frac{\epsilon}{3}; \frac{\epsilon}{3|\Lambda \mathbf{x}|}\right\}$ and $|\Lambda\left(\mathbf{x} - \mathbf{y}\right)| < \min\left\{\frac{\epsilon}{3}; \frac{\epsilon}{3|\alpha|}\right\}$. Then we have $|\Lambda\left(\alpha \mathbf{x} - \beta \mathbf{y}\right)| \leq |\alpha| \cdot |\Lambda\left(\mathbf{x} - \mathbf{y}\right)| + |\alpha - \beta| \cdot |\Lambda \mathbf{x}| + |\alpha - \beta| \cdot |\Lambda\left(\mathbf{x} - \mathbf{y}\right)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon^2}{3} \leq \epsilon$ and hence the **continuity of the multiplication with scalars** (cf. [11, th. 4.3.2]). Since any $\Phi \in X^*$ is **continuous** there are $\Lambda_1, ...\Lambda_n \in \Gamma$ and an $\epsilon > 0$ with $\left\{\max\limits_{1 \leq i \leq n} |\Lambda_i \mathbf{x}| < \epsilon\right\} \subset \{|\Phi \mathbf{x}| < 1\}$. Since it is **linear** we also have $\max\limits_{1 \leq i \leq n} |\Lambda_i \mathbf{x}| = M \Leftrightarrow \max\limits_{1 \leq i \leq n} \left|\Lambda_i\left(\frac{\epsilon \mathbf{x}}{M}\right)\right| < \epsilon \Rightarrow \left|\Phi\left(\frac{\epsilon \mathbf{x}}{M}\right)\right| < 1$ $\Rightarrow |\Phi \mathbf{x}| < \gamma \max\limits_{1 \leq i \leq n} |\Lambda_i \mathbf{x}|$ with $\gamma = \frac{1}{\epsilon}$. From 5.1.2 we infer $\Phi = \sum_{1 \leq i \leq n} \alpha_i \Lambda_i \in \Gamma$ and since obviously $\Gamma \subset X^*$ we obtain $\Gamma = X^*$.

5.3 The weak topology

For a topological vector space $(X;\mathcal{O})$ with a **separating** dual space X^* the initial topology $\mathcal{O}_w = \tau(X^*) \subset \mathcal{O}$ is the **topology** of **weak convergence** or **weak topology**. Hence a filter F **weakly converges** to an $\mathbf{x} \in X$ iff every image filter $\Lambda(F)$ converges to Λx on K. In the subsequent paragraphs we frequently compare the given or **original** topology $\mathcal{O}$ with the weak topology $\mathcal{O}_w$.

5.4 Weakly bounded subsets

A subset $A \subset X$ of a topological vector space X is **weakly bounded** iff all $\Lambda \in X^*$ are **originally bounded** on A.

Proof: According to 1.4.1 the set A is **weakly bounded** iff for every $\Lambda \in X^*$ there is $\tau > 0$ with $A \subset \tau\{|\Lambda| < 1\}$ resp. $|\Lambda x| < \tau\,\forall \mathbf{x} \in A$.

5.5 Weak closure

On a **locally convex** vector space $(X;\mathcal{O})$ the **weak closure** $\overline{A}_w$ of a **convex** set $A \subset X$ coincides with the **original closure** $\overline{A}$.

Proof: Due to 5.3 we have $\mathcal{O}_w \subset \mathcal{O}$ and consequently $\overline{A} \subset \overline{A}_w$. According to 4.4.1 for $\mathbf{x}_0 \notin \overline{A}$ there is a $\Lambda \in X^*$ and a $\gamma \in \mathbb{R}$ with $\mathrm{Re}\Lambda\mathbf{x}_0 < \gamma \leq \mathrm{Re}\Lambda\mathbf{x}$ for all $\mathbf{x} \in \overline{A}$. Hence the set $\{\mathrm{Re}\Lambda\mathbf{x} < \gamma\}$ is a **weak neighbourhood** of $\mathbf{x}_0$ not meeting $\overline{A}$, i.e. $\mathbf{x}_0 \notin \overline{A}_w$.

5.6 Pointwise converging convex combinations

On a **locally convex** vector space X with a **countable local basis** for every sequence $(\mathbf{x}_n)_{n\geq 1} \subset X$ **weakly converging** to a $\mathbf{x} \in X$ there is a second sequence $(\mathbf{y}_n)_{n\geq 1} \subset X$ of **convex combinations** $\mathbf{y}_n = \sum\limits_{j=1}^{i_n} \alpha_j \mathbf{x}_{jn}$ with $1 = \sum\limits_{j=1}^{i_n} \alpha_j$ **originally converging** to $\mathbf{x}$ since for $A = \{\mathbf{x}_1; \mathbf{x}_2; ...\}$ we have $\mathbf{x} \in \overline{\mathrm{co}\, A}_w = \overline{\mathrm{co}\, A}$ owing to 5.5 and the existence of the sequence $(\mathbf{y}_n)_{n\geq 1} \subset \mathrm{co}\, A$ is guaranteed by the **countable basis** $\mathcal{B}(\mathbf{x})$.

5.7 Uniformly converging convex combinations

For every sequence $(f_n)_{n\geq 1} \subset C(K;\mathbb{C})$ of **uniformly bounded**, **continuous** and **complex valued** maps on a **compact Hausdorff space** K **originally** resp. **pointwise converging** to a $f \in C(K;\mathbb{C})$ there is a sequence $g_n = \sum\limits_{j=1}^{i_n} \alpha_j f_{jn}$ with $1 = \sum\limits_{j=1}^{i_n} \alpha_j$ of **convex combinations uniformly converging** to f.

Proof: The sequence $(f_n)_{n\geq 1}$ converges **μ-almost everywhere** to f for any **Borel measure** μ on K (cf. [8, th. 12.1]). According to **Lebesgue's dominated convergence theorem** [8, th. 5.14] the sequence $(f_n)_{n\geq 1}$ converges **in mean** to f. Owing to the **Riesz representation theorem** [8, th. 13.3] for **bounded, continuous** and according to 1.11.4 **uniformly continuous** maps on the **compact** set K the sequence converges **weakly** to f. Due to [11, th. 18.6.4] any functional $\Lambda : C(K;\mathbb{C}) \to \mathbb{C}$ being **continuous** with reference to the given **neighbourhood filter** $\mathcal{W}(\{K\},\mathcal{U})$ of **uniform convergence** is also continuous with reference to the weaker resp. larger neighbourhood system $\mathcal{W}(\mathcal{E};\mathcal{U}) \supset \mathcal{W}(\{K\},\mathcal{U})$ of **pointwise convergence** on $C(K;\mathbb{C})$. Hence weak convergence with regard to the original topology of **pointwise** convergence implies weak convergence with regard to the original topology of **uniform** convergence. But the topology of **uniform convergence** is induced by the **supremum norm** $\|\|$ on K whence $(C(K);\|\|)$ is a **Banach space** satisfiying the hypothesis of the preceding corollary 5.6 which proves the assertion.

5.8 The weak* topology on dual spaces

1. On the **dual vector space** $X^* \subset \mathbb{C}^X$ of linear and continuous functionals $\Lambda : (X; O) \to (\mathbb{C}; d)$ of a topological vector space $(X, \mathcal{O})$ the vector subspace P_X of the **evaluation functionals** resp. **projections** $\pi_{\mathbf{x}} : X^* \to \mathbb{C}$ for $\mathbf{x} \in X$ with $\pi_{\mathbf{x}}\Lambda = \Lambda\mathbf{x}$ is obviously **separating** and **isomorphic** to X. The **local neighbourhoods** $B^*_{\mathbf{x};\epsilon}(0) = \{\Lambda \in X^* : |\Lambda\mathbf{x}| < \epsilon\}$ for $\mathbf{x} \in X$ **and** $\epsilon > 0$ of the null functional 0 form a **subbasis** of its **initial neighbourhood filter**. On account of $\Lambda_1\mathbf{x} + \Lambda_2\mathbf{x} = (\Lambda_1 + \Lambda_2)\,\mathbf{x}$ the neighbourhoods are **translation invariant** with $B^*_{\mathbf{x};\epsilon}(\Lambda) = \Lambda + B^*_{\mathbf{x};\epsilon}(0)$. The corresponding **initial topology** $\mathcal{O}^*_w = \tau(P_X)$ on X^* is the **weak* topology**. Hence a sequence $(\Lambda_n)_{n\geq 1} \subset X^*$ **converges weakly** to a $\Lambda \in X^*$ iff $\lim\limits_{n\to\infty} \Lambda_n\mathbf{x} = \Lambda\mathbf{x}$ **pointwise** for every $\mathbf{x} \in X$. The weak* topology is identical to the **trace topology** of X^* with reference to the **product of the euclidean topologies** $\bigotimes_{x\in X} \tau(d)$ [11, th. 2.2 resp. 4.2] and induced by the local neighbourhoods $B^*_{\mathbf{x};\epsilon}(0) = \{f : X \to \mathbb{C} : f \text{ linear and continuous with } |f(\mathbf{x})| < \epsilon\}$ for $\mathbf{x} \in X$ and $\epsilon > 0$ on the vector space $\mathbb{C}^X$. Due to 5.2 it is **locally convex** and we have $X \simeq P_X = (X^*)^*$, i.e. every **linear** and **weakly* continuous** functional on X^* is an **evaluation** functional of the form $\Lambda \to \Lambda\mathbf{x}$ for some $\mathbf{x} \in X$.
2. According to the **Riesz representation theorem** [8, th. 10.13] a sequence $(\Lambda_n)_{n\geq 1} \subset (C_c(X, \mathbb{C}))^*$ with regard to the **norm** $\|\|^*$ defined in [8, th. 9.13] by $\|\Lambda\|^* = \sup\left\{\left|\Lambda\left(\frac{f}{\|f\|_\infty}\right)\right| : f \in C_c(X, \mathbb{C})\right\}$ **uniformly converges** to a $\Lambda \in (C_c(X, \mathbb{C}))^*$ iff the sequence $(\mu_n)_{n\geq 1} \subset \mathcal{M}_0(\mathcal{L}(X); \mathbb{C})$ with $\Lambda_n f = \int f d\mu_n \, \forall f \in C_c(X, \mathbb{R})$ of **complete and regular complex Borel measures** determined by the **theorem of Lebesgue-Radon-Nikodym** [8, th. 9.8] under the **norm** $\|\|$ defined in [8, th. 9.4] by $\|\mu\| := |\mu|(X)$ **uniformly converges** to $\mu \in \mathcal{M}_0(\mathcal{L}(X); \mathbb{C})$. In comparison with the weak* convergence the Riesz representation theorem implies the stronger assertion of **uniform convergence** on the smaller domain $C_c(X, \mathbb{C})$ of **continuous complex functions with compact support**.
3. According to the **Helly-Bray-theorem** [10, th. 3.9] a sequence $(\mu_n)_{n\geq 1} \subset \mathcal{M}_0(\mathcal{B}(\mathbb{R}); [0;1])$ of **probability resp. bounded measures converges weakly** resp. in **distribution** to a bounded measure $\mu \in \mathcal{M}_0(\mathcal{B}(\mathbb{R}); [0;1])$ iff $\lim\limits_{n\to\infty} \int f d\mu_n = \int f d\mu$ for every **bounded** and **continuous** $f \in \mathcal{C}_b(\mathbb{R}; \mathbb{R})$. Like the weak* convergence the **Helly-Bray theorem** asserts **pointwise convergence** but requesting only **continuous bounded real functions** on the **real line** while its domain is much larger in the sense that the **measures need neither be complete nor regular** and in particular **not λ-continuous** such that there may not exist a **Radon-Nikodym density** with regard to λ and the **continuous character** of the **linear functional** $\Lambda \in (\mathcal{C}_b(\mathbb{R}; \mathbb{R}))^*$ determined by $\Lambda f = \int f d\mu$ is limited to the cases of the **dominated** resp. **monotone convergence theorems** as stated in [8, th. 5.13 and 5.15].

5.9 The Banach-Alaoglu theorem

For every local neighbourhood $U \in \mathcal{U}(\mathbf{0})$ of a topological vector space X the **polar**

$$K_U = \bigcap_{\mathbf{x}\in U} B^*_{\mathbf{x};\epsilon}(0)$$

is **convex**, **balanced** and **weakly*compact**.

Proof: Since the unit disc $\{|\mathbf{z}| < 1\} \subset \mathbb{C}$ is **convex** as well as **balanced** this is also true for K_U. Due to 2.4.3 every local neighbourhood is **absorbing** such that for any $\mathbf{x} \in X$ there is a $\tau_{\mathbf{x}} > 0$ with $\mathbf{x} \in \tau_{\mathbf{x}}U$ resp. $|\Lambda\mathbf{x}| \leq \tau_{\mathbf{x}} \forall \mathbf{x} \in X, \Lambda \in K_U$. Thus we have $K_U \subset X^* \cap P$ with $P = \prod_{\mathbf{x}\in X} \overline{B_{\tau_{\mathbf{x}}}(\mathbf{0})} \subset \mathbb{C}^X$ and according to 5.8 the **product topology** $\mathcal{O}^*_P$ on P coincides on K_U with the **weak* topology** $\mathcal{O}^*_w$ on X^*.

Furthermore K_U is **closed** in $\mathcal{O}^*_P$: Let $f_0 \in \overline{K_U}$ such that for every $\epsilon > 0$ and $\mathbf{x}, \mathbf{y} \in X$ resp. $\alpha, \beta \in \mathbb{C}$ there is a **linear** $\Lambda \in K_U$ with $|(f_0 - \Lambda)(\mathbf{z})| < \epsilon$ for $\mathbf{z} \in \{\mathbf{x}; \mathbf{y}; \alpha\mathbf{x} + \beta\mathbf{y}\}$. On account of $|f_0(\alpha\mathbf{x} + \beta\mathbf{y}) - \alpha f_0(\mathbf{x}) - \beta f_0(\mathbf{y})| = |(f_0 - \Lambda)(\alpha\mathbf{x} + \beta\mathbf{y}) - \alpha(f_0 - \Lambda)(\mathbf{x}) - \beta(f_0 - \Lambda)(\mathbf{y})| <$

$(1+\alpha+\beta)\,\epsilon$ the mapping f_0 is also **linear**. For every $\mathbf{x}\in U$ there is a $\Lambda\in K_U$ with $|\Lambda\mathbf{x}|\le 1$ and $|(f_0-\Lambda)(\mathbf{x})|<\epsilon$, hence $|f_0(\mathbf{x})|\le 1$ and therefore $f_0\in K_U$.

According to **Tychonov's theorem** [11, th. 9.9] the product P is **originally compact** on $\mathbb{C}^X$ and according to [11, th. 9.4] this is also true for the **originally closed** subset $K_U\subset P$. Since $\mathcal{O}_P^*$ and $\mathcal{O}_w^*$ **coincide** on K_U the compactness property also applies to $\mathcal{O}_w^*$ - open covers of K_U.

5.10 Boundedness

A subset $A\subset X$ of a **locally convex** space X is **weakly bounded** iff it is **originally bounded**.

Proof: On account of $\mathcal{O}_w\subset\mathcal{O}$ we only have to show that a **weakly bounded** set A is **originally bounded**. Let U be an originally local neighbourhood. Due to [11, th. 7.7], 1.3.2 and 1.7.2 there is an **originally closed, balanced** and **convex** neighbourhood $\overline{V}\subset U$. We then have $\overline{V}=\bigcap_{\Lambda\in K_V}\{\mathbf{x}\in X:|\Lambda\mathbf{x}|\le 1\}$ with the **polar** K_V according to 5.9 since the right hand side is an **originally closed, balanced** and **closed** set including V with the **dual space** K_V such that on account of 4.5.2 for any $\mathbf{x}_0\notin\overline{V}$ there is a $\Lambda\in K_V$ with $\Lambda\mathbf{x}_0>1$.

Since A is **weakly bounded** for every $\Lambda\in X^*$ there is a $\tau_\Lambda<\infty$ with $|\Lambda\mathbf{x}|\le\tau_\Lambda\,\forall\mathbf{x}\in A$. Hence for every $\mathbf{x}\in A$ the family $K_V^*(\mathbf{x})$ of the **evaluation functionals** $\pi_{\mathbf{x}}:K_V\to\mathbb{C}$ with $\Lambda\to\Lambda\mathbf{x}$ are **weakly* continuous** , **linear** and **pointwise bounded** (by $\tau_\Lambda<\infty$ dependent on $\Lambda\in X^*$ but independent of $\mathbf{x}\in A$!) on the **convex** and due to 5.9 **weakly* compac**t set $K_V\subset X^*$ we can invoke 3.3 to infer that K_V^* is **uniformly bounded** on K_V, i.e. there is a $\tau<\infty$ with $|\Lambda\mathbf{x}|\le\tau\,\forall\mathbf{x}\in A,\Lambda\in K_V$. Due to $\overline{V}=\bigcap_{\Lambda\in K_V}\{|\Lambda\mathbf{x}|\le 1\}$ we infer $\frac{1}{\tau}\mathbf{x}\in\overline{V}\subset U$ for all $\mathbf{x}\in A$. Since V is **balanced** we have $A\subset t\overline{V}\subset tU$ for $t>\tau$ and hence A is **originally bounded**.

5.11 The convex hull

The **convex hull** $\operatorname{co}(A)$ is the set of all **convex combinations** $\sum_{i=1}^{n}\tau_i\mathbf{x}_i$ with $\tau_i\in\mathbb{R}$, $\mathbf{x}_i\in A$, $1\le i\le n$ and $\sum_{i=1}^{n}\tau_i=1$. It is the smallest convex set including A resp. the intersection of all convex sets including A. Some properties are as follows:

1. In a topological vector space X the **convex hull** $\operatorname{co}\left(\bigcup_{i=1}^{n}A_i\right)$ of a family of **compact** and **convex** $A_i\subset X$ with $1\le i\le n$ and $n\ge 1$ is **compact**.
2. In a **locally convex** vector space X the **convex hull** $\operatorname{co}(A)$ of a **precompact** set $A\subset X$ is **precompact**.
3. In a **Fréchet space** X the **convex hull** $\operatorname{co}(A)$ of a **compact** set $A\subset X$ is **closed** and **compact**.
4. Every point $x\in\operatorname{co}(A)$ of $A\subset\mathbb{R}^n$ is a **convex combination** of at most $n+1$ points.

Proof:

1. Let $f:S\times A\to X$ with $S=\left\{(s_1;\dots;s_n)\in\mathbb{R}^n:\sum_{i=1}^{n}s_i=1,s_i\ge 0,1\le i\le n\right\}$ and $A=\bigotimes_{i=1}^{n}A_i$ be defined by $f(\mathbf{s},\mathbf{a})=\sum_{i=1}^{n}s_ia_i$. On account of **Tychonov's theorem** [11, th. 9.9], the **Heine-Borel theorem** [11, th. 9.10] and [11, th. 9.8] the image $f[S\times A]\subset\operatorname{co}\left(\bigcup_{i=1}^{n}A_i\right)$ is **compact** and furthermore **convex** again since for $(\mathbf{s};\mathbf{a}),(\mathbf{t},\mathbf{b})\in S\times A$ and $\alpha,\beta\ge 0$ with $\alpha+\beta=1$ we have $\alpha f(\mathbf{s};\mathbf{a})+\beta f(\mathbf{t};\mathbf{b})=\sum_{i=1}^{n}(\alpha s_ia_i+\beta t_ib_i)=f(\mathbf{u};\mathbf{c})$ with $\mathbf{u}=\alpha\mathbf{s}+\beta\mathbf{t}\in S$ and $c_i=\frac{\alpha s_ia_i+\beta t_ib_i}{\alpha s_i+\beta t_i}\in A_i$ for $1\le i\le n$ due to the convexity of the A_i. Since $a_i\subset f[S\times A]$ for $1\le i\le n$ we have $K=\operatorname{co}\left(\bigcup_{i=1}^{n}A_i\right)$ whence follows the assertion.

2. For a local neighbourhood $U \in \mathcal{U}(0)$ let $V \in \mathcal{U}(0)$ be **convex** with $V + V \subset U$. Due to the hypothesis there is a **finite** $E \subset X$ with $A \subset \bigcup_{\mathbf{x}\in E} U(\mathbf{x}) \subset E + V \subset \mathrm{co}(E) + V$. Since the sum of convex sets is again convex we also have $\mathrm{co}(A) \subset \mathrm{co}(E) + V$. Since owing to 1. resp. [11, th. 17.7] the convex hull $\mathrm{co}(E)$ of the compact set E is again **compact** and particularly **precompact** there is a finite set F with $\mathrm{co}(E) \subset F + V$ resp. $\mathrm{co}(A) \subset F + V + V \subset F + U$.
3. $\mathrm{co}(A)$ is **precompact** due to 2., hence **compact** and **closed** owing of [11, th. 17.3 resp. th. 9.4].
4. For $\mathbf{x} = \sum\limits_{i=1}^{k+1} \tau_i x_i$ with $1 = \sum\limits_{i=1}^{k+1} \tau_i$ w.l.o.g. $\tau_i > 0$, $x_i \in A$ for $1 \le i \le k+1$ and $k > n$ the **kernel** $\ker\Delta$ of the linear mapping $\Delta : \mathbb{R}^{k+1} \to \mathbb{R}^{n+1}$with $\Delta(\tau_1; ...; \tau_{k+1}) = \left(\sum\limits_{i=1}^{k+1} \tau_i x_i; \sum\limits_{i=1}^{k+1} \tau_i\right)$ is at least of dimension $k - n \ge 1$ so that there is a $(t_1; ...; t_{k+1}) \ne 0$ with w.l.o.g. $t_{k+1} = \tau_{k+1}$, $\sum\limits_{i=1}^{k+1} t_i x_i = 0$ and $\sum\limits_{i=1}^{k+1} t_i = 0$ resp. $\sum\limits_{i=1}^{k} t_i = -\tau_{k+1}$. Hence we obtain $x = \sum\limits_{i=1}^{k} (\tau_i - t_i) x_i$ with $1 = \sum\limits_{i=1}^{k} (\tau_i - t_i)$ and since this is true for every $k > n$ we have shown the assertion.

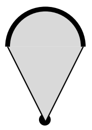
5.12 Separation axioms in dual spaces

For two **non empty, disjoint, convex** and **compact** sets A and B in a topological vector space X with a **separating dual space** X^* there is a $\Lambda \in X^*$ with $\sup\limits_{x\in A} \mathrm{Re}\Lambda x < \inf\limits_{x\in B} \mathrm{Re}\Lambda x$.

Proof: A and B are **closed** due to [11, th. 9.4] resp. 5.2 such that we can invoke 4.4.2 to obtain the desired $\Lambda \in X^*$.

5.13 Extreme sets

An **extreme subset** $S \subset A$ of a **convex** set $A \subset X$ in a topological vector space X is a subset of A containing for every $\mathbf{z} \in S$ the endpoints $\mathbf{x}, \mathbf{y} \in A$ of any line $l(\mathbf{x}; \mathbf{y}) = \{(1-\tau)\mathbf{x} + \tau\mathbf{y} : 0 < \tau < 1\} \subset A$ containing $\mathbf{z} \in l(\mathbf{x}; \mathbf{y})$. Hence A **itself** is an extreme set but also every **vertex, every boundary point** with **tangents outside of** A but also every **complete line segment of the boundary including end points**. An **extreme point** is an extreme set containing exactly one point which consequently does not meet any line in A. Hence extreme points must lie on the **boundary** $\overline{A} \setminus \mathring{A}$ **excluding line segments**. The set of all extreme points of A is $E(A) \subset \overline{A} \setminus \mathring{A}$.



5.14 The Krein-Milman theorem

Any **nonempty, convex and compact** set K in a topological vector space X with **separating dual space** X^* is identical to the **closed convex hull** of its extremal points: $K = \overline{\mathrm{co}}(E(K))$.

Proof:

1. The set $\mathcal{P}$ of all **compact** and **extremal** subsets of K is **not empty** because of $K \in \mathcal{P}$ and also closed concerning **finite intersections**.
2. For every $S \in \mathcal{P}$, $\Lambda \in X^*$ and $\mu_\Lambda = \max\limits_{x\in S} \mathrm{Re}\Lambda x$ we have $S_\Lambda = \{\mathbf{x} \in S : \mathrm{Re}\Lambda\mathbf{x} = \mu_\Lambda\} \in \mathcal{P}$ since for $0 < t < 1$, $\mathbf{x}, \mathbf{y} \in K$ and $t\mathbf{x} + (1-t)\mathbf{y} = \mathbf{z} \in S_\Lambda \subset S$ we have $\mathbf{z} \in S \in \mathcal{P}$, hence $\mathbf{x}, \mathbf{y} \in S$ and therefore $\mathrm{Re}\Lambda\mathbf{x}, \mathrm{Re}\Lambda\mathbf{y} \le \mu_\Lambda$. But since Λ is linear we also have $t\mathrm{Re}\Lambda\mathbf{x} + (1-t)\mathrm{Re}\Lambda\mathbf{y} = \mathrm{Re}\Lambda\mathbf{z} = \mu_\Lambda$, hence $\mathrm{Re}\Lambda\mathbf{x} = \mathrm{Re}\Lambda\mathbf{y} = \mu_\Lambda$ resp. $\mathbf{x}, \mathbf{y} \in S_\Lambda$.
3. Let $\mathcal{P}$ be the family of all **compact** and **extremal** subsets of K. Since $K, K_\Lambda \in \mathcal{P}$ we have $\mathcal{P} \ne \emptyset$ and according to the **Hausdorff maximality principle** [9, th. 14.2.2] there exists a

maximal linearly ordered subfamily $\mathcal{M} \subset \mathcal{P}$ with regard to inclusion. Owing to [11, th. 9.4] and [11, th. 9.2.2] the intersection $M = \bigcap \mathcal{M}$ of its elements is not empty and therefore $M \in \mathcal{P}$. Since $\mathcal{M}$ is maximal no proper subset of M can be contained in $\mathcal{P}$ such that $M \subset S_\Lambda$ for all $\Lambda \in X^*$ due to 1. and 2., i.e. $\Lambda \mathbf{x} = \mu_\Lambda \, \forall \mathbf{x} \in M, \Lambda \in X^*$ and since X^* is **separating** M must be a **single extremal point**. Thus K contains an **extremal point** $M \in E(K)$ which also is an **extremal point of every linear functional** on K.

4. Since K is closed on **account** of [11, th. 9.4] and also **convex** we have $\overline{\mathrm{co}}(E(K)) \subset K$ and again due to [11, th. 9.4] the set $\overline{\mathrm{co}}(E(K))$ is **compact**. For a $\mathbf{x}_0 \in K$ with $\mathbf{x}_0 \notin \overline{\mathrm{co}}(E(K))$ the set $\{\mathbf{x}_0\}$ is **compact** as well as **convex** such that according to 5.12 there is a $\Lambda \in X^*$ with $\mathrm{Re}\Lambda\mathbf{x} < \mathrm{Re}\Lambda\mathbf{x}_0$ for all $\mathbf{x} \in \overline{\mathrm{co}}(E(K))$ contrary to $K_\Lambda \cap E(K) \neq \emptyset$ on account of 3.

5.15 The closed and convex hull of a compact set

The **closed** and **convex hull** $\overline{\mathbf{co}}(K)$ of a **non empty** and **compact** set K in a **locally convex** vector space X with **separating dual space** X^* is identical to the **closed** and **convex hull** of its **extremal points**: $\overline{\mathrm{co}}(K) = \overline{\mathrm{co}}(E(K))$.

Proof: Analogous to the proof of 5.14 with $\overline{\mathrm{co}}(K)$ instead of K (cf. [11, th. 17.7] and 2.4.2).

5.16 The Lebesgue integral as an extreme point

Let $C(I) \subset \mathbb{C}^I$ be the **Banach space** of **complex valued** and **continuous** functions with the **supremum norm** $\|\|$ on the closed real interval $I = [0;1]$ and $M \subset C(I)^*$ the vector space of the **bounded linear** functionals $\Lambda : C(I) \to \mathbb{C}$ with the **weak* topology** induced by the **projections** $p_f^* : M \to \mathbb{C}$ with $p_f^*\Lambda = \Lambda f$ resp. the **subbasis** sets $\{\Lambda \in M : |\Lambda f| < \epsilon\}$ for $f \in C(I)$. The mapping $p : I \to M$ with $p(t) = p_t$ with $p_t(f) = f(t)$ is **weakly* continuous** since the sets $p^{-1}[\{|\Lambda f| < \epsilon\}] = \{t \in I : |f(t)| < \epsilon\}$ for $f \in C(I)$ are **open** in I. Hence due to [11, th. 9.8] the image $K := p[I] = \{p_t : t \in I\}$ is **weakly* compact** in M. The **weak* closure** $\overline{\mathrm{co}}(K)$ of the **convex hull** of K contains every **integral** Λ_μ with $\Lambda_\mu f := \int_I f d\mu$ for **bounded measures** μ on I with $|\mu(I)| = 1$: For every $\epsilon > 0$ and every $f \in C(I)$ there is an $n \geq 1$ such that $|f(\mathbf{x}) - f(\mathbf{y})| < \epsilon$ for $|\mathbf{x} - \mathbf{y}| < \frac{1}{n}$ resp. $\left\| f - \sum\limits_{i=1}^{n} p_{(i-1)/n}(f) \cdot \chi_{I_{i,n}} \right\| < \epsilon$ with $\left| \sum\limits_{i=1}^{n} \mu(I_{i,n}) \right| = 1$ for the **partition** $I = \bigcup\limits_{i=1}^{n} I_{i,n}$ and $I_{i,n} = \left[\frac{i-1}{n}; \frac{i}{n}\right[$, $0 \leq i \leq n-1$ resp. $I_{n,n} = \left[\frac{n-1}{n}; 1\right]$. According to the **dominated convergence theorem** [8, th. 5.14] we obtain $\left| \int_I f d\mu - \sum\limits_{i=1}^{n} \mu(I_{i,n}) \cdot p_{(i-1)/n}(f) \right| < \epsilon$, i.e. $\Lambda_\mu \in \overline{\mathrm{co}}(K)$. The **linear combinations** $c_0\Lambda_\lambda + \sum\limits_{i=1}^{n} c_{t_i} p_{t_i}$ with $c_{t_i} \in \mathbb{C}$, $t_i \in I$ and the **Lebesgue measure** λ generate a vector subspace L with $\mathrm{co}(K) \subset L \subset M$ and hence $\overline{\mathrm{co}}(K) \subset L$ according to 1.5.6. Using L we can show that the **Lebesgue integral** Λ_λ is an **extreme point** of $\overline{\mathrm{co}}(K)$ but not of K: For endpoints $\Lambda_1, \Lambda_2 \in \overline{\mathrm{co}}(K) \subset L$ with $\Lambda_1 = c_1\Lambda_\lambda + \sum\limits_{i=1}^{n} c_{t_t} p_{t_i}$, $\Lambda_2 = c_2\Lambda_\lambda + \sum\limits_{j=1}^{m} d_{s_j} p_{s_j} \in L$ and $\Lambda_\lambda = t\Lambda_1 + (1-t)\Lambda_2$ with $0 < t < 1$ follows $c_1 = c_2 = 1$ since $\Lambda_\lambda \in \overline{\mathrm{co}}(K) \setminus \mathrm{co}(K)$ and hence $t \sum\limits_{i=1}^{n} c_{t_t} p_{t_i} + (1-t) \sum\limits_{j=1}^{m} d_{s_j} p_{s_j} = 0$ such that $t_i = s_j$, $n = m$ and $tc_{t_i} = (1-t) d_{t_j}$ resp. $\sum\limits_{i=1}^{n} c_{t_t} = \sum\limits_{i=1}^{n} d_{t_t} = \frac{t}{1-t} \sum\limits_{i=1}^{n} c_{t_t} = 1$, i.e. $t = \frac{1}{2}$ resp. $c_{t_i} = d_{t_j}$ and hence $\Lambda_1 = \Lambda_2 = \Lambda_\lambda$. Thus we have $\Lambda_\lambda \in E(\overline{\mathrm{co}}(K)) \setminus E(K)$. This case can be excluded if $\overline{\mathrm{co}}(K)$ is also **compact:**

5.17 Milman's theorem

If the **closed convex** hull $\overline{\operatorname{co}}(K)$ of a **compact** set $K \subset X$ in a **locally convex** space X is **compact** it contains all extremal points of K, i.e. $E(\overline{\operatorname{co}}(K)) = E(K)$.

Proof: For a $\mathbf{p} \in E(\overline{\operatorname{co}}(K)) \setminus E(K)$ there is a **balanced convex local** neighbourhood $V \in \mathcal{U}(\mathbf{0})$ with $\left(\mathbf{p}+\overline{V}\right) \cap K = \emptyset$. For the finite cover $K \subset \bigcup\limits_{i=1}^{n} (x_i + V)$ with $\mathbf{x}_i \in K$, $1 \le i \le n$ the sets $A_i = \overline{\operatorname{co}}(K) \cap (\mathbf{x}_i + V) \subset \overline{\operatorname{co}}(K)$ are **convex** as well as **compact** and still cover K. Due to 5.11.1 we have $\overline{\operatorname{co}}(K) \subset \overline{\operatorname{co}}\left(\bigcup\limits_{i=1}^{n} A_i\right) = \operatorname{co}\left(\bigcup\limits_{i=1}^{n} A_i\right)$. On account of $A_i \subset \overline{\operatorname{co}}(K)$ we arrive at $\overline{\operatorname{co}}(K) = \operatorname{co}\left(\bigcup\limits_{i=1}^{n} A_i\right)$ and particularly $\mathbf{p} = \sum\limits_{i=1}^{n} t_i \mathbf{y}_i$ with $t_i > 0$, $\sum\limits_{i=1}^{n} t_i = 1$ and $\mathbf{y}_i \in A_i$. Hence $\mathbf{p} = t_1\mathbf{y}_1 + (1-t_1)\frac{t_2\mathbf{y}_2+\ldots+t_n\mathbf{y}_n}{t_2+\ldots+t_n}$ is a **convex combination** of two points in $\overline{\operatorname{co}}(K)$ such that we conclude that $\mathbf{y}_1 = \mathbf{p}$. But then $\mathbf{p} \in A_i \subset \mathbf{x}_i + \overline{V} \subset K + \overline{V}$ contrary to $\left(\mathbf{p}+\overline{V}\right) \cap K = \emptyset$.

6 Distributions

6.1 The Fréchet space $(\mathcal{C}^\infty(\Omega;\mathbb{R});\mathcal{O}_{c\mathcal{D}})$

In this section we always assume an **open** set $\Omega \subset \mathbb{R}^n$ as domain and the real numbers $\mathbb{R}$as the range of all considered functions until further notice. We start with the vector subspace $\mathcal{C}^\infty \subset \mathcal{C}$ of the **infinitely differentiable real-valued functions** with $D^{\mathbf{p}}f \in \mathcal{C}$ for every $f \in \mathcal{C}^\infty$ and the **differential operator** $D^{\mathbf{p}} := \left(\frac{\partial}{\partial x_1}\right)^{p_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{p_n}$ defined in [6, th. 4.1]. According to 2.6 and 2.8 the **separating** family $\left(\|\ \|_{K_m}\right)_{m\ge0}$ of **seminorms** with $\|f\|_{K_m} = \max\{|D^{\mathbf{p}}f(\mathbf{x})| : \mathbf{x} \in K_m, |\mathbf{p}| \le m\}$ for **compact** $K_m \subset \mathring{K}_{m+1}$, $m \ge 0$ and $\Omega \subset \bigcup_{m\ge0} K_m$ form a **subbasis** $\left\{\|f\|_{K_m} < \frac{1}{m} : f \in \mathcal{C}^\infty\right\}_{m\ge1}$ for the **locally convex** and **metrizable topology** $\mathcal{O}_{c\mathcal{D}}$ **of compact convergence in all derivative**s on the topological vector space $\mathcal{C}^\infty$. This topology obviously includes the weaker **topology** $\mathcal{O}_c \subset \mathcal{O}_{c\mathcal{D}}$ **of compact convergence** and has the following properties:

1. $(\mathcal{C}^\infty;\mathcal{O}_{c\mathcal{D}})$ is **complete** and hence a **Fréchet** space.
2. $(\mathcal{C}^\infty;\mathcal{O}_{c\mathcal{D}})$ has the **Heine-Borel property**.
3. For every **sequence** $(g_n)_{n\ge1} \subset \mathcal{C}^\infty$ **compactly converging in all derivatives** to 0 and any $\varphi \in \mathcal{C}^\infty_K$ with **compact support** $K \subset \Omega$ the **product** $(\varphi g_n)_{n\ge1} \in \mathcal{C}^\infty_K$ **compactly converges in all derivatives** to 0.

Proof:

1. For every $\mathcal{O}_{c\mathcal{D}}$**-Cauchy** sequence $(f_i)_{i\ge1} \subset \mathcal{C}^\infty$ exists a sequence $(k_m)_{m\ge1}$ with $\|f_i - f_j\|_{K_m} < \frac{1}{m}$ for all $i, j \ge k_m$. Since according to 2.13 the space $(\mathcal{C};\mathcal{O}_c)$ is **complete** this implies that every sequence $(D^{\mathbf{p}}f_i)_{i\ge1} \subset \mathcal{C}$ **uniformly** converges on every $K_m \subset \Omega$ to an $f_{\mathbf{p}} \in \mathcal{C}$ and in particular the $(f_i)_{i\ge1}$ uniformly converge on the K_m to an $f \in \mathcal{C}$. According to **Dini's theorem** [11, th. 12.9] the **continuous partial differential quotients** $Q_{j;\delta}f(\mathbf{x}) := \frac{1}{\delta}(f(\mathbf{x}+\delta\mathbf{e}_j) - f(\mathbf{x}))$ for $\delta \to 0$ uniformly converge for every $\mathbf{x} \in K_m \subset \Omega$ to the continuous derivative $D_jf(\mathbf{x})$. The same applies to $Q_{j;\delta}f_i(\mathbf{x}) := \frac{1}{\delta}(f_i(\mathbf{x}+\delta\mathbf{e}_j) - f_i(\mathbf{x})) \overset{\delta\to0}{\to} D_jf_i(\mathbf{x})$ and $Q_{j;\delta}f_i(\mathbf{x}) - Q_{j;\delta}f(\mathbf{x}) \overset{\delta\to0}{\to} D_jf_i(\mathbf{x}) - D_jf(\mathbf{x}) \overset{i\to\infty}{\to} 0$. Hence for every $m \ge 1$ we obtain $|D_jf(\mathbf{x}) - D_jf_i(\mathbf{x})| \le |D_jf(\mathbf{x}) - Q_{j;\delta}f(\mathbf{x})| + |Q_{j;\delta}f(\mathbf{x}) - Q_{j;\delta}f_i(\mathbf{x})| + |Q_{j;\delta}f_i(\mathbf{x}) - D_jf_i(\mathbf{x})| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$ for all $\mathbf{x} \in K_m$, $|\delta| < \delta_m$ and $i \ge k_m$. This implies $D_jf = f_{\mathbf{e}_j}$ an in general $D^{\mathbf{p}}f = f_{\mathbf{p}}$ whence follows $f \in C^\infty$.
2. According to 2.6 the **seminorms** $\|\ \|_{K_m}$ are bounded on the $\mathcal{O}_{c\mathcal{D}}$**-closed** and **bounded** set $\mathcal{F} \subset \mathcal{C}^\infty$ and in particular $|D^{\mathbf{p}}f(\mathbf{x})| \le M_m < \infty$ for all $f \in \mathcal{F}$, $\mathbf{x} \in K_m$ and $|\mathbf{p}| \le m$. The **mean value theorem** [8, th. 11.9] yields $\frac{1}{\delta}|D^{\mathbf{p}}f(\mathbf{x}+\delta\mathbf{e}_j) - D^{\mathbf{p}}f(\mathbf{x})| < \epsilon$ with $\delta := \frac{\epsilon}{M_{m+n}}$ for every

$\epsilon > 0$, $f \in \mathcal{F}$, $\mathbf{x} \in \mathring{K}_m$ and $m \geq 0$, i.e. $\mathcal{F}$ is **equicontinuous** on $\Omega \subset \bigcup_{m\geq 1} K_m$. Since due to [11, th. 9.10] the closure $\overline{\mathcal{F}(\mathbf{x})}$ is **compact** in $\mathbb{C}$ for $\mathbf{x} \in X$ we can apply the **Arzela-Ascoli theorem** [11, th. 19.6] to prove that $\mathcal{F}$ is **compact**.

3. The assertion directly follows from the **Leibniz rule** since for $\varphi \in \mathcal{C}_K^\infty$ there is an $n \geq 1$ with $K \subset K_m$ for every $m \geq n$ and consequently a $C_m < \infty$ with $|D^{\mathbf{q}}\varphi(\mathbf{x})| \leq C_m$ for every multi-index $\mathbf{q} \in \mathbb{N}^n$ with $|\mathbf{q}| \leq m$; $\mathbf{x} \in K_m$ and $m \geq n$. With $\left(\begin{array}{c}\mathbf{p}\\ \mathbf{q}\end{array}\right) \leq D_m < \infty$ for $|\mathbf{p}| \leq m$ this implies
$$\begin{aligned}\|\varphi g_i\|_{K_m} &= \max\left\{\left|\sum_{\mathbf{q}\leq\mathbf{p}} \left(\begin{array}{c}\mathbf{p}\\ \mathbf{q}\end{array}\right) D^{\mathbf{q}}\varphi(\mathbf{x})\, D^{\mathbf{p}-\mathbf{q}} g_i(\mathbf{x})\right| : \mathbf{x} \in K_m, |\mathbf{p}| \leq m\right\}\\ &\leq \max\left\{C_m D_m \sum_{\mathbf{q}\leq\mathbf{p}} \left|D^{\mathbf{p}-\mathbf{q}} g_i(\mathbf{x})\right| : \mathbf{x} \in K_m, |\mathbf{p}| \leq m\right\}\\ &\leq C_m D_m m^n \|g_i\|_{K_m}\end{aligned}$$
and hence the assertion.

6.2 The Fréchet spaces $(\mathcal{C}_K^\infty(\Omega;\mathbb{R});\mathcal{O}_{\mathrm{K}\mathcal{D}})$

For every **compact** $K \subset \Omega$ the family $\mathcal{C}_K^\infty$ of all **infinitely differentiable functions** with **compact support** $\overline{\{f \neq 0\}} \subset K$ is a **closed and hence complete** subspace of $\mathcal{C}^\infty$ relative to $\mathcal{O}_{\mathrm{c}\mathcal{D}}$. The **trace** $\mathcal{O}_{\mathrm{K}\mathcal{D}}$ of the **topology** $\mathcal{O}_{\mathrm{c}\mathcal{D}}$ in $\mathcal{C}_K^\infty \subset \mathcal{C}^\infty$ is the **topology of uniform convergence in all derivatives on the compact set** K.

Proof: Since every $\{\mathbf{x}\} \subset \Omega$ is **compact** and due to [11, th. 18.8] for every open $U \subset \mathbb{C}$ the sets $\{f \in \mathcal{C}^\infty : f(\mathbf{x}) \in U\}$ are open relative to $\mathcal{O}_{\mathrm{c}}$. Hence the **evaluation functional** $\Lambda_{\mathbf{x}} : \mathcal{C}^\infty \to \mathbb{R}$ with $\Lambda_{\mathbf{x}}(f) = f(\mathbf{x})$ for every $\mathbf{x} \in \Omega$ is **continuous** relative to $\mathcal{O}_c$ whence $\ker \Lambda_x$ is **closed** relative to $\mathcal{O}_c$ and especially with reference to the stronger topology $\mathcal{O}_{\mathrm{c}\mathcal{D}}$. Consequently $\mathcal{D}_c = \bigcap_{x\in X\setminus K} \ker \Lambda_x$ is closed in $\mathcal{O}_{\mathrm{c}\mathcal{D}}$ and the assertion then follows from [11, th. 14.2.2].

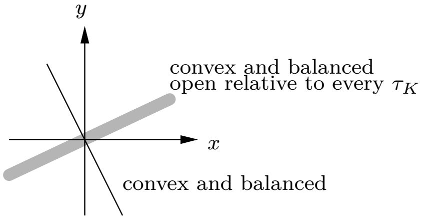
6.3 The space $(\mathcal{C}_c^\infty(\Omega;\mathbb{R});\mathcal{O}_{\mathcal{D}})$

We now consider the **test function space** $\mathcal{C}_c^\infty = \bigcup_{m\geq 1} \mathcal{C}_{K_m}^\infty \subset \mathcal{C}^\infty$ of all infinitely differentiable functions with compact support, i.e. the countable union of all $\mathcal{C}_{K_m}^\infty$ with $K_m = [-\mathbf{m};\mathbf{m}]$. According to 2.6 and 2.8 the **separating** family $(\|\,\|_m)_{m\geq 0}$ of **norms** defined by $\|\varphi\|_m = \max\{|D^{\mathbf{p}}\varphi(\mathbf{x})| : |\mathbf{p}| \leq m\}$ induces a **locally convex** and **metrizable topology** $\mathcal{O}_{\mathcal{D}}$ on $\mathcal{D}$ coinciding on every subspace $\mathcal{C}_K^\infty \subset \mathcal{C}_c^\infty$ with the **trace** $\mathcal{O}_{\mathrm{K}\mathcal{D}}$ of the **topology** $\mathcal{O}_{\mathrm{c}\mathcal{D}}$ induced by the pseudonorms $\|\,\|_{K_m}$ according to 2.14 since for each given K exists an m_0 such that $K \subset K_m$ for every $m \geq m_0$ and for these m the two **nondecreasing** sequences of (semi)norms coincide for every $\varphi \in \mathcal{C}_K^\infty$ whence for every $\epsilon > 0$ and $m \geq m_0$ we have $\left\{\|\varphi\|_{K_m} < \epsilon : \varphi \in \mathcal{C}_K^\infty\right\} = \{\|\varphi\|_m < \epsilon : \varphi \in \mathcal{C}_K^\infty\}$. In particular for **every** $m \in \mathbb{N}$ and $\epsilon > 0$ both $\left\{\|\varphi\|_{K_m} < \epsilon : \varphi \in \mathcal{C}_K^\infty\right\} \supset \{\|\varphi\|_m < \epsilon : \varphi \in \mathcal{C}_K^\infty\}$ are **local neighbourhoods** in $\mathcal{C}_K^\infty$.

This topology is not **complete** any more since e.g. for some **positive** $\varphi \in \mathcal{C}_{[0;1]}^\infty(\mathbb{R};\mathbb{R})$ the limes $\lim\limits_{n\to\infty} \varphi_n$ of the O_D-Cauchy sequence $(\varphi_n)_{n\geq 1} \subset \mathcal{C}_c^\infty(\mathbb{R};\mathbb{R})$ with $\varphi_n(x) = \sum\limits_{k=1}^{n} \frac{1}{k}\varphi(x-k)$ does not have compact support. So in the following section we construct a **stronger** topology $\mathcal{O}_{\mathcal{T}} \supset \mathcal{O}_{\mathcal{D}}$ on $\mathcal{C}_c^\infty$ also coinciding with $\mathcal{O}_{\mathrm{K}\mathcal{D}}$ on every $\mathcal{C}_K^\infty$ which is **complete** but **not metrizable** any more.

6.4 The test function space $(\mathcal{D}; \mathcal{O}_{\mathcal{T}})$

We consider the family $\mathcal{W}$ of all **convex and balanced** sets $W \subset \mathcal{C}_c^\infty$ such that $\mathcal{C}_K^\infty \cap W \in \mathcal{O}_{\mathrm{K}\mathcal{D}}$ for every **compact** $K \subset \Omega$, i.e. for each K the infinitely differentiable functions in W with compact support in K form an **open, convex and balanced neighbourhood of the 0-function** relative to $\mathcal{O}_{\mathrm{K}\mathcal{D}}$. The family of all sets $\varphi + W$ with $\varphi \in \mathcal{C}_c^\infty$ and $W \in \mathcal{W}$ is the **basis** for a **topology** $\mathcal{O}_{\mathcal{T}}$ on the **test function space** $\mathcal{D} = (\mathcal{C}_c^\infty; O_T)$ and $\mathcal{D}$ is a **locally convex vector space**. Also for every **compact** $K \subset \Omega$ the **trace topology** $\mathcal{O}_{\mathcal{T}}|_{\mathcal{C}_K^\infty}$ coincides with $\mathcal{O}_{\mathrm{K}\mathcal{D}}$ on the subspace $\mathcal{D}_K = \left(\mathcal{C}_K^\infty; \mathcal{O}_{\mathcal{T}}|_{\mathcal{C}_K^\infty}\right) = (\mathcal{C}_K^\infty; \mathcal{O}_{\mathrm{K}\mathcal{D}})$.



Proof: For every $\psi \in \bigcup\limits_{i \in I} (\varphi_i + W_i) \cap \bigcup\limits_{j \in J} (\varphi_j + W_j)$ with $\varphi_k \in \mathcal{D}$ resp. $W_k \in \mathcal{W}$ for $k \in I \cup J$ there are $i \in I$ resp. $j \in J$ such that $\psi \in \varphi_k + W_k$ for $k \in \{i; j\}$ and a **compact** $K \subset \Omega$ with $\varphi_k; \psi \in \mathcal{D}_K$. Since $\mathcal{D}_K \cap W_k$ is **open** in $\mathcal{D}_K$ there are $\delta_k > 0$ such that $\psi - \varphi_k \in (1 - \delta_k) W_k$. The **convexity** and **balance** of W_k implies that $\psi - \varphi_k + \delta_k W_k \subset (1 - \delta_k) W_k + \delta_k W_k = W_k$, so that $\psi + \delta_k W_k \subset \varphi_k + W_k$ whence

$$\psi + W \subset (\varphi_i + W_i) \cap (\varphi_j + W_j) \subset \bigcup_{i \in I} (\varphi_i + W_i) \cap \bigcup_{j \in J} (\varphi_j + W_j)$$

with $W = \delta_i W_i \cap \delta_j W_j$. Thus the intersection $\bigcup\limits_{i \in I} (\varphi_i + W_i) \cap \bigcup\limits_{j \in J} (\varphi_j + W_j)$ is **open** in $\mathcal{O}_{\mathcal{T}}$ and $\mathcal{O}_{\mathcal{T}}$ is a **topology**.

The **addition is $\mathcal{O}_{\mathcal{T}}$-continuous** since the **convexity** of every $W \in \mathcal{W}$ implies that $\left(\varphi + \frac{1}{2}W\right) + \left(\psi + \frac{1}{2}W\right) = \varphi + \psi + W$. The **scalar multiplication is continuous** since for $c \in \mathbb{C}$, $\varphi \in \mathcal{D}$ and $W \in \mathcal{W}$ exists a $\delta > 0$ such that $\delta\varphi \in \frac{1}{2}W$ whence for every $b \in B_\delta(c)$ and $\psi \in \varphi + \frac{1}{2(|c| + \delta)}W$ follows $b\psi - c\varphi = b(\psi - \varphi) + (c - b)\varphi \in W$. Due to 1.1 we conclude that $\mathcal{D}_{\mathcal{T}}$ is a **topological vector space.**

For every $\psi; \xi \in \mathcal{D}$ and every **compact** $K \subset \Omega$ the sets $\{\varphi \in \mathcal{D}_K : \|\varphi\|_0 < \|\psi - \xi\|_0\}$ are **open** in $\mathcal{O}_{\mathrm{K}\mathcal{D}}$. This implies $W = \{\varphi \in \mathcal{D}_{\mathcal{K}} : \|\varphi\|_0 < \|\psi - \xi\|_0\} \in \mathcal{U}_{\mathcal{T}}(0)$ with $\psi \notin \xi + W$ whence every **atom** $\{\psi\}$ is **closed** relative to $\mathcal{O}_{\mathcal{T}}$, i.e. $\mathcal{W}$ is a **local convex basis**.

Finally we prove that $\mathcal{O}_{\mathcal{T}}|_{\mathcal{C}_K^\infty} = \mathcal{O}_{\mathrm{K}\mathcal{D}}$: Every $\varphi \in O \cap \mathcal{D}_K$ for an open $O \in \mathcal{O}_{\mathcal{T}}$ has a $\mathcal{O}_{\mathcal{T}}$-open neighbourhood $\varphi + W_\varphi \subset O$ with $W_\varphi \in \mathcal{W}$ whence $\varphi + W_\varphi \cap \mathcal{D}_K \subset O \cap \mathcal{D}_K$ is $\mathcal{O}_{\mathrm{K}\mathcal{D}}$-open and so is $O \cap \mathcal{D}_K$. Conversely for every $\varphi \in E \in \mathcal{O}_{\mathrm{K}\mathcal{D}}$ there are $m_\varphi \in \mathbb{N}$ and $\delta_\varphi > 0$ such that $B_{m_\varphi; \delta_\varphi}(\varphi) \cap \mathcal{D}_K \subset E$ and $V = \bigcup_{\varphi \in E} B_{m_\varphi; \delta_\varphi}(\varphi)$ is a $\mathcal{O}_{\mathcal{T}}$-open set with $V \cap \mathcal{D}_K = E$ whence follows $E \in \mathcal{O}_{\mathcal{T}}|_{\mathcal{D}_K}$.

Note: $\mathcal{O}_{\mathcal{T}}$ is the **strongest** topology on $\mathcal{D}$ such that $(\mathcal{D}; \mathcal{O}_{\mathcal{T}})$ is a **locally convex vector space**; every **injection** $i_K : \mathcal{D}_K \to \mathcal{D}$ is **continuous** and every map $\Lambda : \mathcal{D} \to \mathcal{G}$ into another locally convex vector space is **continuous** iff every $\Lambda \circ i_K : \mathcal{D}_K \to \mathcal{G}$ is continuous on $\mathcal{D}_K$. This construction is sometimes called the **inductive limit** or *Finaltopologie* on $\mathcal{D}$ relative to the subspaces $\mathcal{D}_K$.

6.5 Completeness of the test function space

The topology $\mathcal{O}_{\mathcal{T}}$ of the **test functions** $\mathcal{D}$ has the following properties:

1. $\mathcal{W}$ is a **local basis** of $\mathcal{O}_{\mathcal{T}}$ and contains the **open balls** $B_m(0) = \left\{\varphi \in \mathcal{D} : \|\varphi\|_m < \frac{1}{m}\right\}$ for every $m \in \mathbb{N}$.
2. A **convex balanced** $W \subset \mathcal{D}$ is **open** in $\mathcal{O}_{\mathcal{T}}$ iff $W \in \mathcal{W}$.
3. For every **bounded** $E \subset \mathcal{D}$ there is a **compact** $K \subset \Omega$ such that $E \subset \mathcal{D}_K$ and for every $m \geq 0$ exists a constant $M_m < \infty$ with $\|\varphi\|_m \leq M_m$ for every $\varphi \in E$, i.e. E is **bounded** in $\mathcal{O}_{K\mathcal{D}}$.
4. $\mathcal{D}$ has the **Heine-Borel-property**.

5. For every **Cauchy sequence** $(\varphi_i)_{i\geq 1} \subset \mathcal{D}$ there is a **compact** $K \subset \Omega$ such that $(\varphi_i)_{i\geq 1} \subset \mathcal{D}_{\mathrm{K}}$ and $\lim\limits_{i;j\to\infty} \|\varphi_i - \varphi_j\|_m = 0$ for every $m \geq 0$. Hence $\mathcal{D}$ is **complete** and a **Fréchet** space.
6. The sets $\mathcal{D}_K$ are **nowhere dense** in $\mathcal{D}$ and $\mathcal{D}$ is of **first category**, hence **not metrizable**.

Proof:

1. Obvious from 6.4.
2. If $W \subset \mathcal{D}$ is **open, convex** and **balanced** then for every $\varphi \in W$ there is a $W_\varphi \in \mathcal{W}$ with $\varphi + W_\varphi \subset W$ and $W_\varphi \cap \mathcal{D}_K \in \mathcal{O}_{\mathrm{K}\mathcal{D}}$ whence $W \cap \mathcal{D}_K = \bigcup_{\varphi \in W\cap\mathcal{D}_K} (\varphi + W_\varphi) \in \mathcal{O}_{\mathrm{K}\mathcal{D}}$. The converse is obvious.
3. Assuming there is a set $E \subset \mathcal{D}$ **not** contained in any $\mathcal{D}_K$. Then there are sequences $(\varphi_m)_{m\geq 1} \subset E$ and $(\mathbf{x}_m)_{m\geq 1} \subset \Omega$ without limit point in Ω such that $\varphi_m(\mathbf{x}_m) \neq 0$ for all $m \in \mathbb{N}$. Since every K contains only finitely many $\mathbf{x}_m$ and the sets $\left\{\varphi \in \mathcal{D}_K : |\varphi(\mathbf{x}_m)| < \frac{|\varphi_m(\mathbf{x}_m)|}{m}\right\}$ are open in $\mathcal{D}_{K\mathcal{D}}$ we conclude that $W = \bigcap\limits_{m\geq 1} \left\{\varphi \in \mathcal{D} : |\varphi(\mathbf{x}_m)| < \frac{|\varphi_m(\mathbf{x}_m)|}{m}\right\} \in \mathcal{W}$. Since $\varphi_m \notin mW$ no multiple of W contains E whence according to 1.4 the set E cannot be bounded. Hence we conclude that every bounded $E \subset \mathcal{D}$ is contained in some $\mathcal{D}_K$. By 6.5.3 the set E is also bounded in $\mathcal{D}_K$ whence according to 6.2 follows $\sup\{\|\varphi\|_m : \varphi \in E\} < \infty$ for every $m \in \mathbb{N}$.
4. According to 6.5.3 every **closed** and **bounded** $E \subset \mathcal{D}$ lies in some $\mathcal{D}_K$ where it is **compact** due to 6.1.2. Compactness relative to a **trace topology** always extends to the original topology whence by 6.5.3 the set E is **compact** in $\mathcal{D}$. Conversely a **compact** $E \subset \mathcal{D}$ due to the **Hausdorff property** is **closed** and it is **bounded** according to 6.5.3.
5. Since every Cauchy sequence is **bounded** 6.5.3 means that the complete sequence lies in some $\mathcal{D}_K$ and by 6.5.3 it is a Cauchy sequence relative to $\mathcal{O}_{\mathrm{K}\mathcal{D}}$ whence due to 6.1 it **converges uniformly with all derivatives on** K to a function in $\mathcal{D}_K \subset \mathcal{D}$. By the definition 6.3 this convergence also holds in $\mathcal{O}_{\mathcal{T}}$.
6. According to [8, th. 13.2] the functions $\alpha_{\mathbf{ab}} : \Omega \to [0;1]$ defined by

$$\alpha_{\mathbf{ab}}(\mathbf{x}) = \prod_{i=1}^{n} \alpha_{a_i b_i}(x_i) \text{ for } \alpha_{ab}(x) = \begin{cases} e^{\frac{1}{(x-a)(x-b)}} & \text{if } a \leq x \leq b \\ 0 & \text{else} \end{cases}$$

are **infinitely differentiable** with **compact support** $[\mathbf{a};\mathbf{b}] \subset \Omega$ and **finite norms** $\|\alpha_{\mathbf{ab}}\|_m < \infty$ for $m \in \mathbb{N}$. They can be used to construct an $\varphi + \alpha_{m;\epsilon} \in B_{m;\epsilon}(\varphi) \cap \mathcal{D} \setminus \mathcal{D}_K$ for every $\varphi \in \mathcal{D}_K$, compact $K \subset \Omega$, $\epsilon > 0$ and $m \in \mathbb{N}$ whence $\mathcal{D}_K$ **has no interior points** in $\mathcal{D}$. According to [11, th. 16.1] $\mathcal{D} = \bigcup_{m\geq 1} \mathcal{D}_{K_m}$ for $K_m = [-\mathbf{m};\mathbf{m}]$ is of **first category** relative to $\mathcal{O}_{\mathcal{T}}$ and due to [11, th. 16.2.1] it is not a **Baire space**. Assuming that the complete space $\mathcal{D}$ is metrizable creates a contradiction to **Baire's theorem** [11, th. 16.4].

6.6 Distributions

A **functional** $\Lambda : \mathcal{D} \to \mathbb{C}$ on the topological vector space $\mathcal{D}$ of the **test functions** as defined in 6.3 is called a **distribution** iff one of the following equivalent properties hold:

1. Λ is **continuous**.
2. Λ is **bounded**, i.e. for every **compact** $K_m = [-\mathbf{m};\mathbf{m}]$ exists a $C_m < \infty$ such that $|\Lambda\varphi| \leq C_m \|\varphi\|_m$ for every $\varphi \in \mathcal{D}$.
3. If $(\varphi_n)_{n\geq 1}$ converges to 0 in $\mathcal{D}$ then $(\Lambda\varphi_n)_{n\geq 1}$ converges to 0 in $\mathbb{C}$.
4. The restriction $\Lambda|_{\mathcal{D}_K}$ is **continuous** for every **compact** $K \subset \Omega$.
5. For every **compact** $K \subset \Omega$ exist an $m_K \in \mathbb{N}$ and a $C_{m_K} < \infty$ such that $|\Lambda\varphi| \leq C_{m_K} \|\varphi\|_{m_K}$ for every $\varphi \in \mathcal{D}_K$.

The **vector space of all distributions** is denoted by $\mathcal{D}'$ and as usual we tacitly assume the domain $\mathcal{D} = \left(\mathcal{C}_c^\infty(\Omega;\mathbb{R});\mathcal{O}_T\right)$ of the **test functions** on some open $\Omega \subset \mathbb{R}^n$. If there is a common m_K satisfiying the estimate 5 for **all** comapct $K \subset \Omega$ it is denoted as the **order** of the distribution. If no such m exists the order is **infinite**.

Proof:

1. $\Rightarrow$ 2.: cf. 1.10.3.

2. $\Rightarrow$ 3.: Follows from 6.5.3 and the **metrizability** of the topology $\mathcal{O}_{\mathcal{T}}|_{\mathcal{D}_K} = \mathcal{O}_{\mathrm{K}\mathcal{D}} = \mathcal{O}_{c\mathcal{D}}|_{\mathcal{D}_K}$ according to 6.2, 6.3 and 6.5.3 allowing the application of 1.10.5.

3. $\Rightarrow$ 4.: Every sequence $(\varphi_n)_{n\geq1}$ converging to 0 in $\mathcal{O}_{\mathcal{T}}$ according to 6.5.3 also converges to 0 with regard to $\mathcal{O}_{K\mathcal{D}}$ whence by the hypothesis $(\Lambda\varphi_n)_{n\geq1}$ converges to 0 in $\mathbb{R}$. Since $\mathcal{D}_K$ is **metrizable** we can apply 1.10.1.

4. $\Rightarrow$ 5.: Assuming there is a compact K and $(\varphi_n)_{n\geq1} \subset \mathcal{D}_K$ with $|\Lambda\varphi_n| \geq n\,\|\varphi_n\|_n$ the functions $\psi_n = \frac{\varphi_n}{n\|\varphi_n\|_n} \in \mathcal{D}_K$ obviously converge to 0 on $\mathcal{O}_{K\mathcal{D}}$ such that $\Lambda\psi_n > 1$ for every $n \geq 1$ yields a contradiction to the hypothesis.

5. $\Rightarrow$ 3.: For every $(\varphi_n)_{n\geq1} \subset \mathcal{D}$ converging to 0 relative to $\mathcal{O}_{\mathcal{T}}$ according to 6.5.5 there is a compact K with $(\varphi_n)_{n\geq1} \subset \mathcal{D}_K$. Due to the hypothesis there is a $C_{m_K} < \infty$ such that $|\Lambda\varphi_n| \leq C_{m_K}\,\|\varphi_n\|_{m_K}$ for every $n \geq 1$. Also for every $\epsilon > 0$ there is an $n_{\epsilon;m_K} \geq 1$ such that $\|\varphi_n\|_{m_K} < \frac{\epsilon}{C_m}$ for every $n \geq n_{\epsilon;m_K}$ whence we obtain $|\Lambda\varphi_n| \leq \epsilon$ for every $n \geq n_{\epsilon;m_K}$ thus proving the assertion.

4. $\Rightarrow$ 1.: According to [11, th. 3.2] the **sequential continuity** 3. of the mapping $\Lambda : \mathcal{D} \to \mathbb{R}$ generally implies the **general continuity** 1. iff $\mathcal{D}$ is **first countable**, i.e. due to 2.8 **metrizable** in contradiction to 6.5.6. However in the special case of $\mathcal{D}$ the conclusion is valid since the general continuity 4. on every $\mathcal{D}_K$ implies that every intersection $W_\epsilon \cap \mathcal{D}_K$of the inverse image $W_\epsilon = \{|\Lambda\varphi| < \epsilon : \varphi \in \mathcal{D}\}$ of the **convex balanced** and **local** basis sets $]-\epsilon;\epsilon[$ is **open** in $\mathcal{D}_K$. Due to the **linearity** of Λ it also **convex balanced** and **local** in $\mathcal{D}_K$ and the definition in 6.4 of $\mathcal{O}_{\mathcal{T}}$ shows that W_ϵ is open in $\mathcal{O}_{\mathcal{T}}$. Hence Λ is continuous at the origin and due to 1.10.2 the continuity extends to the entire space $\mathcal{D}$.

6.7 Functions and measures as distributions

1. A **Lebesgue measurable complex-valued** function $f : \Omega \to \mathbb{C}$ is **locally integrable** on Ω iff $\int_K |f|\,d\lambda < \infty$ for every compact $K \subset \Omega$ and the vector space of these functions is denoted as $L^{loc}(\Omega;\mathbb{C})$. For every $f \in L^{loc}(\Omega;\mathbb{C})$ the linear functional $\Lambda_f : \mathcal{D} \to \mathbb{C}$ uniquely determined by $\Lambda_f\varphi = \int \varphi f d\lambda$ is a **distribution of order** 0 with the $\mathcal{O}_{\mathcal{T}}$**-continuity** following from $|\Lambda_f\varphi| \leq \int_K |f|\,d\lambda\cdot\|\varphi\|_0 \leq \int_K |f|\,d\lambda\cdot\|\varphi\|_n$ for K with $\varphi \in \mathcal{D}_K$ and $n \geq 0$ and 6.6.5. Conversely in the case of its existence the representing $f \in L^{loc}(\Omega;\mathbb{C})$ is λ-a.e. determined by the distribution $\Lambda_f \in \mathcal{D}'$ with $\Lambda\varphi = \int \varphi f d\lambda$. Hence we have a **homomorphism** $D : L^{loc}(\Omega;\mathbb{C}) \to \mathcal{D}'$ with $D(f) = \Lambda_f$ which is **neither surjective nor injective**. Its **kernel** contains every λ-a.e. vanishing Borel-measurable function $f : \mathbb{R}^n \to \mathbb{C}$.
   Proof: For every **right-open interval** $[\mathbf{a};\mathbf{b}[\in \mathcal{I}^n$ (cf. [8, th. 7.7]) due to [8, th. 13.3] there is a sequence $(\varphi_k)_{k\geq1} \subset \mathcal{D}$ with $[\mathbf{a};\mathbf{b}] \prec \varphi_k \prec \left]\mathbf{a} - \sum\limits_{i=1}^{n}\frac{\mathbf{e}_i}{k};\mathbf{b} + \sum\limits_{i=1}^{n}\frac{\mathbf{e}_i}{k}\right[$ which λ-a.e. converges to $\chi_{[\mathbf{a};\mathbf{b}[}$ such that for every $f \in L^{loc}(\Omega;\mathbb{C})$ the product $f\cdot\varphi_k$ λ-a.e. converges to $f\cdot\chi_{[\mathbf{a};\mathbf{b}[}$ whence by **dominated convergence** [8, th. 5.14] the hypothesis $\int \varphi f d\lambda = 0$ for every $\varphi \in \mathcal{D}$ implies $\int_{[\mathbf{a};\mathbf{b}[} f d\lambda = 0$ and since the right-open intervals generate the **Borel-σ-algebra** $\mathcal{B}(\mathbb{R}^n)$ we can apply [8, th. 5.23] to conclude λ-a.e. $f = 0$.

2. In the case of a **continuous** $f \in \mathcal{C}(\Omega;\mathbb{C}) \subset L^{loc}(\Omega;\mathbb{C})$ the hypothesis $\int \varphi f d\lambda = 0$ for every $\varphi \in \mathcal{D}$ implies $f = 0$ **everywhere** since the assumption w.l.o.g. $f(\boldsymbol{x}) = \varepsilon > 0$ for some $\boldsymbol{x} \in \Omega$ together with the **continuity** of f in $\mathbf{x}$ and the **locally compact** character of Ω yields a $\delta > 0$ with $f(\mathbf{x}) \geq \frac{\epsilon}{2}\,\forall\mathbf{x} \in \overline{B_\delta}$ such that with [8, th. 13.3] we find a $\varphi \in \mathcal{D}$ with $\overline{B_{\delta/2}} \prec \varphi \prec B_\delta$ resulting in $\Lambda_f\varphi > 0$. Hence in this case the **homomorphism** $D : \mathcal{C}(\Omega;\mathbb{C}) \to \mathcal{D}'$ with $D(f) = \Lambda_f$ is **still not surjective but injective.**

3. According to [8, def. 9.1 and def. 10.1] every complex **Borel measure** $\mu : \mathcal{B}(\Omega) \to \mathbb{C}$ is **finite** and so is every **real** Borel measure $\mu : \mathcal{B}(\Omega) \to [0;\infty]$ on **compact** $K \subset X$. Hence $\Lambda_\mu : \mathcal{D} \to \mathbb{C}$ defined by $\Lambda_\mu \varphi = \int \varphi d\mu$ is a **distribution of order** 0 with the $\mathcal{O}_{\mathcal{T}}$ **-continuity** following from $|\Lambda_\mu \varphi| \leq \mu(K) \cdot \|\varphi\|_0$ for every compact K. Since $\mathcal{D}' \subset (\mathcal{C}_c(\Omega;\mathbb{C}))^*$ is a vector subspace of the **dual space** of the **continuous complex-valued** functions $\varphi : \Omega \to \mathbb{C}$ with **compact support** due to the **Riesz representation theorem** [8, th. 10.13] we have an **injective homomorphism** $M : \mathcal{D}' \to \mathcal{M}_0^*(\mathcal{L}(\Omega);\mathbb{C})$ with $M(\Lambda) = \mu_\Lambda$ determined by $\Lambda\varphi = \int \varphi d\mu = \int \varphi \frac{d\mu}{d\|\mu|} d|\mu|$ into the **Banach space of the complete and regular complex Borel measures** on the **Lebesgue** σ**-algebra** $\mathcal{L}(\Omega) \supset \mathcal{B}(\Omega)$ defined in [8, th. 10.11].

$$\begin{tikzcd}
 & L^{loc}(\Omega;\mathbb{C}) \arrow[dd, "D"] & \\
 & & L^1(\Omega;\mathbb{C}) \arrow[ul, "\iota"'] \\
 & \mathcal{D}' \arrow[ur, "D^{-1}(inj)"'] & \\
C(\Omega;\mathbb{C}) \arrow[uuur, "\iota"] \arrow[ur, "D(inj)"'] & &
\end{tikzcd}$$

3. In the case of a λ**-absolutely continuous measure** μ according to [8, th. 9.8] there is a uniquely determined **Radon-Nikodym derivative** $f \in L^1(\Omega;\mathbb{C})$ such that $\Lambda f = \int \varphi f d\lambda$ for every $\varphi \in \mathcal{D}$. On account of the continuity condition requested by μ the **injective homomorphism** $D^{-1} : \mathcal{D}' \to L^1(\Omega;\mathbb{C})$ is **not surjective** .

6.8 Differentiation of distributions

1. For every **multi-index** $\mathbf{p} \in \mathbb{N}^m$ he **p-th distribution derivative** $D^{\mathbf{p}}\Lambda : \mathcal{D} \to \mathbb{C}$ of a distribution $\Lambda \in \mathcal{D}'$ is defined by $D^{\mathbf{p}}\Lambda\varphi = (-1)^{|\mathbf{p}|}\Lambda D^{\mathbf{p}}\varphi$ for every $\varphi \in \mathcal{D}$. Due to 6.6.5 and since $|D^{\mathbf{p}}\Lambda\varphi| = |\Lambda D^{\mathbf{p}}\varphi| \leq C_m \cdot \|D^{\mathbf{p}}\varphi\|_m \leq C_m \cdot \|\varphi\|_{m+|\mathbf{p}|}$ we conclude that $D^{\mathbf{p}}\Lambda \in \mathcal{D}'$.
2. Also for every $\mathbf{p};\mathbf{q} \in \mathbb{N}^n$ the distributive differential operators **commutate**, i.e.
$$\begin{aligned} D^{\mathbf{p}}D^{\mathbf{q}}\Lambda\varphi &= (-1)^{|\mathbf{q}|} D^{\mathbf{p}}\Lambda D^{\mathbf{q}}\varphi \\ &= (-1)^{|\mathbf{p}|+|\mathbf{q}|}\Lambda D^{\mathbf{p}}D^{\mathbf{q}}\varphi \\ &= (-1)^{|\mathbf{p}+\mathbf{q}|}\Lambda D^{\mathbf{p}+\mathbf{q}}\varphi \\ &= (-1)^{|\mathbf{q}+\mathbf{p}|}\Lambda D^{\mathbf{q}+\mathbf{p}}\varphi \\ &= (-1)^{|\mathbf{p}|} D^{\mathbf{q}}\Lambda D^{\mathbf{p}}\varphi \\ &= D^{\mathbf{q}}D^{\mathbf{p}}\Lambda\varphi. \end{aligned}$$
3. For every p times differentiable function $f \in \mathcal{C}^p(\Omega;\mathbb{C})$ and $\mathbf{p} \in \mathbb{N}^n$ with $|\mathbf{p}| \leq p$ holds $D^{\mathbf{p}}\Lambda_f = \Lambda_{D^{\mathbf{p}}f}$ since for every $\varphi \in \mathcal{D}_K$ with compact support $K \subset [-\mathbf{n};\mathbf{n}]$ **Fubini's theorem** [8, th. 8.5] and an **integration by parts** [6, th. 1.5] applied to the partial derivative in the w.l.o.g. first coordinate yield
$$\begin{aligned} D^{(1;0\ldots0)}\Lambda_f\varphi &= \int \varphi \cdot D^{(1;0\ldots0)} f d\lambda \\ &= \int_{-n}^{n} \ldots \int_{-n}^{n} \varphi(x_1;\ldots;x_n) \cdot \frac{\partial f(x_1;\ldots;x_n)}{\partial x_1} dx_1 \ldots dx_n \\ &= \int_{-n}^{n} \ldots \int_{-n}^{n} \left([\varphi(\mathbf{x}) \cdot \varphi(\mathbf{x})]_{-n}^{n} - \int_{-n}^{n} \frac{\partial\varphi(x_1;\ldots;x_n)}{\partial x_1} \cdot f(x_1;\ldots;x_n) dx_1 \right) dx_2 \ldots dx_n \\ &= -\int_{-n}^{n} \ldots \int_{-n}^{n} \frac{\partial\varphi(x_1;\ldots;x_n)}{\partial x_1} \cdot f(x_1;\ldots;x_n) dx_1 \ldots dx_n \\ &= -\int D^{(1;0\ldots0)}\varphi \cdot f f d\lambda \\ &= \Lambda_{D^{(1;0\ldots0)}f}\varphi \end{aligned}$$
which by induction and the preceding formula 2. lead to the general result.

4. A further generalization yields that for every **differential operator** $D = \sum\limits_{|\mathbf{p}|\leq m} \alpha_{\mathbf{p}} D^{\mathbf{p}}$ of **order** $m \in \mathbb{N}$ with **coefficients** $\alpha_{\mathbf{p}} \in C^{\infty}(\Omega; \mathbb{C})$ exists a **uniquely determined adjoint** D^* defined by $D^*\varphi = \sum\limits_{\mathbf{p}|\leq m} (-1)^{|\mathbf{p}|} D^{\mathbf{p}}(\alpha_{\mathbf{p}}\varphi)$ such that $\int_{\Omega} f(D\varphi)\, d\lambda = \int_{\Omega} (D^* f)\, \varphi d\lambda$ for every $f \in C^{\infty}(\Omega; \mathbb{C})$ and $\varphi \in \mathcal{D}$.

6.9 Functions of bounded variation

According to the **Lebesgue differentiatiom theorem** for complex functions [8, th. 11.7] a **complex-valued left-continuous** function f of **bounded variation** in $\Omega = [a;b] \subset \mathbb{R}$ has a λ-a.e. defined **derivative** Df and due to [8, th. 12.1] it determines a **complex Lebesgue-Stieltjes measure** μ with $\mu([x;y[) = f(y) - f(x)$ for $a \leq x \leq y \leq b$. For these functions holds $D\Lambda_f = \Lambda_{Df}$ iff f is **absolutely continuous**.

Proof: For very $\varphi \in \mathcal{D}$ we have $\varphi(a) = \varphi(b) = 0$ and in particular $\int_a^b \varphi'(x)\, dx = 0$ whence by **Fubini's theorem** [8, th. 8.5]

$$\begin{aligned}
D\Lambda_f \varphi &= -\int_a^b \varphi'(x)\, f(x)\, dx \\
&= \int_a^b f(b)\, \varphi'(x)\, dx - \int_a^b f(x)\, \varphi'(x)\, dx \\
&= \int_a^b \varphi'(x)\, (f(b) - f(x))\, dx \\
&= \int_a^b \left(\int_x^b \varphi'(x)\, \mu dy \right) dx \\
&= \int_{\{a<x<y<b\}} \varphi'(x)\, \mu dy dx \\
&= \int_a^b \left(\int_a^y \varphi'(x)\, dx \right) \mu dy \\
&= \int_a^b (\varphi(y) - \varphi(a))\, \mu dy \\
&= \int_a^b \varphi(y)\, \mu dy \\
&= \Lambda_{\mu}\varphi.
\end{aligned}$$

Hence it remains to prove that $\int \varphi d\mu = \int \varphi D f d\lambda = \int \varphi \frac{df}{d\lambda} d\lambda$ for every $\varphi \in \mathcal{D}$ iff f is absolutely continuous: $\Rightarrow$ directly follows from **fundamental theorem of calculus** [8, th. 12.10.3] since the existence $\frac{df}{d\lambda}$ implies the absolute continuity of f. $\Leftarrow$ is a consequence of the other direction [8, th. 12.10.2] of the same theorem and [8, th. 12.6].

Example: On the one hand the **distribution derivative** of the **Heaviside function** $H = \chi_{[0;\infty[}$ is determined by $D\Lambda_H \varphi = -\int \varphi'(x)\, H(x)\, dx = \varphi(0) - 0$ due to the compact support of $\varphi \in \mathcal{D}$ whence follows $D\Lambda_H = \epsilon_0 = \Lambda_{\delta_{\{0\}}}$ with the **evaluation functional** $\epsilon_0 : D \to \mathbb{R}$ resp. the **dirac measure** $\delta_{\{0\}} = \begin{cases} 1 & \text{for } x = 0 \\ 0 & \text{for } x \neq 0 \end{cases}$ defined by $\epsilon_0 \varphi = \Lambda_{\delta_{\{0\}}} \varphi = \int \varphi d\delta_{\{0\}} = \varphi(0)$. On the other hand the **ordinary derivative** of the Heaviside function is only λ-a.e. defined as $H' = 0$ with $\Lambda_{H'} = 0$.

6.10 Multiplication by a function

For every $g \in \mathcal{C}^{\infty}(\Omega;\mathbb{R})$ and $\Lambda \in \mathcal{D}'$ the **product** $g\Lambda$ defined by $g\Lambda\varphi = \Lambda(g\varphi)$ for every $\varphi \in \mathcal{D}$ is a **distribution** and the **Leibniz rule** holds in the form of $D^{\mathbf{p}}(g\Lambda) = \sum_{\mathbf{q}\leq\mathbf{p}} \binom{\mathbf{p}}{\mathbf{q}} (D^{\mathbf{p}-\mathbf{q}}g)(D^{\mathbf{q}}\Lambda)$ for every $\mathbf{p} \in \mathbb{N}^n$.

Proof: Considering $(-1)^{|\mathbf{q}-\mathbf{r}||} = (-1)^{|\mathbf{q}|+|\mathbf{r}||}$ the **Leibniz rule** [6, t. 4.2] for functions yields

$$\begin{aligned}
\mathbf{u}^{\mathbf{p}} &= [\mathbf{v} + (-\mathbf{v} + \mathbf{u})]^{\mathbf{p}} \\
&= \sum_{\mathbf{q}\leq\mathbf{p}} \binom{\mathbf{p}}{\mathbf{q}} \mathbf{v}^{\mathbf{p}-\mathbf{q}} \sum_{\mathbf{r}\leq\mathbf{q}} \binom{\mathbf{q}}{\mathbf{r}} (-1)^{|\mathbf{q}-\mathbf{r}|} \mathbf{v}^{\mathbf{q}-\mathbf{r}} \mathbf{u}^{\mathbf{r}} \\
&= \sum_{\mathbf{r}\leq\mathbf{p}} (-1)^{|\mathbf{r}|} \mathbf{v}^{\mathbf{p}-\mathbf{q}} \mathbf{u}^{\mathbf{r}} \sum_{\mathbf{r}\leq\mathbf{q}\leq\mathbf{p}} (-1)^{|\mathbf{q}|} \binom{\mathbf{p}}{\mathbf{q}} \binom{\mathbf{q}}{\mathbf{r}}
\end{aligned}$$

whence

$$\sum_{\mathbf{r}|\leq\mathbf{q}\leq\mathbf{p}} (-1)^{|\mathbf{q}|} \binom{\mathbf{p}}{\mathbf{q}} \binom{\mathbf{q}}{\mathbf{r}} = \begin{cases} (-1)^{|\mathbf{p}|} & \text{if } \mathbf{r} = \mathbf{p} \\ 0 & \text{else} \end{cases}.$$

Applying the Leibniz formula to the functional derivative of $\varphi \in \mathcal{D}$ and $g \in C^{\infty}(\Omega;\mathbb{R})$ for $\mathbf{q} \leq \mathbf{p} \in \mathbb{N}^n$ gives

$$D^{\mathbf{q}}(\varphi \cdot D^{\mathbf{p}-\mathbf{q}}g) = \sum_{\mathbf{r}\leq\mathbf{q}} \binom{\mathbf{q}}{\mathbf{r}} (D^{\mathbf{r}}\varphi)\left(D^{\mathbf{p}-\mathbf{q}+\mathbf{q}-\mathbf{r}|}g\right) = \sum_{\gamma\leq\beta} \binom{\mathbf{q}}{\mathbf{r}} (D^{\mathbf{r}}\varphi)(D^{\mathbf{p}-\mathbf{r}}g)$$

whence the formula from above yields

$$\sum_{\mathbf{q}\leq\mathbf{p}} (-1)^{|\mathbf{q}|} \binom{\mathbf{p}}{\mathbf{q}} D^{\mathbf{q}}(\varphi \cdot D^{\mathbf{p}-\mathbf{q}}g) = \sum_{\mathbf{q}\leq\mathbf{p}} (-1)^{|\mathbf{q}|} \binom{\mathbf{p}}{\mathbf{q}} \sum_{\mathbf{r}\leq\mathbf{q}} \binom{\mathbf{q}}{\mathbf{r}} (D^{\mathbf{r}}\varphi)(D^{\mathbf{p}-\mathbf{r}}g) = (-1)^{|\mathbf{p}|} g D^{\mathbf{p}}\varphi.$$

By this equation we finally obtain

$$\begin{aligned}
D^{\mathbf{p}}(g\Lambda)\varphi &= (-1)^{|\mathbf{p}|}(g\Lambda)(D^{\mathbf{p}}\varphi) \\
&= (-1)^{|\mathbf{p}|}\Lambda(gD^{\mathbf{p}}\varphi) \\
&= \sum_{\mathbf{q}\leq\mathbf{p}} (-1)^{|\mathbf{q}|} \binom{\mathbf{p}}{\mathbf{q}} \Lambda\left(D^{\mathbf{q}}(\varphi \cdot D^{\mathbf{p}-\mathbf{q}}g)\right) \\
&= \sum_{\mathbf{q}\leq\mathbf{p}} \binom{\mathbf{p}}{\mathbf{q}} (D^{\mathbf{q}}\Lambda)(\varphi \cdot D^{\mathbf{p}-\mathbf{q}}g) \\
&= \sum_{\mathbf{q}\leq\mathbf{p}} \binom{\mathbf{p}}{\mathbf{q}} \left[(D^{\mathbf{p}-\mathbf{q}}g)(D^{\mathbf{q}}\Lambda)\right]\varphi
\end{aligned}$$

According to 6.6.5 for every compact $K \subset \Omega$ exist $C < \infty$ and $n \geq 1$ with $|\Lambda\varphi| \leq C\|\varphi\|_n$ for every $\varphi \in \mathcal{D}_K$. The Leibniz rule provides a further $C' < \infty$ depending on $g \in \mathcal{C}^{\infty}$, K and n such that $\|g\varphi\|_n \leq C'\|\varphi\|_n$ for every $\varphi \in \mathcal{D}_K$. Hence we obtain $|g\Lambda\varphi| = |\lambda(g\varphi)| \leq C\|g\varphi\|_n \leq C \cdot C'\|\varphi\|_n$ for every $\varphi \in \mathcal{D}_K$ whence $g\Lambda \in \mathcal{D}'$ according to 6.6.5.

6.11 Weak*-convergence of distributions

For a sequence $(\Lambda_n)_{n\geq 1} \subset \mathcal{D}'$ converging **weakly*** to a linear form $\Lambda \in \mathcal{D}^*$ such that $\Lambda f = \lim\limits_{n\to\infty} \Lambda_n f$ for every $f \in \mathcal{D}$ we have

1. $\Lambda \in \mathcal{D}'$.
2. $D^{\mathbf{p}} \Lambda \varphi = \lim\limits_{n\to\infty} D^{\mathbf{p}} \Lambda_n \varphi$ for every $\varphi \in \mathcal{D}$ and every $\mathbf{p} \in \mathbb{N}^n$.
3. $g \Lambda \varphi = \lim\limits_{n\to\infty} g_n \Lambda_n \varphi$ for every $\varphi \in \mathcal{D}$ and every sequence $(g_n)_{n\geq 1} \subset C^\infty$ **compactly converging in all derivatives** to a $g \in C^\infty$ according to 6.1.

Proof:

1. Due to [11, th. 16.2.4 and 16.4.1] the **Banach-Steinhaus theorem** 3.2.2 applied to the Fréchet space $\mathcal{D}_K$ implies that Λ is **continuous** on every subspace $\mathcal{D}_K \subset \mathcal{D}$ whence by 6.6.4 follows the continuity on $\mathcal{D}$, i.e. $\Lambda \in \mathcal{D}'$.
2. Due to 1. for every $\varphi \in \mathcal{D}$ we have $D^{\mathbf{p}} \Lambda \varphi = (-1)^{|\mathbf{p}|} \Lambda (D^{\mathbf{p}} \varphi) = (-1)^{|\mathbf{p}|} \lim\limits_{n\to\infty} \Lambda_n (D^{\mathbf{p}} \varphi) = \lim\limits_{n\to\infty} D^{\mathbf{p}} \Lambda_n \varphi$.
3. For a given $\varphi \in \mathcal{D}$ we define a **separately bilinear** map $B : C^\infty \times \mathcal{D}' \to \mathbb{C}$ by $B(g; \Lambda) = (g\Lambda)$. The assertion then directly follows from 3.7 and the fact that on metric spaces sequential continuity implies general continuity (cf. [11, th. 10.12].

6.12 Smooth partitions of unity

For every open cover Γ of an open set $\Omega \subset \mathbb{R}^n$ with $\Omega \subset \bigcup_{U\in\Gamma} U$ exists a partition of unity $(\psi_n)_{n\geq 1} \subset \mathcal{D}(\mathbb{R}^n; [0;1])$ subordinate to Γ such that $\sum_{n\geq 1} \psi_n = 1$ and for every $n \geq 1$ there is a $U_n \in \Gamma$ with $\mathrm{supp}\psi_n \subset U_n$. Also for every **compact** $K \subset \Omega$ exists an $m \geq 1$ such that $K \prec \sum\limits_{j=1}^{n} \psi_j \prec \Omega$.

Proof: The bijective enumeration $(\delta; \epsilon; \mathbf{z}) : \mathbb{N} \to \{\delta; \epsilon \in \mathbb{Q}^+; \mathbf{z} \in \mathbb{Z}^n : \exists U \in \Gamma : B_\delta(\epsilon \mathbf{z}) \subset U\}$ with $V_n := B_{\delta_n}(\epsilon_n \mathbf{z}_n) \subset U_n$ and $K_n = \overline{B_{\delta_n/2}(\epsilon_n \mathbf{z}_n)}$ defines an **open cover** $(V_n)_{n\geq 1}$ and a **compact cover** $(K_n)_{n\geq 1}$ of Ω. The smooth separation functions developed in [8, th. 13.2] provide a sequence $(\varphi_n)_{n\geq 1} \subset \mathcal{D}(\mathbb{R}^n; [0;1])$ with $K_n \prec \varphi_n \prec V_n$. By $\psi_1 = \varphi_1$ and $\psi_{n+1} = \prod\limits_{j=1}^{n} (1 - \varphi_j) \cdot \varphi_{n+1} = \psi_n \cdot \left(\frac{1}{\varphi_n} - 1\right) \cdot \varphi_{n+1}$ we obtain a further sequence $(\psi_n)_{n\geq 1} \subset \mathcal{D}(\mathbb{R}^n; [0;1])$ with $\mathrm{supp}\psi_n \subset V_n$. Since $\frac{\psi_{n+1}}{\psi_n} = \left(\frac{1}{\varphi_n} - 1\right) \cdot \varphi_{n+1} \Leftrightarrow \frac{\psi_n}{\varphi_n} - \psi_n = \frac{\psi_{n+1}}{\varphi_{n+1}}$ an induction over n shows that $\sum\limits_{j=1}^{n-1} \psi_j = 1 - \prod\limits_{j=1}^{n-1} (1 - \varphi_j)$. Due to $\varphi_j(\mathbf{x}) = 1$ for every $\mathbf{x} \in K_j$ and $j \geq 1$ we conclude that $\bigcup\limits_{j=1}^{n} K_j \prec \sum\limits_{j=1}^{n} \psi_j \prec \bigcup\limits_{j=1}^{n} V_j \subset \Omega$.

6.13 Localization

For every family $(\Lambda_i)_{i\in I} \in \mathcal{D}'(\omega_i)$ of distributions on an open cover $\Gamma = (\omega_i)_{i\in I}$ with $\bigcup_{i\in I} \omega_i = \Omega \subset \mathbb{R}^n$ and $\Lambda_i \varphi = \Lambda_j \varphi$ for every $\varphi \in \mathcal{D}(\omega_i \cap \omega_j)$ exists a uniquely determined $\Lambda \in \mathcal{D}'(\Omega)$ with $\Lambda|_{\mathcal{D}(\omega_j)} = \Lambda_j$ for every $j \in I$.

Proof: Due to the preceding theorem 6.12 exists a smooth partition of unity $(\psi_n)_{n\geq 1} \subset \mathcal{D}(\mathbb{R}^n; [0;1])$ subordinate to Γ such that $\sum_{n\geq 1} \psi_n = 1$ and for every $n \geq 1$ there is an $i_n \in I$ with $\mathrm{supp}\psi_n \subset \omega_{i_n}$. Since for every **compact** $K \subset \Omega$ exists an $m \geq 1$ such that $K \prec \sum\limits_{n=1}^{m} \psi_n \prec \Omega$ every $\varphi \in \mathcal{D}(\Omega)$ can be expressed in the form $\varphi = \sum_{n\geq 1} \psi_n \varphi$ with $\psi_n \varphi \in \mathcal{D}(\omega_{i_n})$ such that $\Lambda \varphi = \sum_{n\geq 1} \Lambda_{i_n}(\psi_n \varphi)$ defines a linear functional $\Lambda : \mathcal{D}(\Omega) \to \mathbb{C}$. According to 6.5.5 for every sequence $(\varphi_j)_{j\geq 1}$ compactly converging in all derivatives to 0 exists a compact $K \subset \Omega$ with $\mathrm{supp}\varphi_j \subset K$ for every $j \geq 1$ whence by the preceding theorem 6.12 there is an $m \geq 1$ such that $\varphi_j = \sum\limits_{n=1}^{m} \psi_n \varphi_j$ resp. $\Lambda \varphi_j = \sum\limits_{n=1}^{m} \Lambda_{i_n} \left(\psi_n \varphi_j\right)$. According to 6.1.3 and 6.6.3 we conclude that $\lim\limits_{j \to \infty} \Lambda \varphi_j = \sum\limits_{n=1}^{m} \lim\limits_{j \to \infty} \Lambda_{i_n} \left(\psi_n \varphi_j\right) = \sum\limits_{n=1}^{m} \Lambda_{i_n} \left(\lim\limits_{j \to \infty} \psi_n \varphi_j\right) = 0$ whence by 6.6.1 follows $\Lambda \in \mathcal{D}'$.

Finally for every $j \in I$ and $\varphi \in \mathcal{D}\left(\omega_j\right)$ follows $\psi_n \varphi \in \mathcal{D}\left(\omega_j \cap \omega_{i_n}\right)$ whence $\Lambda_{i_n}\left(\psi_n \varphi\right) = \Lambda_j\left(\psi_{i_n} \varphi\right)$ such that $\Lambda \varphi = \sum_{n \geq 1} \Lambda_{i_n}\left(\psi_n \varphi\right) = \sum_{n \geq 1} \Lambda_j\left(\psi_{i_n} \varphi\right) = \Lambda_j\left(\left(\sum_{n \geq 1} \psi_{i_n}\right) \varphi\right) = \Lambda_j\left(\varphi\right)$. This assertion implies uniqueness since Γ covers Ω.

6.14 The support of a distribution

A distribution $\Lambda \in \mathcal{D}'\left(\Omega\right)$ **vanishes** on the open set $W = \bigcup \Gamma$ with $\Gamma = \left\{\text{open } \omega \subset \Omega : \Lambda\left[\mathcal{D}\left(\omega\right)\right] = \{0\}\right\}$, i.e. $\Lambda \varphi = 0$ for every $\varphi \in \mathcal{D}\left(V\right)$ on open $V \subset W$ since in the expression $\Lambda \varphi = \sum_{n \geq 1} \Lambda\left(\psi_n \varphi\right)$ based on the smooth **partition of unity** $\left(\psi_n\right)_{n \geq 1}$ with $\psi_n \in \mathcal{D}\left(\omega_n\right)$ subordinate to the cover Γ of W according to 6.12 we have $\psi_n \varphi \in \mathcal{D}\left(\omega_n\right)$ and $\omega_n \in W$ whence $\Lambda\left(\psi_n \varphi\right) = 0$ for every $n \geq 1$. The **support of a distribution** $\Lambda \in \mathcal{D}'_L$ defined as $L = \text{supp}\Lambda = \Omega \setminus W$ has the following properties:

1. For every $\varphi \in \mathcal{D}_K$ with $\text{supp}\Lambda \cap K = \emptyset$ we have $\Lambda \varphi = 0$.
2. $\text{supp}\Lambda = \emptyset \Rightarrow \Lambda = 0$.
3. For every $\psi \in \mathcal{C}^\infty$ with $\psi|_L = 1$ we have $\psi \Lambda = \Lambda$.
4. Every distribution $\Lambda \in \mathcal{D}'_L$ with **compact support** is of **finite order**.
5. Every distribution $\Lambda \in \mathcal{D}'_L$ with **compact support** extends in a **unique way** to an $\mathcal{O}_{c\mathcal{D}}$-**continuous linear functional** $\Gamma \in \left(\mathcal{C}^\infty\right)^*$ with $\Gamma|_{\mathcal{D}} = \Lambda$.
6. The restriction $\Gamma|_{\mathcal{D}}$ of very $\mathcal{O}_{c\mathcal{D}}$-**continuous linear functional** $\Gamma \in \left(\mathcal{C}^\infty\right)^*$ is a **distribution** $\Gamma|_{\mathcal{D}} \in \mathcal{D}'_L$ with **compact support** $L = \text{supp}\Gamma|_D$

Proof:

1. Follows from $\text{supp}\varphi \subset W$.
2. Follows from $W = \Omega$.
3. For $\varphi \in \mathcal{D}_K$ we have $\text{supp}\left(\varphi - \psi\varphi\right) \subset \Omega \setminus L$, i.e. $\text{supp}\Lambda \cap \text{supp}\left(\varphi - \psi\varphi\right) = \emptyset$ whence by 1. follows $\Lambda \varphi = \Lambda\left(\psi \varphi\right) = \left(\psi \Lambda\right) \varphi$.
4. Due to 6.12 we find a $\psi \in \mathcal{D}$ with $\psi|_L = 1$ for $L = \text{supp}\Lambda$. Also we have $\psi \varphi \in \mathcal{D}_L$ and since $\psi|_L = 1$ whence $D^{\mathbf{p}} \psi|_L = 0$ for every $\mathbf{0} \neq \mathbf{p} \in \mathbb{N}^n$ we conclude that by the **Leibniz rule** $\left\|\psi \varphi\right\|_m = \max\left\{\left|D^{\mathbf{p}}\left(\psi \varphi\right)\left(\mathbf{x}\right)\right| : \left|\mathbf{p}\right| \leq m; \mathbf{x} \in L\right\} = \left\|\varphi\right\|_m$. From the preceding proposition 3. and 6.6.5 follows $\left|\Lambda \varphi\right| = \left|\Lambda\left(\psi \varphi\right)\right| \leq C_{m_L} \left\|\psi \varphi\right\|_{m_L} = C_{m_L} \left\|\varphi\right\|_{m_L}$, i.e. Λ is of order m_L.
5. According to 6.12 for $L = \text{supp}\Lambda$ there is a $\psi \in \mathcal{D}_L$ with $\psi|_L = 1$ and $L \prec \psi \prec \Omega$. Then the linear functional $\Gamma \in \left(\mathcal{C}^\infty\right)^*$ defined by $\Gamma f = \Lambda\left(\psi f\right)$ is well defined since $\psi f \in \mathcal{D}_K$ and it is $\mathcal{O}_{c\mathcal{D}}$-continuous since due to 6.1.3 for every sequence $\left(f_n\right)_{n \geq 1} \subset \mathcal{C}^\infty$ compactly converging in all derivatives to 0 the product $\left(\psi f_n\right)_{n \geq 1} \subset \mathcal{D}_K$ also $\mathcal{O}_{c\mathcal{D}}$-converges to 0 whence the continuity of Λ implies $\lim\limits_{n \to \infty} \Gamma f_n = 0$. Since $\mathcal{D}$ is obviously $\mathcal{O}_{c\mathcal{D}}$-dense in $\mathcal{C}^\infty$ this extension is **unique**.
6. According to 1.11.4 and 6.1 for every $\mathcal{O}_{c\mathcal{D}}$-**continuous linear functional** $\Gamma \in \left(\mathcal{C}^\infty\right)^*$ exists a **compact** set $K \subset \Omega$, a **number** $m \in \mathbb{N}$ and a **constant** $C < \infty$ such that $\left|\Gamma\left[B_{K_m}\left(0\right)\right]\right| \leq C$, i.e. $\left|\Gamma f\right| \leq m \cdot C \cdot \max\left\{\left|D^{\mathbf{p}} f\left(\mathbf{x}\right)\right| : \left|\mathbf{p}\right| \leq m; \ \mathbf{x} \in K\right\}$ for **every** $f \in \mathcal{C}^\infty$. Hence Λ is of order at most m and due to 6.12 it follows that $\text{supp}\Lambda \subset K$.

6.15 Distributions with atomic supprt

For every distribution $\Lambda \in \mathcal{D}'_{\{s\}}$ of order m with **atomic support** $\text{supp}\Lambda = \{\mathbf{s}\}$ exists a **differential operator** $D = \sum\limits_{|\mathbf{p}| \leq m} c_{\mathbf{p}} D^{\mathbf{p}}$ with **constant coefficients** $c_{\mathbf{p}} \in \mathbb{C}$ such that $\Lambda = D\delta_{\mathbf{s}}$ with the **evaluation functional** $\delta_{\mathbf{s}} : \mathcal{D}' \to \mathbb{R}$ defined by $\delta_{\mathbf{s}}\left(\varphi\right) = \varphi\left(\mathbf{s}\right)$.

Proof:

Step I: For every $\epsilon > 0$ there is a $\delta > 0$ such that for every $\varphi \in D$; $\mathbf{p} \in \mathbb{N}$ with $|\mathbf{p}| \leq m$ and $\mathbf{x} \in \Omega$ with $|\mathbf{x}| < \delta$ holds

$$\left|D^{\mathbf{p}}\varphi\left(\mathbf{x}\right)\right| < \epsilon \cdot n^{m-|\mathbf{p}|} \left|\mathbf{x}\right|^{m-|\mathbf{p}|} < \epsilon \cdot \left(n \cdot \delta\right)^{m-|\mathbf{p}|}.$$

We prove this statement by **downward** induction over $|\mathbf{p}|$: The case $|\mathbf{p}| = m$ is a direct consequence ot the **continuity** of the derivatives. Assuming the hypothesis for $|\mathbf{p}| = i$ with $1 \leq i \leq m$ implies that for $\mathbf{q} \in \mathbb{N}^n$ with $|\mathbf{q}| = i - 1$ holds

$$\left|\nabla D^{\mathbf{q}}\varphi\left(\mathbf{x}\right)\right| = \left|\sum_{i=1}^{n} D^{\mathbf{e}_i} D^{\mathbf{q}}\varphi\left(\mathbf{x}\right)\right| \leq \sum_{i=1}^{n} \left|D^{\mathbf{e}_i} D^{\mathbf{q}}\varphi\left(\mathbf{x}\right)\right| \leq n \cdot \epsilon \cdot n^{m-i|} \left|\mathbf{x}\right|^{m-i|}.$$

Since $D^{\mathbf{q}}\varphi\left(\mathbf{0}\right) = 0$ the **mean value theorem** [6, th. 1.7] implies that

$$\left|D^{\mathbf{q}}\varphi\left(\mathbf{x}\right)\right| = \left|D^{\mathbf{q}}\varphi\left(\mathbf{x}\right) - D^{\mathbf{q}}\varphi\left(\mathbf{0}\right)\right| \leq \left|\mathbf{x}\right| \cdot \max_{0 \leq t \leq 1} \left\{\left|\nabla D^{\mathbf{q}}\varphi\left(t\mathbf{x}\right)\right|\right\} \leq \epsilon \cdot n^{m-(i-1)|} \left|\mathbf{x}\right|^{m-(i-1|)}$$

and hence the assertion.

Step II: For every $\varphi \in \mathcal{D}$ with $D^{\mathbf{p}}\varphi\left(\mathbf{s}\right) = 0$ for all $\mathbf{p} \in \mathbb{N}^n$ with $|\mathbf{p}| \leq m$ we have $\Lambda\varphi = 0$. According to [8, th. 13.2] for every $0 < \delta$ there is an $f_1 \in \mathcal{D}\left(\Omega\right)$with $\overline{B_{1/2}\left(\mathbf{s}\right)} \prec f_1 \prec B_1\left(\mathbf{s}\right)$ and by scaling we obtain an $f_\delta \in \mathcal{D}$ defined by $f_\delta\left(\boldsymbol{x}\right) = f_1\left(\frac{1}{\delta}\left(\mathbf{x} - \boldsymbol{s}\right) + \mathbf{s}\right)$ with $\overline{B_{\delta/2}\left(\mathbf{s}\right)} \prec f_\delta \prec B_\delta\left(\mathbf{s}\right)$ whence by the **Leibniz rule** [6, th. 4.2] follows $D^{\mathbf{p}}\left(f_\delta\varphi\right)\left(\mathbf{x}\right) = \sum_{\mathbf{q} \leq \mathbf{p}} \begin{pmatrix} \mathbf{p} \\ \mathbf{q} \end{pmatrix} \delta^{|\mathbf{q}|-|\mathbf{p}|}\left(D^{\mathbf{p}-\mathbf{q}}f_\delta\right)\left(\frac{1}{\delta}\left(\mathbf{x} - \boldsymbol{s} + \mathbf{s}\right)\right) \cdot \left(D^{\mathbf{q}}\varphi\right)\left(\mathbf{x}\right)$. **Step I** implies $\left\|f_\delta\varphi\right\|_m \leq \epsilon \cdot n^{2m} \cdot \left\|f_\delta\right\|_m$. Since Λ is of **order** m there is a constant $C < \infty$ with $|\Lambda\psi| \leq C \cdot \left\|\psi\right\|_m$ for every $\psi \in \mathcal{D}_K$ with $K = \overline{B_{\delta/2}\left(\mathbf{s}\right)}$. Hence 6.6.5 implies that from $\mathrm{supp}\Lambda = \{\mathbf{s}\} \subset K$ follows $|\Lambda\varphi| = \left|\Lambda\left(f_\delta\varphi\right)\right| \leq C \cdot \left\|f_\delta\varphi\right\|_m \leq C \cdot \epsilon \cdot n^{2m} \cdot \left\|f_\delta\right\|_m$ and since ϵ was arbitrary we conclude that $\Lambda\varphi = 0$.

Step III: Since for every $\varphi \in \bigcap_{|\mathbf{p}| \leq m} \ker D^{\mathbf{p}}\delta_{\mathbf{s}}$ holds $0 = \left(D^{\mathbf{p}}\delta_{\mathbf{s}}\right)\varphi = (-1)^{|\mathbf{p}|}\delta_{\mathbf{s}}\left(D^{\mathbf{p}}\varphi\right) = (-1)^{|\mathbf{p}|}\left(D^{\mathbf{p}}\varphi\right)\left(\mathbf{s}\right)$ **step II** implies $\Lambda\varphi = 0$ such that we can apply 5.1.1 which proves the assertion.

6.16 Distributions of finite order as derivatives

1. For every distribution $\Lambda \in \mathcal{D}'$ and **compact** $K \subset \Omega$ exists a function $f_K \in \mathcal{C}_K\left(\Omega; \mathbb{C}\right)$ such that $\Lambda|_{\mathcal{D}_K} = D^{\mathbf{m}+\mathbf{2}}f_K$, i.e. for every $\varphi \in \mathcal{D}_K$ holds

$$\Lambda\varphi = (-1)^{n(m+2))} \int_\Omega D^{\mathbf{m}+\mathbf{2}}\varphi \cdot f d\lambda.$$

2. For every distribution $\Lambda \in \mathcal{D}'$ with **compact support** $K = \mathrm{supp}\Lambda \subset \Omega$ und **finite order** m for every multi-index $\mathbf{q} \leq \mathbf{m} + \mathbf{2}$ exist functions $f_{\mathbf{q}} \in \mathcal{C}_K\left(\Omega; \mathbb{C}\right)$ such that $\Lambda = \sum_{\mathbf{q} \leq \mathbf{m}+\mathbf{2}} D^{\mathbf{q}}f_{\mathbf{q}}$, i.e. for every $\varphi \in \mathcal{D}$ holds

$$\Lambda\varphi = \sum_{\mathbf{q} \leq \mathbf{m}+\mathbf{2}} (-1)^{|\mathbf{q}|} \int_\Omega D^{\mathbf{q}}\varphi \cdot f_{\mathbf{q}} d\lambda.$$

Note: We abbreviate $\mathbf{q} = (q_1; ...; q_n) \in \mathbb{N}^n$ and $\mathbf{M} = (M; ...; M)$ with $M \in \mathbb{N}$ resp. $\mathbf{2} = (2; ...; 2)$.

Proof:

1. If not stated otherwise in this proof we always assume $\varphi \in \mathcal{D}_K$. Due to the **Heine-Borel-property** of $\mathbb{R}^n$ there is an $M \in \mathbb{N}$ such that $K \subset [-\mathbf{M};\mathbf{M}]$ and the **mean value theorem** [6, th. 1.7] implies that $|\varphi| \le \max\limits_{-\mathbf{M}\le\mathbf{x}\le\mathbf{M}} M \cdot |(D_i\varphi)(\mathbf{x})|$. For the differential operator $T = D_1 \circ ... \circ D_n$ **Fubini's theorem** [8, th. 8.5] and the **fundamental theorem of calculus** [8, th. 12.10] yield $\varphi(\mathbf{y}) = \int_{-\mathbf{M}\le\mathbf{x}\le\mathbf{y}} (T\varphi)(\mathbf{x})\,d\mathbf{x}$. Applying the inequality **above** to successive derivatives of φ and invoking 6.6.5 gives
$$\begin{aligned}|\Lambda\varphi| &\le C \cdot \|\varphi\|_m \\ &\le C \cdot M^{n\cdot m} \cdot \max_{-\mathbf{M}\le\mathbf{x}\le\mathbf{M}} |(T^m\varphi)(\mathbf{x})| \\ &\le C \cdot M^{n\cdot m} \cdot \int_{-\mathbf{M}\le\mathbf{x}\le\mathbf{y}} \left|\left(T^{m+1}\varphi\right)(\mathbf{x})\right| d\mathbf{x}.\end{aligned}$$
   Due to the **fundamental theorem of calculus** as formulated above the map $T : \mathcal{D}_K \to \mathcal{D}_K$ is **bijective** and this is also true for $T^{m+1} : \mathcal{D}_K \to \mathcal{D}_K$. Hence we may define a linear functional $\Lambda_1 : T^{m+1}[\mathcal{D}_K] \to \mathbb{C}$ by $\Lambda_1 T^{m+1}\varphi = \Lambda\varphi$ and from the inequality above follows $|\Lambda_1\varphi| \le C \cdot \int_K |\varphi(\mathbf{x})|\,d\mathbf{x}$ for every $\varphi \in T^{m+1}[\mathcal{D}_K]$, i.e. Λ_1 is **bounded** on $T^{m+1}[\mathcal{D}_K] \subset L^1(K;\mathbb{C})$ with regard to the norm $\|\ \|^* : \left(L^1(K;\mathbb{C})\right)^* \to \mathbb{R}_0^+$ defined by $\|\Lambda\|^* = \sup\left\{\left|\Lambda\left(\frac{\varphi}{\|\varphi\|_1}\right)\right| : \varphi \in L^1(K;\mathbb{C})\right\}$. According to the **Hahn-Banach theorem** 4.2 Λ_1 can be extended to a **bounded linear functional** $\Lambda_1 : L^1(K;\mathbb{C}) \to \mathbb{C}$. By the representation theorem [8, th. 9.13] for the dual space $\left(L^1(K;\mathbb{C})\right)^*$ resp the theorem of of **Radon-Nikodym** [8, th. 9.8] there is a **bounded** function $g \in L^1(K;\mathbb{C})$ with $\Lambda\varphi = \Lambda_1 T^{m+1}\varphi = \int_K g(\mathbf{y})\left(T^{m+1}\varphi\right)(\mathbf{y})\,d\mathbf{y}$. If we extend g to $\mathbb{R}^n$ by defining $g(\mathbf{y}) = 0$ for $\mathbf{y} \in \mathbb{R}^n \setminus K$ the function $f_K : \mathbb{R}^n \to \mathbb{C}$ given by $f_K(\mathbf{x}) = (-1)^{n(m+1)} \int_{-\infty}^{x_1} ... \int_{-\infty}^{x_n} g(\mathbf{y})\,dy_n...dy_1$ is obviously **continuous** with $\operatorname{supp} f_K = K$. Finally n integrations by parts show that $\Lambda\varphi = (-1)^n \int_\Omega \left(D^{\mathbf{m}+\mathbf{2}}\varphi\right)(\mathbf{x}) \cdot f_K(\mathbf{x})\,d\mathbf{x}$, i.e. the assertion.

2. According to [11, th. 10.5] there is an **open** W with **compact closure** $\overline{W}$ such that $K \subset W \subset \overline{W} \subset \Omega$ and due to 6.12 we find a $\psi \in D(\Omega)$ with $K \prec \psi \prec W$. Hence for every $\varphi \in \mathcal{D}$ we have $\psi\varphi \in D_K$ so that due to the preceding result 6.16.1 there is an $f_K \in \mathcal{C}_K(\Omega;\mathbb{C})$ with the **Leibniz rule** [6, th. 4.2] and 6.14.3 yield
$$\begin{aligned}\Lambda\varphi &= \Lambda(\psi\varphi) \\ &= (-1)^n \int_\Omega D^{\mathbf{m}+\mathbf{2}}(\psi\varphi) \cdot f_K d\lambda \\ &= (-1)^n \int_\Omega \sum_{\mathbf{q}\le\mathbf{m}+\mathbf{2}} f_K \cdot \begin{pmatrix}\mathbf{m}+\mathbf{2} \\ \mathbf{q}\end{pmatrix} \left(D^{\mathbf{m}+\mathbf{2}-\mathbf{q}}\psi\right)(D^{\mathbf{q}}\varphi)\,d\lambda \\ &= \sum_{\mathbf{q}\le\mathbf{m}+\mathbf{2}} (-1)^{|\mathbf{q}|} \int_\Omega D^{\mathbf{q}}\varphi \cdot f_{\mathbf{q}} d\lambda\end{aligned}$$
   with $f_{\mathbf{q}} = (-1)^{n-|\mathbf{q}|} \cdot \begin{pmatrix}\mathbf{m}+\mathbf{2} \\ \mathbf{q}\end{pmatrix} \cdot f_K \cdot D^{\mathbf{m}+\mathbf{2}-\mathbf{q}}\psi \in \mathcal{C}_K(\Omega;\mathbb{C})$.

6.17 General distributions as derivatives

For every distribution $\Lambda \in \mathcal{D}'$ and every $\mathbf{q} \in \mathbb{N}^n$ exist functions $f_{\mathbf{q}} \in \mathcal{C}_c(\Omega;\mathbb{C})$ such that each **compact** $K \subset \Omega$ intersects the supports of only finitely many $f_{\mathbf{q}}$ and $\Lambda\varphi = \sum\limits_{\mathbf{q}\ge 0} D^{\mathbf{q}} f_{\mathbf{q}}$, i.e. for every $\varphi \in \mathcal{D}_K$ holds
$$\Lambda\varphi = \sum_{\mathbf{q}\ge 0} (-1)^{|\mathbf{q}|} \int_K D^{\mathbf{q}}\varphi \cdot f_{\mathbf{q}} d\lambda.$$

If Λ is of **finite order** the functions $f_{\mathbf{q}}$ can be chosen so that only finitely many are different from 0.

Proof: According to 6.12 exists a **partition of unity** $(\psi_n)_{n\ge 1} \subset \mathcal{D}(\mathbb{R}^n;[0;1])$ subordinate to the cover $\Gamma = \{[-\mathbf{m};\mathbf{m}] : \mathbf{m} \ge \mathbf{1}\}$ of Ω such that $\sum_{n\ge 1}\psi_n = 1$ and for every $n \ge 1$ there is a $[-\mathbf{m};\mathbf{m}] \in \Gamma$

with supp$\psi_n \subset [-\mathbf{n};\mathbf{n}]$. Also for every **compact** $K \subset \Omega$ exists an $m \geq 1$ such that $K \prec \sum\limits_{j=1}^{n} \psi_j \prec \Omega$. Since the preceding theorem 6.16 applies to every product $\psi_n \Lambda$ defined according to 6.10 there are finitely may $f_{n;\mathbf{q}} \in \mathcal{C}_{[-\mathbf{n};\mathbf{n}]}(\Omega;\mathbb{C})$ such that $\psi_n \Lambda = \sum\limits_{\mathbf{q}\geq 0} D^{\mathbf{q}} f_{n;\mathbf{q}}$. Then the sums $f_{\mathbf{q}} = \sum\limits_{n\geq 1} f_{n;\mathbf{q}}$ are finite whence $f_{\mathbf{q}} \in \mathcal{C}_c(\Omega;\mathbb{C})$ and since we have $\varphi = \sum\limits_{n\geq 1} \psi_n \varphi$ for every $\varphi \in \mathcal{D}$ we have $\Lambda = \sum\limits_{n\geq 1} \psi_n \Lambda$ whence by the preceding theorem 6.16 follows the assertion.

6.18 Convolutions with test functions

In the context of **convolutions** we always assume $\Omega = \mathbb{R}^n$. Also we use the **translation** $\tau_{\mathbf{x}} : \mathcal{F}(\mathbb{R}^n;\mathbb{C}) \to \mathcal{F}(\mathbb{R}^n;\mathbb{C})$ defined by $\tau_{\mathbf{x}} f(\mathbf{y}) = f(\mathbf{y}-\mathbf{x})$ for $\mathbf{x} \in \mathbb{R}^n$ and the **reflection** $\check{}: \mathcal{F}(\mathbb{R}^n;\mathbb{C}) \to \mathcal{F}(\mathbb{R}^n;\mathbb{C})$ defined by $\check{f}(\mathbf{y}) = f(-\mathbf{y})$. The relation $\int (\tau_{\mathbf{x}} f) \cdot \varphi d\lambda = \int f \cdot (\tau_{-\mathbf{x}}\varphi)\, d\lambda$ leads to the extension of the definition to **distributions** $\Lambda \in \mathcal{D}'$ by $(\tau_{\mathbf{x}}\Lambda)\varphi = \Lambda(\tau_{-\mathbf{x}}\varphi)$ for $\varphi \in \mathcal{D}$ and $\mathbf{x} \in \mathbb{R}^n$. Obviously the translate of a distribution is continuous according to 6.6.5 and hence a distribution. By $\left(\tau_{\mathbf{x}}\check{f}\right)(\mathbf{y}) = f(-(\mathbf{y}-\mathbf{x})) = f(\mathbf{x}-\mathbf{y}) = f(-\mathbf{y}+\mathbf{x}) = (\tau_{-\mathbf{x}} f)\check{}(\mathbf{y})$ the **convolution** of two integrable functions $f \in L^1$ and $g \in L^p$ with $1 \leq p \leq \infty$ according to the definition in [6, th. 5.1] takes the form $(f * g)(\mathbf{x}) = \int f(\mathbf{y}) \cdot g(\mathbf{x}-\mathbf{y})\, d\mathbf{y} = \int f \cdot \tau_{\mathbf{x}}\check{g} d\lambda$. Extending this definition to **distributions** we obtain $(\Lambda * \varphi)(\mathbf{x}) = \Lambda(\tau_{\mathbf{x}}\check{\varphi})$ for $\Lambda \in \mathcal{D}'$; $\varphi \in \mathcal{D}$ and $\mathbf{x} \in \mathbb{R}^n$. The convolution has the following properties for $\Lambda \in \mathcal{D}'$;$\varphi; \psi \in \mathcal{D}$; $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{p} \in \mathbb{N}^n$:

1. supp$(\psi * \varphi) =$ supp$\psi +$ suppφ.
2. supp$(\Lambda * \varphi) =$ supp$\Lambda +$ suppφ.
3. $\tau_{\mathbf{x}}(\Lambda * \varphi) = (\tau_{\mathbf{x}}\Lambda) * \varphi = \Lambda * (\tau_{\mathbf{x}}\varphi)$.
4. $\psi * \varphi \in \mathcal{D}$ with $D^{\mathbf{p}}(\psi * \varphi) = (D^{\mathbf{p}}\psi) * \varphi = \psi * (D^{\mathbf{p}}\varphi)$.
5. $\Lambda * \varphi \in \mathcal{D}$ with $D^{\mathbf{p}}(\Lambda * \varphi) = (D^{\mathbf{p}}\Lambda) * \varphi = \Lambda * (D^{\mathbf{p}}\varphi)$.
6. $\Lambda * (\varphi * \psi) = (\Lambda * \varphi) * \psi$.

Proof:

1. Directly follows from the definitions.
2. Dito
3. Dito.
4. Follows from [6, th. 1.15].
5. For every $1 \leq i \leq n$ we have $D^{\mathbf{e}_i}(\Lambda * \varphi)(\mathbf{x}) = D^{\mathbf{e}_i}\Lambda(\tau_{\mathbf{x}}\check{\varphi}) = -\Lambda(D^{\mathbf{e}_i}(\tau_{\mathbf{x}}\check{\varphi})) = \Lambda(\tau_{\mathbf{x}}(D^{\mathbf{e}_i}\varphi)\check{}) = \Lambda * (D^{\mathbf{e}_i}\varphi)$ which generalizes to $D^{\mathbf{p}}(\Lambda * \varphi) = \Lambda * (D^{\mathbf{p}}\varphi)$. The first equality follows from $\Lambda * (D^{\mathbf{p}}\varphi) = \Lambda(\tau_{\mathbf{x}}(D^{\mathbf{p}}\varphi)\check{}) = \Lambda\left((-1)^{|\mathbf{p}|} D^{\mathbf{p}}(\tau_{\mathbf{x}}\check{\varphi})\right) = (D^{\mathbf{p}}\Lambda) * \varphi$.
6. Using the **auxiliary variable $\boldsymbol{\xi}$**, **differentiating under the integral sign** according to [6, th. 1.15], **changing the order of integration** due to **Fubini's theorem** [8, th. 8.5] and finally the **variable $\boldsymbol{\eta} = \mathbf{x} - \mathbf{y}$** the preceding theorem 6.17 gives

$$\begin{aligned}(\Lambda * (\varphi * \psi))(\mathbf{x}) &= (\Lambda * (\psi * \varphi))(\mathbf{x}) \\ &= \Lambda_{\boldsymbol{\xi}} (\psi * \varphi)(\mathbf{x} - \boldsymbol{\xi}) \\ &= \Lambda_{\boldsymbol{\xi}} \int \psi(\boldsymbol{\eta}) \varphi(\mathbf{x} - \boldsymbol{\xi} - \boldsymbol{\eta}) \, d\boldsymbol{\eta} \\ &= \sum_{\mathbf{q} \geq 0} (-1)^{|\mathbf{q}|} \int f_{\mathbf{q}}(\boldsymbol{\xi}) \cdot D^{\mathbf{q}} \Big(\int \psi(\boldsymbol{\eta}) \varphi(\mathbf{x} - \boldsymbol{\xi} - \boldsymbol{\eta}) \, d\boldsymbol{\eta} \Big) d\boldsymbol{\xi} \\ &= \sum_{\mathbf{q} \geq 0} (-1)^{|\mathbf{q}|} \int f_{\mathbf{q}}(\boldsymbol{\xi}) \cdot \Big(\int \psi(\boldsymbol{\eta}) D^{\mathbf{q}} \varphi(\mathbf{x} - \boldsymbol{\xi} - \boldsymbol{\eta}) \, d\boldsymbol{\eta} \Big) d\boldsymbol{\xi} \\ &= \sum_{\mathbf{q} \geq 0} \int_K f_{\mathbf{q}}(\boldsymbol{\xi}) \cdot \Big(\int \psi(\boldsymbol{\eta}) \cdot (D^{\mathbf{q}} \varphi)(\mathbf{x} - \boldsymbol{\xi} - \boldsymbol{\eta}) \, d\boldsymbol{\eta} \Big) d\boldsymbol{\xi} \\ &= \int \sum_{\mathbf{q} \geq 0} \int f_{\mathbf{q}}(\boldsymbol{\xi}) \cdot (D^{\mathbf{q}} \varphi)(\mathbf{y} - \boldsymbol{\xi}) \, d\boldsymbol{\xi} \cdot \psi(\mathbf{x} - \mathbf{y}) \, d\mathbf{y} \\ &= \int \sum_{\mathbf{q} \geq 0} (-1)^{|\mathbf{q}|} \int f_{\mathbf{q}}(\boldsymbol{\xi}) \cdot D^{\mathbf{q}} (\varphi(\mathbf{y} - \boldsymbol{\xi}))(\boldsymbol{\xi}) \, d\boldsymbol{\xi} \cdot \psi(\mathbf{x} - \mathbf{y}) \, d\mathbf{y} \\ &= \int \Lambda_{\boldsymbol{\xi}} \varphi(\mathbf{y} - \boldsymbol{\xi}) \cdot \psi(\mathbf{x} - \mathbf{y}) \, d\mathbf{y} \\ &= \int (\Lambda * \varphi)(\mathbf{y}) \cdot \psi(\mathbf{x} - \mathbf{y}) \, d\mathbf{y} \\ &= ((\Lambda * \varphi) * \psi)(\mathbf{x}) .\end{aligned}$$

6.19 Approximate identities

For every **approximate identity** or **mollifier** $(\rho_k)_{k \geq 1} \subset \mathcal{D}$ with $\rho_k(\mathbf{x}) = k^n \rho(k\mathbf{x})$ for $\mathbf{x} \in \mathbb{R}^n$ derived from a **positive** $\rho \in \mathcal{D}(\Omega; \mathbb{R}^+)$ with $\int \rho d\lambda = 1$ as e.g. $\rho = \frac{\alpha_{1;0}}{\int \alpha_{1;0} d\lambda}$ with $\alpha_{1;0}$ defined in [6, th. 13.2], every $\varphi \in \mathcal{D}$ and every $\Lambda \in \mathcal{D}'$ we have

1. $\lim\limits_{k \to \infty} \varphi * \rho_k = \varphi$ with **compact convergence in all derivatives** in $\mathcal{D}$
2. $\lim\limits_{k \to \infty} \Lambda * \rho_k = \Lambda$ with **weak*-convergence** in $\mathcal{D}'$

Proof:

1. For every simply **continuous** $\varphi \in C$ with $\mathbf{x} \in \mathbb{R}^n$ and $K = \operatorname{supp} \rho$ we have $\frac{1}{k} K = \operatorname{supp} \rho_k$ and $\int k^n \rho(k(\mathbf{x} - \mathbf{y})) \, d\mathbf{y} = 1$ whence $|\varphi(\mathbf{x}) - (\varphi * \rho_k)(\mathbf{x})| = \left| \int (\varphi(\mathbf{x}) - \varphi(\mathbf{y})) \cdot k^n \rho(k(\mathbf{x} - \mathbf{y})) \, d\mathbf{y} \right| \leq \sup\limits_{\mathbf{x} - \mathbf{y} \in \frac{1}{k} K} |\varphi(\mathbf{x}) - \varphi(\mathbf{y})|$, i.e. **uniform convergence** of $\lim\limits_{k \to \infty} \varphi * \rho_k = \varphi$. Owing to 6.18.4 we can extend this result to $\lim\limits_{k \to \infty} D^{\mathbf{p}} (\varphi * \rho_k) = \lim\limits_{k \to \infty} (D^{\mathbf{p}} \varphi * \rho_k) = D^{\mathbf{p}} \varphi$ for every $\mathbf{p} \in \mathbb{N}^n$ whence follows **uniform convergence in all derivatives**.
2. The assertion follows from 6.6.3, 6.19.1 and 6.18.6 since $\Lambda \check{\varphi} = (\Lambda * \varphi)(\mathbf{0}) = \lim\limits_{k \to \infty} (\Lambda * (\rho_k * \varphi))(\mathbf{0})$ $= \lim\limits_{k \to \infty} ((\Lambda * \rho_k) * \varphi)(\mathbf{0}) = \lim\limits_{k \to \infty} (\Lambda * \rho_k) \check{\varphi}$.

6.20 Representation of linear maps as convolutions

For any **linear** map $L : \mathcal{D} \to \mathcal{C}(\mathbb{R}^n)$ exists a $\Lambda \in \mathcal{D}$ such that $L\varphi = \Lambda * \varphi$ for every $\varphi \in \mathcal{D}$ iff L is $\mathcal{O}_T$**-continuous** and one of the two following equivalent conditions hold:

1. $\tau_{\mathbf{x}} L = L \tau_{\mathbf{x}}$ for every $\mathbf{x} \in \mathbb{R}^n$.
2. $D^{\mathbf{p}} L \varphi = L D^{\mathbf{p}} \varphi$ for every $\mathbf{p} \in \mathbb{N}^n$ and $\varphi \in \mathcal{D}$.

Proof:

1. ⇒: For every compact $K \subset \mathbb{R}^n$ and every sequence $(\varphi_n)_{n\geq 1} \subset \mathcal{D}_K$ with $\lim\limits_{k\to\infty}\varphi_n = \varphi \in \mathcal{D}_K$ and $\lim\limits_{k\to\infty}(\Lambda * \varphi_n) = f \in \mathcal{C}^\infty$ on account of the continuity 6.6.1 of Λ we have $f(\mathbf{x}) = \lim\limits_{k\to\infty}(\Lambda*\varphi_n)(\mathbf{x}) = \lim\limits_{k\to\infty}\Lambda(\tau_{\mathbf{x}}\check{\varphi}_n) = \Lambda(\tau_{\mathbf{x}}\check{\varphi}) = (\Lambda*\varphi)(\mathbf{x})$, i.e. $\lim\limits_{k\to\infty}(\varphi_n;\Lambda*\varphi_n) = (\varphi;\Lambda*\varphi)$. Hence the **closed graph theorem** 3.6 implies that L is **continuous**. Furthermore we have $\tau_{\mathbf{x}}(\Lambda_{\boldsymbol{\xi}}*\varphi)(\mathbf{y}) = (\Lambda_{\boldsymbol{\xi}}*\varphi)(\mathbf{y}-\mathbf{x}) = \Lambda_{\boldsymbol{\xi}}\varphi(\mathbf{y}-\mathbf{x}-\boldsymbol{\xi}) = \Lambda_{\boldsymbol{\xi}}(\tau_x(\varphi(\mathbf{y}-\boldsymbol{\xi}))) = \Lambda_{\boldsymbol{\xi}}*(\tau_{\mathbf{x}}\varphi)$ whence $\tau_{\mathbf{x}}L = L\tau_{\mathbf{x}}$.
   ⇐: Since $\varphi\to\check{\varphi}$ is continuous on $\mathcal{D}$ and the evaluation at $\mathbf{0}$ is continuous on C the functional $\Lambda:\mathcal{D}\to\mathbb{R}$ defined by $\Lambda\varphi = (L\check{\varphi})(\mathbf{0})$ is a **distribution**. According to the hypothesis we have $(L\varphi)(\mathbf{x}) = (\tau_{-\mathbf{x}}L\varphi)(\mathbf{0}) = (L\tau_{-\mathbf{x}}\varphi)(\mathbf{0}) = \Lambda(\tau_{-\mathbf{x}}\varphi)\check{} = \Lambda(\tau_{\mathbf{x}}\check{\varphi}) = (\Lambda*\varphi)(\mathbf{x})$ whence its **convolution** represents L. The representation is **uniquely determined** since for every $\Lambda\in\mathcal{D}'$ with $\Lambda*\varphi = 0$ for every $\varphi\in\mathcal{D}$ follows $\Lambda(\check{\varphi}) = (\Lambda*\varphi)(\mathbf{0}) = 0$ whence $\Lambda = 0$.

2. ⇒: Follows from 6.18.5.
   ⇐: For $\varphi\in D$ define $h_\varphi\in C^\infty(\mathbb{R}^n)$ by $h_\varphi(\mathbf{x}) = (\tau_{-\mathbf{x}}L\tau_{\mathbf{x}}\varphi)(0) = (L\tau_{\mathbf{x}}\varphi)(\mathbf{x})$ for $\mathbf{x}\in\mathbb{R}^n$. Then the hypothesis for every basis vector $\mathbf{e}_j\in\mathbb{R}^n$ with $1\leq j\leq n$ implies
   $$\begin{aligned}(D^{\mathbf{e}_j}h_\varphi)(\mathbf{x}) &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left((L\tau_{\mathbf{x}+\epsilon}\varphi)(\mathbf{x}+\epsilon) - (L\tau_{\mathbf{x}}\varphi)(\mathbf{x})\right)\\ &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left((L\tau_{\mathbf{x}+\epsilon}\varphi)(\mathbf{x}+\epsilon) - (L\tau_{\mathbf{x}}\varphi)(\mathbf{x}+\epsilon) + (L\tau_{\mathbf{x}}\varphi)(\mathbf{x}+\epsilon) - (L\tau_{\mathbf{x}}\varphi)(\mathbf{x})\right)\\ &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left((L(\tau_{\mathbf{x}+\epsilon}\varphi - \tau_{\mathbf{x}}\varphi))(\mathbf{x}+\epsilon)\right) + \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(L\tau_{\mathbf{x}}(\varphi(\mathbf{x}+\epsilon) - \varphi(\mathbf{x}))\right)\\ &= (D^{\mathbf{e}_j}L\tau_{\mathbf{x}}\varphi)(\mathbf{x}) - (L\tau_{\mathbf{x}}D^{\mathbf{e}_j}\varphi)(\mathbf{x})\\ &= 0\end{aligned}$$
   whence follows $h_\varphi(\mathbf{x}) = h_\varphi(\mathbf{0})$. This is equivalent to $\tau_{\mathbf{x}}L = L\tau_{\mathbf{x}}$whence by 2. follows the assertion.

6.21 Convolution with smooth functions

Owing to 6.14.5 every distribution $\Lambda\in\mathcal{D}'_L$ with compact support $L\subset\mathbb{R}^n$ extends in a unique way to a $\Gamma\in(C^\infty(\mathbb{R}^n;\mathbb{R}))^*$ defined by $\Gamma f = \Lambda(\psi f)$ with $\psi\in\mathcal{D}_L(\Omega)$ with $\psi|_L = 1$ and $L\prec\psi\prec\Omega$ according to 6.12. Due to the **Hahn-Banach theorem** subsec:The Hahn-Banach-theorem resp. 6.6.2 the **bounded character** of Λ extends to C^∞ such that for every **compact** $K_m = [-\mathbf{m};\mathbf{m}]$ exists a $C_m<\infty$ such that $|\Lambda\varphi|\leq C_m\|f\|_m$ for every $f\in C^\infty$. Hence according to 6.18.5 the **convolution** $\Lambda*f\in\mathcal{C}^\infty$ defined by $(\Lambda*f)(\mathbf{x}) = \Gamma\left(\tau_{\mathbf{x}}\check{f}\right) = \Lambda(\tau_{\mathbf{x}}(\psi f)\check{})$ for $\Lambda\in\mathcal{D}'$ with compact support $L\subset\mathbb{R}^n$; $f\in C^\infty$; $\varphi\in\mathcal{D}_K$ and $\mathbf{x}\in\mathbb{R}^n$ has the following properties:

1. $\tau_{\mathbf{x}}(\Lambda*f) = (\tau_{\mathbf{x}}\Lambda)*f = \Lambda*(\tau_{\mathbf{x}}f)$.
2. $D^{\mathbf{p}}(\Lambda*f) = (D^{\mathbf{p}}\Lambda)*f = \Lambda*(D^{\mathbf{p}}f)$.
3. $\Lambda*(f*\psi) = (\Lambda*f)*\psi = (\Lambda*\psi)*f\in\mathcal{D}$.

Proof:

1. Directly follows from the definitions as in 6.18.3

2. Directly follows from the definitions as in 6.18.4

3. Owing to 6.12 for every $\mathbf{x}\in\mathbb{R}^n$ there is a $\varphi_{\mathbf{x}}\in\mathcal{D}$ with $\varphi_{\mathbf{x}}|_{\mathbf{x}-K-L} = f$ whence for every $\boldsymbol{\xi}\in L$ and $\boldsymbol{\eta}\in\mathbf{x}-\boldsymbol{\xi}-K\in\mathbf{x}-K-L$ follows $\mathbf{x}-\boldsymbol{\xi}-\boldsymbol{\eta}\in K$ and consequently $(f*\psi)(\mathbf{x}-\boldsymbol{\xi}) = \int f(\boldsymbol{\eta})\psi(\mathbf{x}-\boldsymbol{\xi}-\boldsymbol{\eta})\,d\boldsymbol{\eta} = (\varphi_{\mathbf{x}}*\psi)(\mathbf{x}-\boldsymbol{\xi})$, so that
   $$\Lambda*(f*\psi)(\mathbf{x}) = \Lambda_{\boldsymbol{\xi}}(f*\psi)(\mathbf{x}-\boldsymbol{\xi}) = \Lambda_{\boldsymbol{\xi}}(\varphi_{\mathbf{x}}*\psi)(\mathbf{x}-\boldsymbol{\xi}) = \Lambda*(\varphi_{\mathbf{x}}*\psi)(\mathbf{x}).\ \text{(I)}$$

   Also for $\boldsymbol{\xi}\in\mathbf{x}-K$ and $\boldsymbol{\eta}\in L$ we have $f(\boldsymbol{\xi}-\boldsymbol{\eta}) = \varphi_{\mathbf{x}}(\boldsymbol{\xi}-\boldsymbol{\eta})$ whence $(\Lambda*f)(\boldsymbol{\xi}) = \Lambda_{\boldsymbol{\eta}}f(\boldsymbol{\xi}-\boldsymbol{\eta})$

$= \Lambda_{\boldsymbol{\eta}}\varphi(\boldsymbol{\xi} - \boldsymbol{\eta}) = (\Lambda * \varphi_{\mathbf{x}})(\boldsymbol{\xi})$ and thus

$$((\Lambda * f) * \psi)(\mathbf{x}) = \int (\Lambda * f)(\boldsymbol{\xi})\,\psi(\mathbf{x} - \boldsymbol{\xi})\,d\boldsymbol{\xi} = \int (\Lambda * \varphi_{\mathbf{x}})(\boldsymbol{\xi})\,\psi(\mathbf{x} - \boldsymbol{\xi})\,d\boldsymbol{\xi}\,((\Lambda * \varphi_{\mathbf{x}}) * \psi)(\mathbf{x})\,. \text{ (II)}$$

Finally due to 6.18.2 we have $\operatorname{supp}(\Lambda * \psi) \subset L + K$ whence $\operatorname{supp}(\Lambda * \psi)^{\vee} \subset \mathbf{x} - K - L$ and consequently

$$((\Lambda * \psi) * f)(\mathbf{x}) = ((\Lambda * \psi) * \varphi_{\mathbf{x}})(\mathbf{x})\,. \text{ (III)}$$

According to 6.18.6 and the **commutativity** of the convolution the right sides of (I) - (III) coincide which proves the assertion.

6.22 Convolution with distributions

If at least one of the distributions $\Lambda; \Gamma \in \mathcal{D}'$ has a **compact support** the **convolution** $\Lambda * \Gamma \in \mathcal{D}'$ defined by $(\Lambda * \Gamma)\varphi = \Lambda * (\Gamma * \check{\varphi})(\mathbf{0})$ for every $\varphi \in \mathcal{D}$ has the following properties:

1. The distribution $\Lambda * \Gamma \in \mathcal{D}'$ is **uniquely determined** by $(\Lambda * \Gamma) * \varphi = \Lambda * (\Gamma * \varphi)$.
2. $\Lambda * \Gamma = \Gamma * \Lambda$.
3. $\operatorname{supp}(\Lambda * \Gamma) \subset \operatorname{supp}\Lambda + \operatorname{supp}\Gamma$.
4. If at least **two** of $\Lambda; \Gamma; \Delta \in \mathcal{D}'$ have a **compact support** we have $(\Lambda * \Gamma) * \Delta = \Gamma * (\Lambda * \Delta)$.
5. For every $\mathbf{p} \in \mathbb{N}^n$we have $D^{\mathbf{p}}\Lambda = (D^{\mathbf{p}}\delta) * \Lambda$ and in particular $\Lambda = \delta * \Lambda$.
6. For every $\mathbf{p} \in \mathbb{N}^n$we have $D^{\mathbf{p}}(\Lambda * \Gamma) = (D^{\mathbf{p}}\Lambda) * \Gamma = \Lambda * (D^{\mathbf{p}}\Gamma)$.

Proof:

1. $\Lambda * \Gamma$ is well defined since for **compact** $\operatorname{supp}\Gamma$ theorem 6.18.5 implies $\Gamma * \check{\varphi} \in \mathcal{D}$ and $\Lambda * (\Gamma * \check{\varphi}) \in \mathcal{C}^{\infty}$ while for **compact** $\operatorname{supp}\Lambda$ 6.18.5 implies $\Gamma * \check{\varphi} \in \mathcal{C}^{\infty}$ and 6.21 assures that $\Lambda * (\Gamma * \check{\varphi}) \in \mathcal{C}^{\infty}$. In any case the map $\Lambda * \Gamma : \mathcal{D} \to \mathbb{R}$ is **well defined** and **linear** with $(\tau_{\mathbf{x}}(\Lambda * \Gamma))\varphi = \Lambda * (\Gamma * \check{\varphi})(\mathbf{0} - \mathbf{x}) = \Lambda_{\boldsymbol{\eta}}(\Gamma * \check{\varphi})(-\mathbf{x} - \boldsymbol{\eta}) = \Lambda_{\boldsymbol{\eta}}(\Gamma_{\boldsymbol{\xi}}\varphi(\mathbf{x} + \boldsymbol{\eta} - \boldsymbol{\xi})) = \Lambda_{\boldsymbol{\eta}}(\Gamma * (\tau_{\mathbf{x}}\check{\varphi})(\mathbf{0} - \boldsymbol{\eta})) = \Lambda * (\Gamma * (\tau_{\mathbf{x}}\check{\varphi}))(\mathbf{0}) = (\Lambda * \Gamma)\tau_{\mathbf{x}}$ whence $\tau_{\mathbf{x}}(\Lambda * \Gamma) = (\Lambda * \Gamma)\tau_{\mathbf{x}}$ for every $\mathbf{x} \in \mathbb{R}^n$. Hence 6.20 implies that $\Lambda * \Gamma \in \mathcal{D}'$ is **uniquely determined**.
2. For $\varphi; \psi \in \mathcal{D}$ theorem 6.18.6 implies that $(\Lambda * \Gamma) * (\varphi * \psi) = \Lambda * (\Gamma * (\varphi * \psi)) = \Lambda * ((\Gamma * \varphi) * \psi) = \Lambda * (\psi * (\Gamma * \psi))$. Applying 6.21.3 for **compact** $\operatorname{supp}\Gamma$ resp. 6.18.5 in the case of **compact** $\operatorname{supp}\Lambda$ gives $(\Lambda * \Gamma) * (\varphi * \psi) = (\Lambda * \psi) * (\Gamma * \varphi)$ and repeating the computation for with exchanged φ and ψ resp. Λ and Γ yields $(\Gamma * \Lambda) * (\varphi * \psi) = (\Gamma * \varphi) * (\Lambda * \psi)$. Finally a third invocation of the commutativity of the concolution of functions obtains $(\Lambda * \Gamma) * (\varphi * \psi) = (\Gamma * \Lambda) * (\varphi * \psi)$ and the assertion follows from the uniqueness argument at the end of the proof of 6.20.1.
3. By 2. we may assume that $\operatorname{supp}\Gamma$ is **compact** such that as in the proof of 6.21.3 an application of 6.18.2 yields $\operatorname{supp}(\Gamma * \check{\varphi}) \subset \operatorname{supp}\Gamma - \operatorname{supp}\varphi$. Hence $(\Lambda * \Gamma)\varphi = \Lambda\left((\Gamma * \check{\varphi})^{\vee}\right) \neq 0 \Leftrightarrow \operatorname{supp}\Lambda \cap (\operatorname{supp}\varphi - \operatorname{supp}\Gamma) \neq \emptyset \Leftrightarrow (\operatorname{supp}\Lambda + \operatorname{supp}\Gamma) \cap \operatorname{supp}\varphi \neq \emptyset$.
4. We conclude form 3. that both $(\Lambda * \Gamma) * \Delta$ and $\Lambda * (\Gamma * \Delta)$ are defined if two of their supports are compact. For every $\varphi \in \mathcal{D}$ the definition gives $(\Lambda * (\Gamma * \Delta)) * \varphi = \Lambda * ((\Gamma * \Delta) * \varphi) = \Lambda * (\Gamma * (\Delta * \varphi))$. In the case of a **compact** $\operatorname{supp}\Delta$ and because $\Delta * \varphi \in \mathcal{D}$ theorem 6.21.3 yields $((\Lambda * \Gamma) * \Delta) * \varphi = (\Lambda * \Gamma) * (\Delta * \varphi) = \Lambda * (\Gamma * (\Delta * \varphi))$ whence follows the assertion in this case. If $\operatorname{supp}\Delta$ is **not compact** then $\operatorname{supp}\lambda$ is **compact** and the preceding case combined with the **commutativity** proved in 2. gives $\Lambda * (\Gamma * \Delta) = \Lambda * (\Delta * \Gamma) = (\Delta * \Gamma) * \Lambda = \Delta * (\Gamma * \Lambda) = \Delta * (\Lambda * \Gamma) = (\Lambda * \Gamma) * \Delta$.

5. For every $\varphi \in \mathcal{D}$ and $\mathbf{x} \in \mathbb{R}^n$ we have $(\delta * \varphi)(\mathbf{x}) = \delta(\tau_{\mathbf{x}}\check{\varphi}) = (\tau_{\mathbf{x}}\check{\varphi})(\mathbf{0}) = \check{\varphi}(-\mathbf{x}) = \varphi(\mathbf{x})$ whence $\delta * \varphi = \varphi$. With 4. and 6.18.5 follows $(D^{\mathbf{p}}\Lambda) * \varphi = \Lambda * D^{\mathbf{p}}\varphi = \Lambda * D^{\mathbf{p}}(\delta * \varphi) = \Lambda * (D^{\mathbf{p}}\delta) * \varphi$.
6. With 2., 4. resp. 5. we obtain $D^{\mathbf{p}}(\Lambda * \Gamma) = (D^{\mathbf{p}}\delta) * (\Lambda * \Gamma) = ((D^{\mathbf{p}}\delta) * \Lambda) * \Gamma = (D^{\mathbf{p}}\Lambda) * \Gamma$ and $((D^{\mathbf{p}}\delta) * \Lambda) * \Gamma = (\Lambda * D^{\mathbf{p}}\delta) * \Gamma = \Lambda * D^{\mathbf{p}}\Gamma$.

7 Duality in Banach spaces

7.1 The norm of a bounded linear operator

The function $\| \, \| : \mathcal{B}(X;Y) \to [0;\infty[$ defined by $\|\Lambda\| = \sup\left\{\frac{\|\Lambda\mathbf{x}\|_Y}{\|\mathbf{x}\|_X} : \mathbf{x} \in X\right\} = \sup\left\{\|\Lambda\mathbf{x}\|_Y : \|\mathbf{x}\|_X = 1\right\}$ is a **norm** on the vector space $\mathcal{B}(X;Y)$ of all **bounded linear maps** $\Lambda : X \to Y$ between the **normed** vector spaces X and Y. If Y is a **Banach** space so is $\mathcal{B}(X;Y)$. In the case of **infinitely dimensional function spaces** X res. Y the linear map Λ is called an **operator** (usually a **differential operator**) and in the case of $Y = \mathbb{C}$ it is a **functional** (very often represented as an **integral**)

Proof: By the hypothesis we have $\|\Lambda\| < \infty$ and the norm on Y implies $\|\alpha\Lambda\| = |\alpha| \cdot \|\Lambda\|$. Hence for every $\mathbf{x} \in X$ we have

$$\begin{aligned} \|(\Lambda_1 + \mathbf{\Lambda}_2)\mathbf{x}\|_Y &= \|\Lambda_1\mathbf{x} + \Lambda_2\mathbf{x}\|_Y \\ &\leq \|\Lambda_1\mathbf{x}\|_Y + \|\Lambda_2\mathbf{x}\|_Y \\ &\leq (\|\Lambda_1\| + \|\Lambda_2\|) \cdot \|\mathbf{x}\|_X \end{aligned}$$

whence follows $\|\Lambda_1 + \mathbf{\Lambda}_2\| \leq \|\Lambda_1\| + \|\Lambda_2\|$. Finally for every $\Lambda \neq \mathbf{0}$ there is an $\mathbf{x} \neq \mathbf{0}$ with $\Lambda\mathbf{x} \neq 0$ whence $\|\Lambda\| \geq \frac{\|\Lambda\mathbf{x}\|_Y}{\|\mathbf{x}\|_X} > 0$ which shows that $\| \, \| : \mathcal{B}(X;Y) \to [0;\infty[$ is a **norm**.

For a $\| \, \|$-Cauchy sequence $(\Lambda_n)_{n\in\mathbb{N}} \subset \mathcal{B}(X;Y)$ there isn an $M \in \mathbb{N}$ such that for $n;m \geq M$ we have $\|\Lambda_n - \Lambda_m\| < \epsilon$ and every **fixed** $\mathbf{x} \in X$ holds

$$\|\Lambda_n\mathbf{x} - \Lambda_m\mathbf{x}\|_Y \leq \|\Lambda_n - \Lambda_m\| \cdot \|\mathbf{x}\|_X < \epsilon \cdot \|\mathbf{x}\|_X \, ,$$

i.e. $(\mathbf{f}_n(\mathbf{x}))_{n\in\mathbb{N}} \subset Y$ is a $\| \, \|_Y$-Cauchy sequence. In the case of a **complete** space Y follows

$$\lim_{n\to\infty} \|\Lambda\mathbf{y} - \Lambda_n\mathbf{y}\|_Y = 0$$

for a uniquely determined $\Lambda\mathbf{x} \in Y$. Owing to the **continuity of multiplication and addition** the function $\Lambda : X \to Y$ is **linear** such that for every $\epsilon > 0$ there is an $M_x \geq M$ with

$$\|\Lambda\mathbf{x} - \Lambda_n\mathbf{x}\|_Y \leq \epsilon \, \|\mathbf{x}\|_X$$

for $n \geq M_x$. This implies $\frac{\|\Lambda\mathbf{x}\|_Y}{\|\mathbf{x}\|_X} \leq \|\Lambda_n\| + \epsilon$ for every $\mathbf{x} \in X$ and $n \geq M_x$ and since the $\mathbf{f}_n$ due to the $\| \, \|$-Cauchy character of the sequence are **uniformly bounded** we conclude that $\Lambda \in \mathcal{B}(X;Y)$. Due to 2.10.2 the **bounded** linear maps Λ resp. $\mathbf{f}_M$ are **uniformly continuous** such that for every $\mathbf{x} \in \delta B_1(\mathbf{0})$ there is a neighbourhood $U(\mathbf{x})$ with $\|\Lambda\mathbf{x} - \Lambda_{M_x}\mathbf{x}\|_Y \leq 2\epsilon$ for $\mathbf{x} \in U(\mathbf{x}) \cap \delta B_1(\mathbf{0})$. For $n;m \geq M$ we have $\|\Lambda\mathbf{x}_n - \Lambda_{M_x}\mathbf{x}\|_Y \leq \epsilon$ whence $\|\Lambda\mathbf{x} - \Lambda_n\mathbf{x}\|_Y \leq 3\epsilon$ for $n \geq M_x$. The finitely many $(U(\mathbf{x}_i))_{1\leq i\leq k}$ covering the **compact** set $\delta B_1(\mathbf{0})$ provide an $N = \max\{M_{x_1};...;M_{x_k}\}$ such that $\|\Lambda\mathbf{x} - \Lambda_n\mathbf{x}\|_Y \leq 3\epsilon$ for $\mathbf{x} \in \delta B_1(\mathbf{0})$ and $n \geq N$ whence $\|\Lambda - \Lambda_n\| \leq 3\epsilon$, i.e. $\lim\limits_{n\to\infty} \|\Lambda_n - \Lambda\| = 0$.

7.2 The norm* of the dual space

According to the preceding paragraph 7.1 the **norm*** $\| \, \|^* : X^* \to [0;\infty[$ defined by $\|\mathbf{x}^*\|^* = \sup\{|\langle\mathbf{y};\mathbf{x}^*\rangle| : \|\mathbf{y}\| = 1\}$ with the usual notation $\langle\mathbf{y};\mathbf{x}^*\rangle = \mathbf{x}^*\mathbf{y}$ on the dual space X^* of the **continuous** and due to 2.10.2 **bounded** linear functionals $\mathbf{x}^* : X \to \mathbb{C}$ is **complete** such that $(X^*; \| \, \|^*)$

is a **Banach space**. According to [7, th. 3.10] in the **finite dimensional case** X is **isomorphic** (and **isometric**) to its dual space X^*. As the theorems of **Lebesgue-Radon-Nikodym** [8, th. 9.8] resp. **Riesz** [8, th. 10.11] show in the general case this is not true so that we have to confine ourselves to subspaces.

According to 5.8 the **weak* topology** generated by the neighbourhoods

$$U^*_{\mathbf{y};\epsilon}(0^*) = \{\mathbf{x}^* \in X^* : |\langle \mathbf{y}; \mathbf{x}^* \rangle| < \epsilon\}$$

for $\mathbf{y} \in X$ **and** $\epsilon > 0$ is the smallest topology on X^* such that all **linear evaluation functionals** $\mathbf{x} : X^* \to \mathbb{C}$ defined by $\mathbf{x}\mathbf{y}^* = \mathbf{y}^*\mathbf{x} = \langle \mathbf{x}; \mathbf{y}^* \rangle$ for $\mathbf{y}^* \in X^*$ are **continuous**. Due to $\{\mathbf{y}^* \in X^* : \|\mathbf{y}^*\|^* < \epsilon\} \subset \left\{\mathbf{y}^* \in X^* : \left|\left\langle \frac{\mathbf{x}}{\|\mathbf{x}\|}; \mathbf{y}^* \right\rangle\right| < \epsilon\right\} = \{|\mathbf{x}| < \epsilon \cdot \|\mathbf{x}\|\}$ the evaluation functionals are also **originally continuous** and obviously **originally bounded** with referencet to the original norm $\| \|^*$. Hence **the weak* topology is included in the original norm* topology** on X^*.

7.3 The norm of the bidual space

The vector subspace $C^*(X^*; \mathbb{C}) \subset X^{**}$ of the **weakly* continuous** and due to 2.10.2 **bounded linear evaluation functionals** $\mathbf{x}^{**} : X^* \to \mathbb{C}$ by $\varphi : X \to C^*(X^*; \mathbb{C})$ with $\mathbf{y}^*\varphi(\mathbf{x}) = \mathbf{x}\mathbf{y}^* = \langle \mathbf{x}; \mathbf{y}^* \rangle$ for every $\mathbf{y}^* \in X^*$ is **isomorphic** to X.

The **norm** $\| \|^{**} : C^*(X^*; \mathbb{C}) \to [0; \infty[$ defined by

$$\|\varphi(\mathbf{x})\|^{**} = \sup\left\{\left|\left\langle \mathbf{x}; \frac{\mathbf{y}^*}{\|\mathbf{y}^*\|^*} \right\rangle\right| : \mathbf{y}^* \in X^*\right\} = \sup\{|\langle \mathbf{x}; \mathbf{y}^* \rangle| : \|\mathbf{y}^*\|^* = 1\}$$

coincides with $\|\mathbf{x}\| = \|\varphi(\mathbf{x})\|^{**}$ so that $(C^*(X^*; \mathbb{C}); \| \|^{**})$ is also **isometric** to $(X; \| \|)$.

Proof: According to the corollary 4.3 to the **Hahn-Banach theorem** for every $\mathbf{x} \in X$ there is an $\mathbf{y}^* \in X^*$ with $\langle \mathbf{x}; \mathbf{y}^* \rangle = \|\mathbf{x}\|$ and $\langle \mathbf{z}; \mathbf{y}^* \rangle \leq \|\mathbf{z}\| \; \forall \mathbf{z} \in X$ whence $\|\mathbf{y}^*\|^* = 1$ such that we infer $\|\varphi(\mathbf{x})\|^{**} \geq \|\mathbf{x}\|$. The definition of the dual norm implies $\left|\left\langle \frac{\mathbf{x}}{\|\mathbf{x}\|}; \mathbf{y}^* \right\rangle\right| \leq \|\mathbf{y}^*\|^*$ resp. $\left|\left\langle \mathbf{x}; \frac{\mathbf{y}^*}{\|\mathbf{y}^*\|^*} \right\rangle\right| \leq \|\mathbf{x}\|$ for every $\mathbf{y}^* \in Y$ whence follows $\|\varphi(\mathbf{x})\|^{**} \leq \|\mathbf{x}\|$ and hence the assertion.

Note: According to [11, th. 14.2.3] the **complete** vector subspace $C^*(X^*; \mathbb{C}) = \varphi[X]$ is **closed** in X^{**}. A Banach space X is called **reflexive** iff $C^*(X^*; \mathbb{C}) = X^{**}$. The most important examples are the **Lebesgue spaces** $L^p(\lambda)$ for $1 < p < \infty$.

7.4 Annihilators

According to [7, th. 3.13] the **annihilators** of the vector subspaces $M \subset X$ resp. $N \subset X^*$ of a Banach space X are defined as the vector subspaces

$$M^0 = \{\mathbf{x}^* \in X^* : \langle \mathbf{y}; \mathbf{x}^* \rangle = 0 \, \forall \mathbf{y} \in M\} = \bigcap_{\mathbf{y} \in M} \ker \varphi(\mathbf{y}) \subset X^*$$

$${}^0N = \{\mathbf{x} \in X : \langle \mathbf{x}; \mathbf{y}^* \rangle = 0 \, \forall \mathbf{y}^* \in N\} = \bigcap_{\mathbf{y}^* \in N} \ker \mathbf{y}^* \subset X$$

Note that

1. M^0 is **weakly* closed** hence **norm* closed** in X^* since the $\varphi(\mathbf{y}) \in X^{**}$ are both weakly* and norm* continuous.
2. ${}^0N$ is **weakly closed** hence **norm closed** in X since the $\mathbf{y}^* \in N$ are both weakly and norm continuous.

7.5 The closure of the annihilator

For vector subspaces $M \subset X$ resp. $N \subset X^*$ of a Banach space X

1. $\overline{M} = \overline{M}_w = {}^0\left(M^0\right)$
2. $\overline{N} = \overline{N}_w = \left({}^0N\right)^0$

Proof: Due to 5.5 and since vector subspaces are **convex** the **weak** and **norm** closures of $\overline{M} = \overline{M}_w \subset X$ resp. the **weak*** and **norm*** closures of $\overline{N} = \overline{N}_w \subset X^*$ coincide.

For $\mathbf{x} \in {}^0\left(M^0\right)$ we have $\langle \mathbf{x}; \mathbf{y}^* \rangle = 0 \,\forall \mathbf{y}^* \in M^0$ whence follows $\mathbf{x} \in M$. Furthermore ${}^0\left(M^0\right)$ is **weakly closed** hence **norm closed** in X whence is includes the closure $\overline{M} = \overline{M}_w \subset {}^0\left(M^0\right)$. Due to the corollary 4.4 of the **Hahn-Banach theorem** for every $\mathbf{x} \notin \overline{M}$ there is an $\mathbf{y}^* \in X^*$ with $\langle \mathbf{x}; \mathbf{y}^* \rangle = 1$ and $\langle \mathbf{z}; \mathbf{y}^* \rangle = 0 \,\forall \mathbf{z} \in M$, i.e. $\mathbf{y}^* \in M^0$ hence $\mathbf{x} \notin {}^0\left(M^0\right)$. The second proposition is proved analogously.

7.6 The dual of the quotient space

For every **closed** vector subspace $M \subset X$ of a **Banach** space X the maps

1. $\varphi : M^* \to X^*/M^0$ defined by $\varphi\left(\mathbf{m}^*\right) = \mathbf{x}^*_{\mathbf{m}} + M^0$ with the **extension** $\mathbf{x}^*_{\mathbf{m}} \in X^*$ of every $\mathbf{m}^* = \mathbf{x}^*_{\mathbf{m}}|_M \in M^*$ according to the **Hahn-Banach-theorem** 4.2 and
2. $\psi : (X/M)^* \to M^0$ defined by $\psi\left(\mathbf{z}^*\right) = \mathbf{z}^* \circ \pi$ with the **canonical projection** $\pi : X \to X/M$

are both **isometric isomorphisms**.

Proof: For the sake of clarity and deviating from 7.2 in this proof we use the same notation for the three norms on X, X^* and X/Y .

1. Directly from the definition we infer that φ is **well defined**, **injective**, **surjective** and **linear**. According to 2.10.5 on the one hand we have $\left\|\varphi\left(\mathbf{m}^*\right)\right\| = \inf\left\{\left\|\mathbf{x}^*_{\mathbf{m}} + \mathbf{n}^*\right\| : \mathbf{n}^* \in M^0\right\} \leq \left\|\mathbf{x}^*_{\mathbf{m}}\right\|$ whereas on the other hand owing to the definition 7.2 of the dual norm holds
$$\begin{aligned}\left\|\mathbf{m}^*\right\| &= \sup\left\{\left|\langle \mathbf{y}; \mathbf{m}^* \rangle\right| : \mathbf{y} \in M \vee \|\mathbf{y}\| = 1\right\} \\ &\leq \sup\left\{\left|\langle \mathbf{y}; \mathbf{x}^*_{\mathbf{m}} \rangle + \langle \mathbf{y}; \mathbf{n}^* \rangle\right| : \|\mathbf{y}\| = 1\right\} \\ &= \left\|\mathbf{x}^*_{\mathbf{m}} + \mathbf{n}^*\right\|\end{aligned}$$
such that we obtain $\left\|\mathbf{m}^*\right\| \leq \left\|\varphi\left(\mathbf{m}^*\right)\right\| \leq \left\|\mathbf{x}^*_{\mathbf{m}}\right\|$. Due to the **Hahn-Banach-theorem** 4.2 there is an extension $\mathbf{x}^*_{\mathbf{m}} \in X^*$ of $\mathbf{m}^*$ with $\left\|\mathbf{x}^*_{\mathbf{m}}\right\| \leq \left\|\mathbf{m}^*\right\|$ which completes the proof.
2. The map ψ is obviously **well defined, injective** and **linear**. For every $\mathbf{y}^* \in M^0$ and since $M \subset \ker \mathbf{y}^*$ there is an $\mathbf{z}^* \in (X/M)^*$ with $\mathbf{z}^* \circ \pi = \mathbf{y}^*$. Since $\ker \mathbf{z}^* = \pi\left(\ker \mathbf{y}^*\right)$ is a closed vector subspace of X/M and due to 1.11.2 the map $\mathbf{z}^*$ is **continuous** whence $\mathbf{z}^* \in (X/M)^*$. Hence ψ is **surjective**. Due to the proof of 2.10.5 we have $\|\mathbf{x}\| = 1 \Leftrightarrow \left\|\pi\left(\mathbf{x}\right)\right\| = 1$ whence $\left\|\psi\left(\mathbf{z}^*\right)\right\| = \left\|\mathbf{z}^* \circ \pi\right\| = \sup\left\{\left|\langle \pi\left(\mathbf{x}\right); \mathbf{z}^* \rangle\right| : \mathbf{x} \in X \vee \|\mathbf{x}\| = 1\right\} = \sup\left\{\left|\langle \mathbf{y}; \mathbf{z}^* \rangle\right| : \mathbf{y} \in X/Y \vee \|\mathbf{y}\| = 1\right\} = \left\|\mathbf{z}^*\right\|$ which proves that ψ is an **isometry**.

7.7 The adjoint of a bounded operator

Analogously to the finite dimensional case from [7, th. 6.11] for every $\Lambda \in \mathcal{B}(X; Y)$ between Banach spaces $X; Y$ the **adjoint** map $\Lambda^* : Y^* \to X^*$ defined by $\Lambda^* \mathbf{y}^* = \mathbf{y}^* \circ \mathbf{f}$ is **linear and bounded** with $\langle \Lambda \mathbf{x}; \mathbf{y}^* \rangle = \langle \mathbf{x}; \Lambda^* \mathbf{y}^* \rangle$ and $\|\Lambda^*\| = \|\Lambda\|$.

Proof: The map Λ^* is obviously **well defined** and **linear** with $\langle \Lambda \mathbf{x}; \mathbf{y}^* \rangle = \langle \mathbf{x}; \mathbf{y}^* \circ \Lambda \rangle = \langle \mathbf{x}; \Lambda^* \mathbf{y}^* \rangle$. Also by 7.1 and 7.3 we have $\|\Lambda \mathbf{x}\| = \sup\limits_{\|\mathbf{y}^*\|=1} |\langle \Lambda \mathbf{x}; \mathbf{y}^* \rangle|$ such that

$$\begin{aligned}
\|\Lambda^*\| &= \sup_{\|\mathbf{y}^*\|=1} \|\Lambda^* \mathbf{y}^*\| \\
&= \sup_{\|\mathbf{y}^*\|=1} \sup_{\|\mathbf{x}\|=1} |\langle \mathbf{x}; \Lambda^* \mathbf{y}^* \rangle| \\
&= \sup_{\|\mathbf{x}\|=1} \sup_{\|\mathbf{y}^*\|=1} |\langle \Lambda \mathbf{x}; \mathbf{y}^* \rangle| \\
&= \sup_{\|\mathbf{x}\|=1} \|\Lambda \mathbf{x}\| \\
&= \|\Lambda\| .
\end{aligned}$$

7.8 Properties of the adjoint operator

For $\Lambda \in \mathcal{B}(X; Y)$ on Banach spaces $X; Y$ we have

1. $\ker \Lambda^* = (\Lambda [X])^0$.
2. $\ker \Lambda = {}^0 (\Lambda^* [Y^*])$.
3. $\ker \Lambda^* \subset Y^*$ is **weakly* closed.**
4. Λ^* is **bijective** iff $\Lambda [X]$ is **dense** in Y.
5. Λ is **bijective** iff $\Lambda^* [Y^*]$ is **weakly* dense** in X^*.

Proof:

1. $\mathbf{y}^* \in \ker \Lambda^* \Leftrightarrow \Lambda^* \circ \mathbf{y}^* = \mathbf{0} \Leftrightarrow \langle \mathbf{x}; \Lambda^* \mathbf{y}^* \rangle = \langle \Lambda \mathbf{x}; \mathbf{y}^* \rangle = 0 \, \forall \mathbf{x} \in X \Leftrightarrow \mathbf{y}^* \in (\Lambda [X])^0$
2. $\mathbf{x} \in \ker \Lambda \Leftrightarrow \Lambda \mathbf{x} = 0 \Leftrightarrow \langle \Lambda \mathbf{x}; \mathbf{y}^* \rangle = \langle \mathbf{x}; \Lambda^* \mathbf{y}^* \rangle = 0 \, \forall \mathbf{y}^* \in Y^* \Leftrightarrow \mathbf{x} \in {}^0 (\Lambda^* [Y^*])$
3. 8.4.1 with $M = \ker \Lambda^*$
4. 8.5.1 with $M = \Lambda [X]$
5. 8.5.2 with $N = \Lambda^* [Y^*]$

7.9 Open and surjective operators

For $\Lambda \in \mathcal{B}(X; Y)$ on Banach spaces X and Y the following statements are equivalent:

1. There is an $\delta > 0$ such that $\|\Lambda^* \circ \mathbf{y}^*\| \geq \delta \|\mathbf{y}^*\|$ for every $\mathbf{y}^* \in Y^*$
2. There is an $\delta > 0$ such that $B_\delta(\mathbf{0}_Y) \subset \overline{\Lambda [B_1(\mathbf{0}_X)]}$
3. There is an $\delta > 0$ such that $B_\delta(\mathbf{0}_Y) \subset \Lambda [B_1(\mathbf{0}_X)]$, i.e. **f** is an **open mapping**
4. $Y = \Lambda [X]$, i.e. Λ is **surjective**

Proof:

1. ⇒ 2.: According to 4.5.2 for $\mathbf{y}_0 \notin \overline{\Lambda [B_1(\mathbf{0}_X)]}$ there is an $\mathbf{y}^*$such that $|\langle \mathbf{y}_0; \mathbf{y}^* \rangle| > 1$ but $|\langle \mathbf{y}; \mathbf{y}^* \rangle| \leq 1$ for every $\mathbf{y} \in \overline{\Lambda [B_1(\mathbf{0}_X)]}$. Hence for every $\mathbf{x} \in B_1(\mathbf{0}_X)$ we have $|\langle \mathbf{x}; \Lambda^* \circ \mathbf{y}^* \rangle| = |\langle \Lambda \mathbf{x}; \mathbf{y}^* \rangle| \leq 1$ whence owing to the continuity of $\mathbf{x} \mapsto |\langle \Lambda \mathbf{x}; \mathbf{y}^* \rangle|$ follows $\|\Lambda^* \circ \mathbf{y}^*\| \leq 1$. The hypothesis then implies $\delta < \delta |\langle \mathbf{y}_0; \mathbf{y}^* \rangle| \leq \delta \|\mathbf{y}_0\| \cdot \|\Lambda^*\| \leq \|\mathbf{y}_0\| \cdot \|\Lambda^* \circ \mathbf{y}^*\| \leq \|\mathbf{y}_0\|$ whence we infer that $\mathbf{y} \in \overline{\Lambda [B_1(\mathbf{0}_X)]}$ for every $\mathbf{y} \in Y$ with $\|\mathbf{y}\| < \delta$.

2. ⇒ 3.: This step specializes the main part of the proof of the **open mapping theorem** 3.4: We proceed by **induction** and for $\mathbf{y}_1 \in B_1(\mathbf{0}_Y)$ we choose a sequence $(\epsilon_n)_{n\geq 1}$ such that $\sum_{n\geq 1} \epsilon_n < 1 - \|\mathbf{y}_1\|$. For given y_n and w.l.o.g. $\delta = 1$ the hypothesis implies the existence of an $\mathbf{x}_n \in X$ with $\|\mathbf{x}_n\| \leq \|\mathbf{y}_n\|$ and $\|\mathbf{y}_n - \Lambda \mathbf{x}_n\| < \epsilon_n$. By $\mathbf{y}_{n+1} = \mathbf{y}_n - \Lambda \mathbf{x}_n$ we obtain two sequences $(\mathbf{x}_n)_{n\geq 1}$ and

$(\mathbf{y}_n)_{n\geq 1}$ with $\|\mathbf{x}_{n+1}\| \leq \|\mathbf{y}_{n+1}\| = \|\mathbf{y}_n - \mathbf{f}(\mathbf{x}_n)\| < \epsilon_n$. Hence follows $\sum_{n\geq 1}\|\mathbf{x}_n\| \leq \|\mathbf{x}_1\| + \sum_{n\geq 1}\epsilon_n \leq \|\mathbf{y}_1\| + \sum_{n\geq 1}\epsilon_n < 1$ such that by the **completeness** of X resp. the **triangle equation** we infer that $\mathbf{x} = \sum_{n\geq 1}\mathbf{x}_n \in B_1(\mathbf{0}_X)$. From the **continuity** of Λ and $\lim\limits_{n\to\infty}\mathbf{y}_n = 0$ follows $\Lambda\mathbf{x} = \lim\limits_{N\to\infty}\sum\limits_{n=1}^{N}\Lambda\mathbf{x}_n = \lim\limits_{N\to\infty}\sum\limits_{n=1}^{N}(\mathbf{y}_n - \mathbf{y}_{n+1}) = \mathbf{y}_1$, i.e. the assertion.

3. $\Rightarrow$ 4.: Follows from the linearity of Λ.

4. $\Rightarrow$ 1.: By the **open mapping theorem** 3.4 there is a $\delta > 0$ such that $B_\delta(\mathbf{0}_Y) \subset \Lambda[B_1(\mathbf{0}_X)]$. Owing to the continuity of $\mathbf{x} \mapsto |\langle\mathbf{x};\Lambda^*\mathbf{y}^*\rangle|$

$$\begin{aligned}\|\Lambda^*\circ\mathbf{y}^*\| &= \sup\{|\langle\mathbf{x};\Lambda^*\mathbf{y}^*\rangle| : \mathbf{x}\in B_1(\mathbf{0}_X)\}\\ &= \sup\{|\langle\Lambda\mathbf{x};\mathbf{y}^*\rangle| : \mathbf{x}\in B_1(\mathbf{0}_X)\}\\ &\geq \sup\{|\langle\mathbf{y};\mathbf{y}^*\rangle| : \mathbf{y}\in B_\delta(\mathbf{0}_Y)\}\\ &= \delta\|\mathbf{y}^*\|.\end{aligned}$$

7.10 The closed range theorem

For $\mathbf{f}\in\mathcal{B}(X;Y)$ on Banach spaces X and Y the following three statements are equivalent:

1. $\mathbf{f}^*[Y^*]$ is **closed** in X^*.
2. $\mathbf{f}[X]$ is **closed** in Y.
3. $\mathbf{f}^*[Y^*]$ is **weakly* closed** in X^*.

Proof:

1. $\Rightarrow$ 2.: By $\Gamma\mathbf{x} = \Lambda\mathbf{x}$ for every $\mathbf{x}\in X$ we define a $\Gamma : X \to \overline{\Lambda[X]}$ coinciding with Λ on X and proceed to show that Γ is **surjective** whence follows the assertion: Since $\Gamma[X]$ is **dense** in $\overline{\Lambda[X]}$theorem 8.8.4 implies that $\Gamma^* : \overline{\Lambda[X]}^* \to X^*$ is **bijective**. For every $\mathbf{y}^*\in\overline{\Lambda[X]}^*$ the **Hahn-Banach theorem** 4.2 furnishes an extension $\overline{\mathbf{y}}^*\in Y^*$ with $\langle\mathbf{x};\Lambda\overline{\mathbf{y}}^*\rangle = \langle\Lambda\mathbf{x};\overline{\mathbf{y}}^*\rangle = \langle\Gamma\mathbf{x};\mathbf{y}^*\rangle = \langle\mathbf{x};\Gamma^*\mathbf{y}^*\rangle$. Hence $\Gamma^*\mathbf{y}^* = \Lambda^*\overline{\mathbf{y}}^*$ coincide for every $\mathbf{x}\in X$, i.e. Λ^* and Γ^* must have identical ranges $\Gamma^*\left[\overline{\Lambda[X]}^*\right] = \Lambda^*[Y^*]$. From the hypothesis follows that $\Gamma^*\left[\overline{\Lambda[X]}^*\right]$ is **closed**, hence **complete** such that the **open mapping theorem** 3.4 applies to Γ^*. Since Γ^* is bijective the open character of Γ^* implies that there exists a $c > 0$ such that $B_c(\mathbf{0}^*_{X^*}) \subset \Gamma^*[B_1(\mathbf{0}^*_{Y^*})]$, i.e. $\|\mathbf{x}^*\| \leq c\|\Gamma^*\mathbf{x}^*\|$ for every $\mathbf{x}^*\in X^*$. By the preceding theorem 7.9 follows $\Lambda[X] = \Gamma[X] = \overline{\Lambda[X]}$ and hence the assertion.

2. $\Rightarrow$ 3.: Due to [11, th. 14.2.2] the image $\Lambda[X]$ is **complete**, and owing to **Baire's theorem** [11, 16.4.2 resp. 16.2.4] it is of **second category** such that the **open mapping theorem** 3.4 applies. Hence there exists an $K > 0$ such that $B_{1/K}(\mathbf{0}_Y) \subset \Lambda[B_1(\mathbf{0}_X)]$, i.e. for every $\mathbf{y} = \Lambda(\mathbf{x}) \in \Lambda[X]$ there is an $\mathbf{x}\in X$ with $\|\mathbf{x}\| \leq K\|\Lambda\mathbf{x}\| = K\|\mathbf{y}\|$. Thus the **linear functional** $\Gamma : \Lambda[X] \to \mathbb{C}$ defined for every $\mathbf{x}^*\in(\ker\Lambda)^0$ by $\Gamma\Lambda\mathbf{x} = \langle\mathbf{x};\mathbf{x}^*\rangle$ is **continuous** since $|\Gamma\mathbf{y}| = |\Gamma\Lambda\mathbf{x}| = |\langle\mathbf{x};\mathbf{x}^*\rangle| \leq K\|\mathbf{y}\|\|\mathbf{x}^*\|$. By the **Hahn-Banach theorem** 4.2 some $\mathbf{y}^*\in Y^*$ extends Γ with $\langle\Lambda\mathbf{x};\mathbf{y}^*\rangle = \Gamma\Lambda\mathbf{x} = \langle\mathbf{x};\mathbf{x}^*\rangle$ whence the definition 7.7 of the adjoint map implies $\mathbf{x}^* = \Lambda^*\mathbf{y}^*$. Hence we have shown that $(\ker\Lambda)^0 \subset \Lambda^*[Y^*]$ and the assertion follows from 8.5.2 and 8.8.2 since $\overline{\Lambda^*[Y^*]}_w = \left({}^0(\Lambda^*[Y^*])\right)^0 = (\ker\Lambda)^0$.

3. $\Rightarrow$ 1.: follows directly from the definitions in 5.3.

7.11 Surjective operators

A **linear**, **continuous** and **bounded** map $\Lambda\in\mathcal{B}(X;Y)$ on Banach spaces X and Y is **surjective** iff its **adjoint** Λ^* is **bijective** and $\mathbf{f}^*[Y^*]$ is **closed** in X^*.

Proof:

$\Rightarrow$: The **bijectivity** of Λ^* follows from 8.8.4 and due to 8.9.1 the **dilation principle** [11, th. 14.11] applies to yield its **closed** character.

$\Leftarrow$: The image $\Lambda[X]$ is **dense** by 8.8.4 is and **closed** by 8.10.2.

7.12 Compact operators

A **linear**, **continuous** and **bounded** map $\Lambda \in \mathcal{B}(X;Y)$ on Banach spaces X and Y is **compact** iff one of the following equivalent conditions is satisfied:

1. $\overline{\Lambda[B_1(\mathbf{0}_X)]}$is **sequentially compact** .
2. $\overline{\Lambda[B_1(\mathbf{0}_X)]}$ is **compact**.
3. $\Lambda[B_1(\mathbf{0}_X)]$ is **totally bounded**.
4. For every bounded sequence $(\mathbf{x}_n)_{n\in\mathbb{N}} \subset X$ exists a subsequence $(\mathbf{x}_{n_k})_{k\in\mathbb{N}}$ such that its image $(\Lambda\mathbf{x}_{n_k})_{k\in\mathbb{N}}$ converges to an $\mathbf{y} \in Y$.

Also

5. The vector subspace $\mathcal{C}(X;Y) \subset \mathcal{B}(X;Y)$ of the **compact** operators is **norm closed.**

Proof:

1. ⇔ 2.: follows from [11, th. 10.12] since Y is **metric** and **complete**.

1. ⇔ 3.: follows from [11, th. 17.5] since Y is **metric** and **complete**.

1. ⇔ 4.: follows from the definition [11, def. 10.11] since X and Y are **vector spaces** and **f** is **linear**.

5.: For **compact** $\Lambda;\Gamma \in \mathcal{B}(X;Y)$ resp. $\alpha;\beta \in \mathbb{C}$ and due to 8.12.3 the set $(\alpha\Lambda+\beta\Gamma)[B_1(\mathbf{0}_X)]$ is obviously **totally bounded** such that the compact operators form a **vector subspace** $\mathcal{C}(X;Y) \subset \mathcal{B}(X;Y)$. For any $\Lambda \in \overline{\mathcal{C}}$ and $r>0$ there is a $\Gamma \in \mathcal{C}$ with $\|\Lambda-\Gamma\|<r$ and since $\Gamma[B_1(\mathbf{0}_X)]$ is totally bounded there are $(\mathbf{x}_i)_{1\le i\le n} \subset B_1(\mathbf{0}_X)$ such that $\Gamma[B_1(\mathbf{0}_X)] \subset \bigcup\limits_{i=1}^{n} B_r(\Gamma\mathbf{x}_i)$. Since $\|\Lambda\mathbf{x}-\Gamma\mathbf{x}\|<r$ for every $\mathbf{x} \in B_1(\mathbf{0}_X)$ it follows that $\Lambda[B_1(\mathbf{0}_X)] \subset \bigcup\limits_{i=1}^{n} B_{3r}(\Lambda\mathbf{x}_i)$ whence $\Lambda \in \mathcal{C}(X;Y)$ by 8.12.3.

7.13 Operators with finite dimensional range

1. If $\dim Y<\infty$ then every $\Lambda \in \mathcal{B}(X;Y)$ is **compact**.
2. If $\Lambda \in \mathcal{B}(X;Y)$ is **compact** with a **closed image** $\Lambda[X]$ then $\dim Y<\infty$.

Proof:

1. Due to 1.13 the neighbourhood $\overline{B_1(\mathbf{0}_X)}$ is **compact**. Owing to [11, th. 9.8] its continuous image $\Lambda\left[\overline{B_1(\mathbf{0}_X)}\right]$ is compact hence totally bounded and since $\Lambda[B_1(\mathbf{0}_X)] \subset \Lambda\left[\overline{B_1(\mathbf{0}_X)}\right]$ the assertion follows from 8.12.3.
2. The **closed** image $\Lambda[X]$ is also complete in the Banach space Y such that the **open mapping theorem** 3.4 applies whence Λ is a **homeomorphism**. In this case condition 8.12.2 implies that $\Lambda[X]$ is **locally compact** whence by 1.13 the image has a **finite dimension** $\dim \operatorname{im}\Lambda<\infty$.

7.14 Compact adjoints

A **linear**, **continuous** and **bounded** operator $\Lambda \in \mathcal{B}(X;Y)$ on Banach spaces X and Y is **compact** iff its **adjoint** Λ^* is compact.

Proof:

⇒: Any sequence $(\mathbf{y}_n^*)_{n\in\mathbb{N}} \subset B_1(\mathbf{0}_{Y^*})$ is **equicontinuous** since for every $\mathbf{y};\mathbf{y}' \in Y$ we have $|\mathbf{y}_n^*\mathbf{y}-\mathbf{y}_n^*\mathbf{y}'| = |\mathbf{y}_n^*(\mathbf{y}-\mathbf{y}')| \le \|\mathbf{y}-\mathbf{y}'\|$. Since according to the hypothesis $\overline{\Lambda[B_1(\mathbf{0}_X)]}$ is compact the **Arzela-Ascoli theorem** [11, th. 19.6] implies the existence of a **uniformly converging** subsequence $\left(\mathbf{y}_{n_k}^*\right)_{k\in\mathbb{N}} \subset \overline{\Lambda[B_1(\mathbf{0}_X)]}$. Due to $\left\|\Lambda^*\mathbf{y}_{n_k}^*-\Lambda^*\mathbf{y}_{n_l}^*\right\| = \sup\limits_{\|\mathbf{x}\|=1}\left|\left\langle\Lambda\mathbf{x};\mathbf{y}_{n_k}^*-\mathbf{y}_{n_l}^*\right\rangle\right| = \sup\limits_{\|\mathbf{x}\|=1}\left|\mathbf{y}_{n_k}^*\Lambda\mathbf{x}-\mathbf{y}_{n_l}^*\Lambda\mathbf{x}\right|$

the sequence $\left(\Lambda^* \mathbf{y}^*_{n_k}\right)_{k\in\mathbb{N}} \subset X^*$ is Cauchy whence the assertion follows due to 8.12.4 and since X^* is **complete**.

$\Leftarrow$: According to7.3 the **isometries**

$\varphi : X \to C^*(X^*;\mathbb{C}) \subset X^{**}$ defined by $\mathbf{y}^*\varphi(\mathbf{x}) = \mathbf{x}\mathbf{y}^* = \langle \mathbf{x};\mathbf{y}^*\rangle$ for every $\mathbf{x} \in X$ resp.

$\psi : Y \to C^*(Y^*;\mathbb{C}) \subset Y^{**}$ defined by $\mathbf{x}^*\psi(\mathbf{y}) = \mathbf{y}\mathbf{x}^* = \langle \mathbf{y};\mathbf{x}^*\rangle$ for every $\mathbf{y} \in Y$

allow the application of the first part to the adjoint Λ^*: For $\mathbf{x} \in X$ and $\mathbf{y}^* \in Y^*$ we have

$\langle \mathbf{y}^*;\psi\Lambda\mathbf{x}\rangle = \langle \Lambda\mathbf{x};\mathbf{y}^*\rangle = \langle \mathbf{x};\Lambda^*\mathbf{y}^*\rangle = \langle \Lambda^*\mathbf{y}^*;\varphi\mathbf{x}\rangle = \langle \mathbf{y}^*;\Lambda^{**}\varphi\mathbf{x}\rangle$

so that $\psi\Lambda = \Lambda^{**}\varphi$. For $\mathbf{x} \in B_1(\mathbf{0}_X)$ follows $\varphi(\mathbf{x}) \in B_1(\mathbf{0}_{X^{**}})$ whence $(\psi \circ \Lambda)[B_1(\mathbf{0}_X)] \subset \Lambda^{**}[B_1(\mathbf{0}_{X^{**}})]$. The first part of the theorem yields that Λ^{**} is compact and $\Lambda^{**}[B_1(\mathbf{0}_{X^{**}})]$ is totally bounded. This property obviously extends to the subset $(\psi \circ \Lambda)[B_1(\mathbf{0}_X)]$ and further to $\Lambda[B_1(\mathbf{0}_X)]$ since ψ is an isometry. Hence Λ is compact.

7.15 Direct sums

A **closed** vector subspace $M \subset X$ of a topological vector space X has a **closed complement** N such that $X = M \oplus N$ if

1. $\dim M < \infty$ and X is **locally convex** or
2. $\dim X/M < \infty$.

Proof:

1. For $M = \operatorname{span}\left\{(\mathbf{e}_i)_{1\le i\le n}\right\}$ and every $\mathbf{x} = \sum\limits_{i=1}^{n} a_i(\mathbf{x})\,\mathbf{e}_i$ the **coefficients** $a_i : M \to \mathbb{C}$ are **linear** and **continuous functionals** which by the **Hahn-Banach theorem** 4.2 can be extended to linear and continuous $\overline{a}_i : X \to \mathbb{C}$. Then $N = \bigcap\limits_{i=1}^{n} \ker \overline{a}_i$ is the desired complement.
2. For $X/M = \operatorname{span}\left\{(\mathbf{e}_i + M)_{1\le i\le n}\right\}$ the desired complement is obtained by $N = \operatorname{span}\left\{(\mathbf{e}_i)_{1\le i\le n}\right\}$.

7.16 The Banach algebra $\mathcal{B}(X)$

An **algebra** is a complex vector space $(B;+;\cdot)$ with a **multiplication** $\circ : B \times B \to B$ and the usual rules of compliance for $\Lambda;\Gamma;\Theta \in B$ resp. $\lambda \in \mathbb{C}$ such that $(B;+;\circ)$ is a **ring**:

1. **associativity**: $\Lambda \circ (\Gamma \circ \Theta) = (\Lambda \circ \Gamma) \circ \Theta$ and $\lambda \cdot (\Lambda \circ \Gamma) = (\lambda \cdot \Lambda) \circ \Gamma$
2. **distributivity**: $\Lambda \circ (\Gamma + \Theta) = \Lambda \circ \Gamma + \Lambda \circ \Theta$
3. **a neutral element** $\exists I \in X : \Lambda \circ I = I \circ \Lambda = \Lambda$

An element $\Lambda \in X$ is **invertible** iff there is an

4. **inverse** $\Lambda^{-1} \in X$ with $\mathbf{\Lambda}^{-1} \circ \Lambda = \Lambda \circ \Lambda^{-1} = I$

By the **open mapping theorem** 3.4 $\Lambda \in X$ is invertible iff it is **bijective**, i.e. $\ker \Lambda = \{0\}$ and $\Lambda[X] = X$. In the following paragraphs we examine the **Banach algebra** of **linear**, **continuous** and **bounded** operators $\mathcal{B}(X) = \mathcal{B}(X;X)$ on a Banach space X with the **composition** $\circ$. For $\Lambda;\Gamma \in \mathcal{B}(X)$ we have the additional property

5. $\|\Lambda \circ \Gamma\| \le \|\Lambda\| \cdot \|\Gamma\|$

7.17 Compact operators in $\mathcal{B}(X)$

For **compact** $\Lambda; \Gamma \in \mathcal{B}(X)$ on a Banach space X we have:

1. $\dim\ker(\Lambda - \lambda I) < \infty$ for every $\lambda \neq 0$
2. $\mathbf{0} \in \sigma(\Lambda)$ if $\dim X = \infty$
3. $\Lambda \circ \Gamma; \Gamma \circ \Lambda \in \mathcal{B}(X)$

Proof:

1. The restriction $\Lambda|_Y : Y \to Y$ on the **closed** subspace $Y = \ker(\Lambda - \lambda I)$ is again **compact** so that the assertion follows from 8.13.2.
2. If $\mathbf{0} \notin \sigma(\Lambda)$ then Λ is bijective hence $\Lambda[X] = X$ so that 8.13.2 applies again.
3. obvious.

7.18 The image of a compact operator

For every **compact** $\Lambda \in \mathcal{C}(X)$ and $\lambda \neq 0$ the image $\operatorname{im}(\Lambda - \lambda I)$ is a **closed** set.

Proof: By the preceding theorem 8.17.1 we have $\dim\ker(\Lambda - \lambda I) < \infty$ and by 8.15.1 follows the existence of a closed subspace M with $X = \ker(\Lambda - \lambda I) \oplus M$. Then $\Gamma : M \to X$ with $\Gamma\mathbf{x} = \Lambda\mathbf{x} - \lambda\mathbf{x}$ is bijective with $\operatorname{im}\Gamma = \operatorname{im}(\Lambda\mathbf{x} - \lambda\mathbf{x})$. Assuming the existence of a sequence $(\mathbf{x}_n)_{n\in\mathbb{N}} \subset M \cap \delta B_1(\mathbf{0}_X)$ with $\lim\limits_{n\to\infty} \Gamma\mathbf{x}_n = 0$. Since Λ is compact and due to 8.12.1 a subsequence $(\Lambda x_{n_k})_{k\in\mathbb{N}} \subset \overline{\Lambda[B_1(\mathbf{0}_X)]}$ converges to some $\mathbf{x}_0 \in X$ whence $\lim\limits_{k\to\infty} \lambda\mathbf{x}_{n_k} = \mathbf{x}_0$. Since $M \cap \delta B_1(\mathbf{0}_X)$ is a **closed** subspace we conclude that $\mathbf{x}_0 \in M$ whence the **continuity** of Γ implies $\Gamma\mathbf{x}_0 = \lim\limits_{k\to\infty} \lambda\Gamma\mathbf{x}_{n_k} = 0$ and from the bijectivity Γ follows $\mathbf{x}_0 = \mathbf{0}$ in contradiction to $\|\mathbf{x}_n\| = 1\,\forall n \in \mathbb{N}$. Hence there is no such sequence, i.e. there exists an $r > 0$ with $\|\mathbf{x}\| \geq 1 \Rightarrow \|\Gamma\mathbf{x}\| \geq r$ resp. $\|\Gamma\mathbf{x}\| \geq r\,\|\mathbf{x}\|$ for all $\mathbf{x} \in M$ such that the closed character of $\operatorname{im}\Gamma = \operatorname{im}(\Lambda\mathbf{x} - \lambda\mathbf{x})$ follows from the **dilation principle** [11, th. 14.11] .

7.19 Eigenvalues

A scalar $\lambda \in \mathbb{C}$ is an **eigenvalue** of an operator $\Lambda \in \mathcal{B}(X)$ iff there are **eigenvectors** $\mathbf{0} \neq \mathbf{x} \in X$ with $\Lambda\mathbf{x} = \lambda\mathbf{x}$. Hence $\lambda \in \mathbb{C}$ is an eigenvector of $\Lambda \in B(X)$ iff its **eigenspace** $\ker(\Lambda\mathbf{x} - \lambda\mathbf{x}) \neq \{\mathbf{0}\}$ resp. iff $\operatorname{im}(\Lambda\mathbf{x} - \lambda\mathbf{x}) \subsetneq X$. The **spectrum** $\sigma(\Lambda)$is the set of all $\lambda \in \mathbb{C}$ such that $\Lambda - \lambda I$ is not **invertible**, i.e. it is

1. not **surjective**
2. not **injective**

In the case if $\dim X < \infty$ the two conditions are **equivalent**. (cf. [7, th 3.8]). The number λ is an **eigenvalue** of Λ iff 2. is satisfied.

A set $E \subset \sigma(\Lambda)$ of eigenvalues of a **compact** $\Lambda \in \mathcal{C}(X; Y)$ with $|\lambda| > r$ for some $r > 0$ and every $\lambda \in E$ has the following properties:

1. $\operatorname{im}(\Lambda - \lambda I) \neq X$ for every $\lambda \in E$
2. E is finite.

Proof: We show that if either 1. or 2. is false then there exist closed subspaces $M_n \subset X$ and scalars $\lambda_n \in E$ such that for every $n \geq 1$

a) $M_n \subsetneq M_{n+1}$

b) $\Lambda[M_n] \subset M_n$

c) $(\Lambda - \lambda_{n+1} I)[M_{n+1}] \subset M_n$

The proof wil be completed by showing that this contradicts the compactness of Λ:

1. Assuming the existence of an $\lambda_0 \in E$ with $\text{im}\,(\Lambda - \lambda_0 I) = X$ we define the **closed** subspace $M_n = \ker \Gamma^n$ for $\Gamma = \Lambda - \lambda_0 I$. Then there exists an **eigenvector** $0 \neq \mathbf{x}_1 \in M_1$. Since $\text{im}\Gamma = X$ there is a sequence $(\mathbf{x}_n)_{n \geq 1} \subset X$ such that $\Gamma \mathbf{x}_{n+1} = \mathbf{x}_n$ for $n \geq 1$. Consequently on the one hand we have $\Gamma^n \mathbf{x}_{n+1} = \mathbf{x}_1 \neq 0$ but on the other hand $\Gamma^{n+1} \mathbf{x}_{n+1} = \Gamma \mathbf{x}_1 = 0$. Hence $\mathbf{x}_{n+1} \in M_{n+1} \backslash M_n$ and we have shown that a) holds. Since in this case we have $\Gamma \circ \Lambda = \Lambda \circ \Gamma$ and in particular $\Lambda \circ \Gamma^n = \Gamma^n \circ \Lambda$ condition b) is also satisfied while c) is obvious.
2. Assuming that there is an infinite sequence $(\lambda_n)_{n \geq 1} \subset E$ of distinct **eigenvalues** with corresponding **eigenvectors** $\mathbf{e}_n$ define the **finite dimensional** hence **closed** subspaces $M_n = \text{span}\,\{\mathbf{e}_1; ...; \mathbf{e}_n\}$. Since the λ_n are distinct the $\mathbf{e}_n$ are **linearly independent** whence follows a). For $\mathbf{x} = \sum\limits_{i=1}^{n} \alpha_i \mathbf{e}_i \in M_n$ we obviously have $\Lambda \mathbf{x} \in M_n$ and $(\Lambda - \lambda_n I)\,\mathbf{x} = \sum\limits_{i=1}^{n-1} \alpha_i\,(\lambda_i - \lambda_n)\,\mathbf{e}_i \in M_{n-1}$ which shows b) and c).
3. Assuming a) theorem subsec:Closed-subspaces provides $\mathbf{y}_n \in M_n$ with $\|\mathbf{y}_n\| \leq 2$ and $d\,(\mathbf{y}_n; M_{n-1})$ for $n \geq 2$. Conditions b) and c) then imply that $z_m = \Lambda y_m - (\Lambda - \lambda_n I)\,\mathbf{y}_n \in M_{n-1}$ whence $\|\Lambda \mathbf{y}_n - \Lambda \mathbf{y}_m\| = \|\lambda_n \mathbf{y}_n - \mathbf{z}_m\| = |\lambda_n| \cdot \left\|\mathbf{y}_n - \frac{1}{\lambda_n} \mathbf{z}_n\right\| \geq |\lambda_n| > r$. Hence $(\Lambda \mathbf{y}_n)_{n \geq 1}$ has no convergent subsequence although $(\mathbf{y}_n)_{n \geq 1}$ is bounded. This contradicts the assumed compact character of Λ hence both 1. and 2. must be true.

7.20 The spectrum of a compact operator

For every **compact** $\Lambda \in \mathcal{C}\,(X)$ and $0 \neq \lambda \in \sigma\,(\Lambda)$ the following statements hold:

1. The following four numbers are finite and equal:
   $\alpha = \dim \ker\,(\Lambda - \lambda I)$
   $\beta = \dim X/\text{im}\,(\Lambda - \lambda I)$
   $\alpha^* = \dim \ker\,(\Lambda^* - \lambda I)$
   $\beta^* = \dim X^*/\text{im}\,(\Lambda^* - \lambda I)$
2. λ is an **eigenvalue** of Λ and of Λ^*.
3. $\sigma\,(\Lambda)$ is **compact**, **countable** and has at most one **limit point**, namely, 0.

Note: A **linear** and **continuous** $\Gamma : X \to Y$ map between **Banach** spaces X and Y is called **Fredholm**, iff $\dim \ker \Gamma < \infty$ and $\dim X/\text{im}\Gamma < \infty$.

Proof:

Proposition 1.:

Step I: For tvery **closed** subspace $M \subset Y$ of a **locally convex** space Y holds $\dim Y/M \leq \dim M^0$: For every positive integer $k \leq \dim Y/M$ exist linearly independent $\mathbf{y}_1; ...; \mathbf{y}_k \in Y \setminus M$ and due to 1.5.7 each $M_i = \text{span}\,\{\mathbf{y}_1; ...; \mathbf{y}_i\} \oplus M$ is closed. By 4.5.1 there are **continuous** and linearly independent functionals $\Lambda_1; ...; \Lambda_k$ on Y such that $\Lambda_i y_i = 1$ and $\Lambda_i\,[M_{i-1}] \subset \{0\}$.

Step II: $\beta \leq \alpha^*$: This follows from **step I** with $Y = X$, $M = \text{im}\,(\Lambda - \lambda I)$ since by subsec:The image of a compact operator zhe space M is **closed** and by 8.8.1 we have $\ker\,(\Lambda^* - \lambda I) = (\text{im}\,(\Lambda - \lambda I))^0$.

Step III: $\beta^* \leq \alpha$: Apply **step I** with $Y = X^*$ with weak* topology and $M = \text{im}\,(\Lambda^* - \lambda I)$ since due to subsec:The image of a compact operator resp. 8.10.3 M is weakly* closed while theorem 8.8.2 yields $\ker\,(\Lambda - \lambda I) = {}^0\,(\text{im}\,(\Lambda^* - \lambda I))$.

Step IV: $\alpha \leq \beta$: We assume $\alpha > \beta$ whence according to 8.17.1 and subsec:Direct-sums there are closed complements $M; N$ such that $X = \ker\,(\Lambda - \lambda I) \oplus M = \text{im}\,(\Lambda - \lambda I) \oplus N$. Since $\dim N = \dim X/\text{im}\,(\Lambda - \lambda I)$ there is a linear $\varphi : \ker\,(\Lambda - \lambda I) \to N$ with $\dim \ker \varphi \geq 1$. Owing to 8.13.1 and since $\dim N = \beta < \infty$ the composition $\varphi \circ \pi : X \to N$ with the canonical and in particular **continuous projection** $\pi : X \to \ker\,(\Lambda - \lambda I)$ is **compact**. and so is $\Phi = \Lambda + \varphi \circ \pi$. Since $\pi\,[M] = \{0\}$ we have $(\Phi - \lambda I)\,[M] = \text{im}\,(\Lambda - \lambda I)$ and

$$(\Phi - \lambda I)\left[\ker(\Lambda - \lambda I)\right] = (\Lambda - \lambda I + \varphi \circ \pi)\left[\ker(\Lambda - \lambda I)\right] = (\varphi \circ \pi)\left[\ker(\Lambda - \lambda I)\right] = \varphi\left[\ker(\Lambda - \lambda I)\right] = N.$$

This implies

$$X = \operatorname{im}(\Lambda - \lambda I) \oplus N \subset \operatorname{im}(\Phi - \lambda I).$$

Also for every $\mathbf{x}_0 \in \ker\varphi$ we have $\varphi\mathbf{x}_0 = (\Phi - \lambda I)\mathbf{x}_0 = 0$, i.e. λ is an eigenvalue of Φ and since Φ is **compact** the preceding theorem subsec:Eigenvalues states that $\operatorname{im}(\Phi - \lambda I) \neq X$ in contradiction to the result above. Hence the assertion a) is proved.

Proposition 2.: Assuming that λ is **not** an eigenvalue of Λ implies $\alpha(\Lambda) = 0$, i.e. **injectivity**, whence $\beta(\Lambda) = 0$ by 1., i.e. **surjectivity**. Thus $\Lambda - \lambda I$ is **invertible** and $\lambda \notin \sigma(\Lambda)$.

Proposition 3.: The preceding theorem 8.19.2 shows that $\mathbf{0}$ is the only possible limit point of the spectrum $\sigma(\Lambda)$, that $\sigma(\Lambda)$ is at most **countable** and that $\sigma(\Lambda) \cup \{\mathbf{0}\}$ is **compact**. In the case of $\dim X < \infty$ there are at most finitely many eigenvalues for a finite number of linearly independent eigenvectors (cf. [7, th 5.1]). In the case of $\dim X = \infty$ theorem 8.17.2 assures that $\mathbf{0} \in \sigma(\Lambda)$ such that in both cases we have shown that $\sigma(\Lambda)$ is compact.

8 Umordnung von Vektorsummen

8.1 Einleitung

Nach dem Riemannsche Umordnungssatz lässt sich eine bedingt konvergent Reihe reeller Zahlen so umordnen, dass die Partialsummen gegen jede beliebige reelle Zahl konvergieren. Da die Partialsummen einer bedingt konvergenten Reihe zwar beliebig groß werden, die Beträge der Summanden aber gegen Null konvergieren, kann man eine Umordnung konstruieren, die eine beliebige reelle Zahl approximiert. Die Verallgemeinerung auf endlichdimensionale Banachräume und insbesondere die komplexen Zahlen und den $\mathbb{R}^n$ gelang 1905 P. Lévy bzw. 1913 E. Steinitz [5]. Der elementare und sehr aufwendige Beweis wurde von W. Gross [2], I. Halperin [3] und P. Rosenthal [4] vereinfacht. Der mehr topologisch motivierte aber ebenfalls konstruktive Ansatz von T. Banakh [1] aus dem Jahr 2017 verkürzt den Beweis deutlich. Abgesehen vom Satz von **Hahn-Banach** 4.2 setzt der Beweis nur elementare linear Algebra und Topologie voraus.

Nach einem ersten Abschnitt mit bekannten Ausssagen zu bedingt und absolut konvergenten Reihen wird im zweiten Abschnitt der Beweis des ersten Teils des Satzes von Lévy-Steinitz nach P. Rosenthal behandelt: Die durch Umordnung möglichen Grenzwerte einer bedingt konvergenten Vektorreihe bilden einen affiner Unterraum. Der Beweis enthält Ergebnisse zur Umordnung endlicher Vektorketten, die auch für sich von Interesse sind. Im dritten Abschnitt wird der Beweis des Satzes von Lévy-Steinitz einnschließlich der Aussagen zur Lage des affinen Unterraums nach T. Banakh dargestellt.

8.2 Absolute Konvergenz von Vektorsummen

Sind die Partialsummen einer Vektorfolge $(v_i)_{i\in\mathbb{N}}$ mit $v_i \in \mathbb{R}^n$ **absolut** konvergent mit $\sum_{i\in\mathbb{N}} |v_i| < \infty$, so ist der Grenzwert $\sum_{j\in\mathbb{N}} v_{p(j)} = v$ unabhängig von der Permutation p.

Beweis: Für jedes $\epsilon > 0$ und jede Permutation $p : \mathbb{N} \to \mathbb{N}$ gibt es ein $i_0 \in \mathbb{N}$ mit $\sum_{i\geq i_0} \|v_i\| < \epsilon$ und für $i_1 = \max\{p^{-1}(j) : 0 \leq j \leq i_0\}$ gilt demnach $\left\|\sum_{i\geq 0} v_i - \sum_{j\geq 0} v_{p(j)}\right\| \leq \left\|\sum_{0<i\leq i_1} v_i - \sum_{0<j\leq i_1} v_{p(j)}\right\| + \sum_{i>i_1} \|v_i\| + \sum_{j>i_1} \left\|v_{p(j)}\right\| \leq \left\|\sum_{i_0<i\leq i_1} v_i - \sum_{\substack{0<j\leq i_1 \\ p(j)>i_0}} v_{p(j)}\right\| + 2\epsilon \leq 4\epsilon$. Dabei wurde im vorletzten Schritt die endliche Summe $\sum_{0<j\leq i_1} v_{p(j)}$ so umgeordnet, dass die „großen“ $v_{p(j)}$ mit $p(j) = i$ für $0 \leq i < i_0$ zu Beginn stehen und sich mit den entsprechenden „großen“ Summanden in $\sum_{0<i\leq i_1} v_i$ aufheben. Die übrigen „kleinen“ Summanden $v_i$mit $i > i_0$ und $v_{p(j)}$ mit $p(j) > i_0$ heben sich nicht gegenseitig auf, lassen sich aber wegen der absoluten Konvergenz durch jeweils ϵ abschätzen.

8.3 Riemannscher Umordnungssatz

Sind die Partialsummen einer **Zahlenfolge** $(x_i)_{i\in\mathbb{N}}$ mit $x_i \in \mathbb{R}$ **bedingt** konvergent mit $\sum_{i\in\mathbb{N}} x_i < \infty$, aber $\sum_{i\in\mathbb{N}} |x_i| = \infty$, so gibt es für jedes $x \in \mathbb{R}$ eine Permutation $p : \mathbb{N} \to \mathbb{N}$ mit $\sum_{j\in\mathbb{N}} x_{p(j)} = x$.

Beweis: Wegen $|x_i| = x_i^+ + x_i^-$ und $x_i = x_i^+ - x_i^-$ sind die Reihen der Positivteile $x_i^+ = \frac{1}{2}\left(|x_i| + x_i\right) = \max\{x_i; 0\}$ und der Negativteile $x_i^- = \frac{1}{2}\left(|x_i| - x_i\right) = \min\{x_i; 0\}$ **divergent**: $\sum_{i\in\mathbb{N}} x_i^+ = \sum_{i\in\mathbb{N}} x_i^- = \infty$. Für o.B.d.A. $x > 0$ und $n \geq 1$ definiere zunächst $i_0 = j_0 = 0$ und $i_1 = \min\left\{ i'_1 \geq 1 : \sum\limits_{i=1}^{i'_1} x_i^+ \geq v \right\}$ sowie $j_1 = \min\left\{ j'_1 \geq 1 : \sum\limits_{i=1}^{i_1} x_i^+ - \sum\limits_{j=1}^{j'_1} x_j^- \leq x \right\}$. Anschließend setzt man die Umordnung induktiv fort mit

$$i_{n+1} = \min\left\{ i'_{n+1} \geq i_n : \sum_{m=0}^{n-1} \left(\sum_{i=i_m+1}^{i'_{m+1}} x_i^+ - \sum_{j=j_m+1}^{j_{m+1}} x_j^- \right) + \sum_{i=i_n+1}^{i'_{n+1}} x_i^+ \geq x \right\}$$

sowie

$$j_{n+1} = \min\left\{ j'_{n+1} \geq j_n : \sum_{m=0}^{n} \left(\sum_{i=i_m+1}^{i_{m+1}} x_i^+ - \sum_{j=j_m+1}^{j'_{m+1}} x_j^- \right) \leq x \right\}$$

Für die neugeordneten Partialsummen gilt

$$\left| \sum_{m=0}^{n} \left(\sum_{i=i_{m-1}+1}^{i_m} x_i^+ - \sum_{j=j_{m-1}+1}^{j_m} x_j^- \right) - x \right| \leq \max\left\{ x_{i_n}^+ ; x_{j_n}^- \right\}$$

Da wegen der Konvergenz der Gesamtreihe $\lim\limits_{i\to\infty} |x_i| = 0$, folgt daraus die Behauptung.

8.4 Darstellung affiner Unterräumen als Grenzwerte von Vektorsummen

Jeder affine Unterraum $v + \Gamma$ des $\mathbb{R}^n$ mit $v \in \mathbb{R}^n$ und einem Untervektorraum $\Gamma \subset \mathbb{R}^n$ lässt sich als Menge Σ der Grenzwerte der Partialsummen $\sum_{i\in\mathbb{N}} v_{p(i)}$ aller möglichen Permutationen $p : \mathbb{N} \to \mathbb{N}$ einer Folge $(v_i)_{i\in\mathbb{N}} \subset \mathbb{R}^n$ von Vektoren darstellen, denn nach dem Riemannschen Umordnungssatz gibt es für jedes $1 \leq j \leq m$ eine Permutation $p_j : \mathbb{N} \to \mathbb{N}$ der Folge $(x_i)_{i\in\mathbb{N}}$ mit $x_i = \frac{(-1)^i}{i}$, so dass $\sum_{i\in\mathbb{N}} x_{p_j(i)} = x_j$.

8.5 Eingrenzung von Polygonen

Für jede **endliche geschlossene Vektorkette** $(v_i)_{i\in I} \subset \mathbb{R}^n$ für $I = \{1; ...; m\}$ mit $\sum\limits_{i=1}^{m} v_i = 0$ und Beträgen $\|v_i\| \leq 1, i \in I$ gibt es eine Permutation $p : I \to I$ mit $p(1) = 1$, so dass $\forall j \in I$ gilt $\left\| \sum\limits_{i=1}^{j} v_{p(i)} \right\| \leq C_n$ mit $C_1 = 1$ und $C_n \leq \sqrt{4C_{n-1}^2 + 1}$.

Beweis durch Induktion über n: Für $n = 1$ und o.B.d.A $v_1 > 0$ wähle die folgenden $v_{p(2)}, v_{p(3)}, ... < 0$, bis die Summe im Bereich $-1 \leq v_1 + v_{p(2)} + v_{p(3)} + ... < 0$ liegt. Anschließend wähle wieder positive $v_{p(i)}$, bis die Summe wieder im poitiven Bereich liegt, usw., bis alle m Vektoren verbraucht sind. Wegen $\|v_i\| \leq 1$ kann man die $v_{p(i)}$ so wählen, dass die Sume die Bereich $[-1; 1]$ nicht verlässt, womit sich die Behauptung mit $C_1 = 1$ ergibt. Für $n > 1$ wähle unter den 2^{m-1} möglichen Kombinationen die Summe $v = v_1 + u_1 + ... + u_s$ mit $\{u_1; ...; u_s\} \subset \{v_2; ...; v_m\}$ und maximaler Länge $\|v\|$. Man betrachtet den Vektor v als positive Bezugsrichtung und zeigt zunächst mit Hife des **inneren Produktes** $\langle \ldots, \ldots \rangle$, dass die $v_1, u_1, ..., u_s$ in Richtung v ,d.h. $\langle v_1, v\rangle, \langle u_i, v\rangle \geq 0$, und die übrigen Vektoren $\{w_1; ...; w_t\} := \{v_2; ...; v_m\} \setminus \{u_1; ...; u_s\}$ mit $1 + s + t = m$ und $v = -w_1 - ... - w_t$ in Richtung $-v$, d.h.$\langle w_j, v\rangle \leq 0$, orientiert sind:

Angenommen, $\langle v_1, v\rangle < 0$, dann wäre $\|v_1 + w_1 + w_2 + ...\| \geq \left\langle (v_1 + w_1 + w_2 + ...), \frac{-v}{\|v\|}\right\rangle = \frac{-\langle v_1,v\rangle}{\|v\|} + \|v\| > \|v\|$ und damit $v_1 + w_1 + w_2 + ...$ eine längere Vektorkette als L.

Angenommen, $\langle u_i, v\rangle < 0$, dann wäre $\|v - u_i\| \geq \left\langle (v - u_i), \frac{v}{\|v\|}\right\rangle = \frac{-\langle u_i,v\rangle}{\|v\|} + \|v\| > \|v\|$ und damit $v - u_i$ eine längere Vektorkette als L.

Angenommen, $\langle w_j, v\rangle > 0$, dann wäre $\|v + w_j\| \geq \left\langle (v + w_j), \frac{v}{\|v\|}\right\rangle = \frac{\langle w_j,v\rangle}{\|v\|} + \|v\| > \|v\|$ und damit $v + w_j$ eine längere Vektorkette als L.

Sei nun $u' := u - \left\langle u, \frac{v}{\|v\|}\right\rangle \frac{v}{\|v\|}$ die Komponente des Vektors u in dem $n-1$-dimensionalen Unterraum $\{v\}^{\perp} := \left\{u \in \mathbb{R}^n : \left\langle u, \frac{v}{\|v\|}\right\rangle = 0\right\}$ orthogonal zu $\{v\} := \left\{u \in \mathbb{R}^n : \left\langle u, \frac{v}{\|v\|}\right\rangle = u\right\}$. Wegen $v' = v'_1 + u'_1 + ... + u'_s = -w'_1 - ... - w'_t = 0$ gibt es nach Induktionsvoraussetzung eine Permutation q auf $\{1;...;s\}$, so dass $\forall 1 \leq j \leq s$ gilt $\left\|v'_1 + \sum\limits_{i=1}^{j} u'_{q(i)}\right\| \leq C_{n-1}$ und eine Permutation r auf $\{1;...;t\}$ mit $q(1) = 1$, so dass $\forall 1 \leq j \leq t$ gilt $\left\|\sum\limits_{i=1}^{j} w'_{r(i)}\right\| \leq C_{n-1}$. Die durch q bzw. r vorgegebene Reihenfolge der $u_{q(i)}$ bzw. $w_{r(i)}$ gewährleistet, dass die Vektorketten **beliebiger Länge** orthogonal zu v nicht länger als C_{n-1} werden. Man sucht nun analog zum Beweis für $n = 1$ passende Teilketten abwechselnd aus den $u_{q(i)}$ bzw. $w_{r(i)}$ so heraus, dass auch ihre Längen **parallel** zu v nicht länger als 1 werden: Man beginnt in positiver Richtung mit v_1 mit $0 \leq \left\langle v_1, \frac{v}{\|v\|}\right\rangle \leq 1$ und findet wegen $\left\langle w_j, \frac{v}{\|v\|}\right\rangle \leq 1$ bzw. $\sum\limits_{j=1}^{t} \left\langle w_j, \frac{v}{\|v\|}\right\rangle = \|v\|$ ein $1 \leq t_1 \leq t$, so dass $-1 \leq \left\langle v_1, \frac{v}{\|v\|}\right\rangle + \sum\limits_{i=1}^{t_1} \left\langle w_{r(i)}, \frac{v}{\|v\|}\right\rangle \leq 0$. Anschließend sucht man ein $1 \leq s_1 \leq s$, so dass $0 \leq \left\langle v_1, \frac{v}{\|v\|}\right\rangle + \sum\limits_{i=1}^{t_1} \left\langle w_{r(i)}, \frac{v}{\|v\|}\right\rangle + \sum\limits_{i=1}^{s_1} \left\langle u_{q(i)}, \frac{v}{\|v\|}\right\rangle \leq 1$. Als nächstes kommt wieder ein $t_1 < t_2 \leq t$, so dass $-1 \leq \left\langle v_1, \frac{v}{\|v\|}\right\rangle + \sum\limits_{i=1}^{t_1} \left\langle w_{r(i)}, \frac{v}{\|v\|}\right\rangle + \sum\limits_{i=1}^{s_1} \left\langle u_{q(i)}, \frac{v}{\|v\|}\right\rangle + \sum\limits_{i=t_1+1}^{t_2} \left\langle w_{r(i)}, \frac{v}{\|v\|}\right\rangle \leq 0$, usw., bis alle $u_{q(i)}$ bzw. $w_{r(i)}$ verbraucht sind. Die gesuchte Anordnung ist also $\left(v_1; w_{r(1)}, ..., w_{r(t_1)}, u_{q(1)}, ..., u_{q(s_1)}, w_{r(t_1+1)}, ..., w_{r(t_2)}, ...\right)$. Die Länge der Vektorkette in Richtung v ist höchstens 1 und orthogonal dazu in Richtung $v^{\perp}$ für die $u_{q(i)}$ bzw. $w_{r(i)}$ höchstens $C_{n-1} + C_{n-1}$. Insgesamt gilt also $C_n \leq \sqrt{4C_{n-1}^2 + 1}$.

8.6 Umordnung endlicher Vektorketten I

Für jede **endliche Vektorkette** $(v_i)_{i\in I} \subset \mathbb{R}^n$ mit $I = \{1;...;m\}$ und $\left\|\sum\limits_{i=1}^{m} v_i\right\| \leq \epsilon$ sowie Beträgen $\|v_i\| \leq \epsilon, i \in I$ gibt es eine Permutation $p : I \to I$ mit $p(1) = 1$, so dass $\forall j \in I$ gilt $\left\|\sum\limits_{i=1}^{j} v_{p(i)}\right\| \leq \epsilon\,(C_n + 1)$, denn die v_i lassen sich durch $v_{m+1} := -\sum\limits_{i=1}^{m} v_i$ zu einer geschlossenen Vektorkette gemäß 2.1 ergänzen, so dass $\left\|\sum\limits_{i=1}^{j} v_{p(i)}\right\| \leq \epsilon C_n$ und damit $\left\|\sum\limits_{\substack{1\leq i\leq j\\ p(i)\neq m+1}} v_{p(i)}\right\| \leq \epsilon C_n + \epsilon$.

8.7 Umordnung endlicher Vektorketten II

Für jede **endliche Vektorkette** $(v_i)_{i\in I} \subset \mathbb{R}^n$ mit $I = \{1;...;m\}$ und Beträgen $\|v_i\| \leq \epsilon, i \in I$ sowie jedes $0 < t < 1$ gibt es eine Permutation $p : I \to I$ mit $p(1) = 1$, und ein $j \in I$, so dass $\left\|\sum\limits_{i=1}^{j} v_{p(i)} - tv\right\| \leq \epsilon\sqrt{C_{n-1}^2 + 1}$ mit $v := \sum\limits_{i=1}^{m} v_i$.

Beweis: Sei zunächst $n = 1$ und o.B.d.A $v > 0$ sowie $1 \leq j \leq m$ der kleinste Index mit $\sum\limits_{i=1}^{j} v_i - tv > 0$, dann ist $\sum\limits_{i=1}^{j-1} v_i - tv < 0$ und wegen $v_j < \epsilon$ folgt $\left|\sum\limits_{i=1}^{j} v_i - tv\right| < \epsilon$, womit die Behauptung für $C_0 := 0$ erfüllt ist. Im Fall $n > 1$ betrachtet man wie im Beweis zu 20.4 die Projektionen $v_i' := v_i - \left(v_i, \frac{v}{\|v\|}\right)\frac{v}{\|v\|}$ auf den orthogonalen Unterraum $\{v\}^{\perp}$. Wegen $\sum\limits_{i=1}^{j} v_i' = v' = 0$ und $\|v_i'\| \leq \epsilon$ lässt sich 2.1 anwenden und liefert eine Permutation $p : I \to I$ mit $p(1) = 1$, so dass $\forall j \in I$ gilt $\left\|\sum\limits_{i=1}^{j} v_{p(i)}'\right\| \leq \epsilon C_{n-1}$. Für die Komponenten $\left\langle v_i, \frac{v}{\|v\|}\right\rangle \frac{v}{\|v\|}$ parallel zu v gilt $\left\langle v_i, \frac{v}{\|v\|}\right\rangle < \epsilon$ und $\left\|\sum\limits_{i=1}^{m} \left\langle v_{p(i)}, \frac{v}{\|v\|}\right\rangle\right\| = \|v\| = v, \frac{v}{\|v\|}$, so dass sich wie im Beweis für $n = 1$ ein $j \in I$ finden lässt mit $\left\|\sum\limits_{i=1}^{j} \left\langle v_{p(i)}, \frac{v}{\|v\|}\right\rangle - t\|v\|\right\| < \epsilon$. Die Differenz $\sum\limits_{i=1}^{j} v_{p(i)}' - tv$ der beiden Vektorketten und in Richtung v ist höchstens ϵ und orthogonal dazu in Richtung $v^{\perp}$ höchstens ϵC_{n-1}. Insgesamt gilt also $\left\|\sum\limits_{i=1}^{j} v_{p(i)} - tv\right\| \leq \epsilon\sqrt{C_{n-1}^2 + 1}$.

8.8 Umordnungssatz

Besitzt die Folge $(S_m)_{m\geq 1}$ der Partialsummen $S_m = \sum\limits_{i=1}^{m} v_i$ einer Folge $(v_i)_{i\in\mathbb{N}} \subset \mathbb{R}^n$ von Vektoren eine Teilfolge $(S_{m_k})_{k\geq 1}$, die gegen ein $S \in \mathbb{R}^n$ konvergiert, so lässt sich die Gesamtreihe durch eine Bijektion $p : \mathbb{N} \to \mathbb{N}$ so umordnen, dass sie gegen S konvergiert: $\lim\limits_{m\to\infty} \left\|S_{p(m)} - S\right\| = 0$.

Beweis: Man verwendet 8.2, um die zwischen den Gliedern der konvergierenden Teilfolge liegenden Vektoren $v_{m_k+1}, ..., v_{m_{k+1}-1}$ so umzuordnen, dass ihre Partialsummen und damit die Abweichungen von S_{m_k} minimal werden: Für $\delta_k := \|S_{m_k} - S\|$ und $\epsilon_k := \max\{\delta_k + \delta_{k+1}, \sup\{\|v_i\| : i \geq m_k\}\}$ gilt $\left\|\sum\limits_{i=m_k+1}^{m_{k+1}-1} v_i\right\| = \left\|\left(\sum\limits_{i=1}^{m_{k+1}} v_i - S\right) - \left(\sum\limits_{i=1}^{m_k} v_i - S\right) - v_{m_{k+1}}\right\| < \delta_{k+1} + \delta_k + \|v_{k+1}\| < 2\epsilon_k$. Gemäß 8.5 existiert eine Permutation p_k von $\{m_k + 1; ..., ; m - 1\}$ mit $\left\|\sum\limits_{i=m_k+1}^{j} v_{p_k(i)}\right\| \leq 2\epsilon_k (C_n + 1)\, \forall m_k + 1 \leq j \leq m_{k+1} - 1$. Setzt man $p(m) := p_k(i)$ für $m_k + 1 \leq i \leq m_{k+1} - 1$ und $p(m_k) := m_k$ sonst, so folgt $\left\|S_{p(m)} - S_{m_k}\right\| \leq 2\epsilon_k (C_n + 1) \to 0$ und wegen $\|S_{m_k} - S\| = \delta_k \to 0$ schließlich die Behauptung.

8.9 Satz von Lévy und Steinitz I

Die Menge Σ der Grenzwerte der Partialsummen $\sum_{i\in\mathbb{N}} v_{p(i)}$ aller möglichen Permutationen $p : \mathbb{N} \to \mathbb{N}$ einer gegebenen Folge $(v_i)_{i\in\mathbb{N}} \subset \mathbb{R}^n$ von Vektoren ist ein affiner Unterraum der Gestalt $\Sigma = \Sigma + \Gamma$ mit einem Untervektorraum $\Gamma \subset \mathbb{R}^n$.

Beweis: Für $\Sigma = \emptyset$ ist nichts zu zeigen. Sei also $v \in \Sigma \neq \emptyset$, dann kann o.B.d.A. v_1 durch $v_1 - v$ ersetzt werden und durch diese Verschiebung der gesamten Summe erhält man $0 \in S$. Es genügt nun zu zeigen, dass Σ ein Untervektorraum ist:

$s_1, s_2 \in \Sigma \Rightarrow s_1 + s_2 \in \Sigma :$ Da es drei verschiedene Anordnungen gibt, die **unabhängig von endlichen Umordnungen** gegen s_1, 0 bzw. s_2 konvergieren, existieren für jedes $m \geq 1$ **endliche** Indexmengen $\{1; ...; m\} \subset I_m \subset J_m \subset K_m \subset I_{m+1} \subset ...$ mit $\left\|\sum_{i\in I_m} v_i - s_1\right\| < 2^{-m}$, $\left\|\sum_{i\in J_m} v_i - 0\right\| < 2^{-m}$ und $\left\|\sum_{i\in K_m} v_i - s_2\right\| < 2^{-m}$. Die Summe bewegt sich also auf I_m in Richtung s_1, dann auf $J_m \setminus I_m$ wieder zurück in Richtung $-s_1$ bzw. 0 und schließlich auf $K_m \setminus J_m$ in Richtung s_2. Wenn man die Summanden im Bereich $J_m \setminus I_m$ entfernt, werden die Restsummen gegen $s_1 + s_2$ streben: Mittels einer endlichen Permutation $p : \mathbb{N} \to \mathbb{N}$ lassen sich die Indexmengen I_m, J_m, K_m und I_{m+1} aufsteigend sortieren, so dass es Indizes $i_m < j_m < k_m < i_{m+1} < ...$ gibt mit $I_m = p\left[\{1; ...; i_m\}\right]$, $J_m = p\left[\{1; ...; j_m\}\right]$, usw. und damit $\left\|\sum\limits_{i=1}^{i_m} v_{p(i)} - s_1\right\| < \frac{1}{m}$, $\left\|\sum\limits_{i=1}^{j_m} v_{p(i)}\right\| < \frac{1}{m}$ und $\left\|\sum\limits_{i=1}^{k_m} v_{p(i)} - s_2\right\| < \frac{1}{m}$. Wegen $\left\|\sum\limits_{i=j_m+1}^{k_m} v_{p(i)} - s_2\right\| = \left\|\sum\limits_{i=1}^{k_m} v_{p(i)} - s_2 - \sum\limits_{i=1}^{j_m} v_{p(i)}\right\| < \frac{2}{m}$ folgt daraus für den Rest $\left\|\sum\limits_{i=1}^{i_m} v_{p(i)} + \sum\limits_{i=j_m+1}^{k_m} v_{p(i)} - (s_1 + s_2)\right\| < \frac{3}{m}$. Wenn man den hinteren Abschnitt mittels $p'(p(i)) = \begin{cases} p(i + j_m - i_m) & \text{für } i_m < i \leq i_m + k_m - j_m \\ p(i - j_m + i_m) & \text{für } i_m + k_m - j_m < i \leq k_m \\ p(i) & \text{sonst} \end{cases}$ nach vorne in die Lücke schiebt, ergibt sich eine Partialsumme $\left\|\sum\limits_{i=1}^{i_m+k_m-j_m} v_{p' \circ p(i)} - (s_1 + s_2)\right\| < \frac{3}{m}$ und damit eine gegen $s_1 + s_2$ konvergierende Teilfolge, woraus nach 8.4 die Behauptung folgt.

$s \in \Sigma \Rightarrow ts \in \Sigma \forall t \in \mathbb{R}$: Man verwendet o.B.d.A s_2 mit den Anordnungen aus Teil 1. und nutzt die eben bewiesene Additivität, um sich auf $0 < t < 1$ zu beschränken und damit 8.3 anwenden zu können: Mit $\delta_m = \sup\left\{\left\|v_{p(i)}\right\| : i = j_m + 1, ..., k_m\right\}$ gibt es eine Permutation q_m von $\{p(j_m + 1), ..., p(k_m)\}$, so dass $\left\|\sum\limits_{i=j_m+1}^{k_m} v_{q_m(p(i))} - t \cdot \sum\limits_{i=j_m+1}^{k_m} v_{p(i)}\right\| \leq \delta_m \sqrt{C_{n-1}^2 + 1}$. Wegen $\left\|t \cdot \sum\limits_{i=j_m+1}^{k_m} v_{p(i)} - ts_2\right\| < \frac{2}{m}$ und $\left\|\sum\limits_{i=1}^{j_m} v_{p(i)}\right\| < \frac{1}{m}$ folgt $\left\|\sum\limits_{i=1}^{j_m} v_{p(i)} + \sum\limits_{i=j_m+1}^{k_m} v_{q_m(p(i))} - ts_2\right\| \leq \delta_m \sqrt{C_{n-1}^2 + 1} + \frac{3}{m}$. Aus der Annahme $\Sigma \neq \emptyset$ folgt $\lim\limits_{m \to \infty} \delta_m = 0$, so dass die Teilfolge der so umgeordneten Partialsummen gegen ts_2 konvergiert und aus 8.4 folgt wieder die Behauptung.

$s \in \Sigma \Rightarrow -s \in \Sigma$: Man verwendet wieder s_2 mit den Anordnungen aus Teil 1. und betrachtet $\left\|\sum\limits_{i=1}^{j_m} v_{p(i)} + \sum\limits_{i=k_m+1}^{j_{m+1}} v_{p(i)} - (-s_2)\right\| = \left\|\sum\limits_{i=1}^{j_m} v_{p(i)} + \sum\limits_{i=1}^{j_{m+1}} v_{p(i)} - \left(\sum\limits_{i=1}^{k_m} v_{p(i)} - s_2\right)\right\| < \frac{1}{m} + \frac{1}{m+1} + \frac{1}{m}$. Analog zu 1. lässt sich durch die Rückverschiebung der letzten $j_{m+1} - (k_m + 1)$ Glieder eine gegen $-s_2$ konvergierende Teilfolge von Partialsummen bilden, so dass sich mit 8.4 erneut die Behauptung ergibt.

8.10 Komponentenweise bedingte Konvergenz

Eine Reihe $\sum_{i \in \mathbb{N}} v_i$ mit $v_i \in \mathbb{R}^n$ heißt **bedingt konvergent**, wenn es eine Permutation $p : \mathbb{N} \to \mathbb{N}$ gibt mit $\left\|\sum_{i \in \mathbb{N}} v_{p(i)}\right\| < \infty$ und **komponentenweise bedingt konvergent**, wenn die Summen $\sum_{i \in \mathbb{N}} \langle v_i, w \rangle$ der Projektionen in alle Richtungen $w \in \mathbb{R}^n$ und insbesondere die Summen $\sum_{i \in \mathbb{N}} v_{ij}$ der Komponenten $v_{ij} = \langle v_i, e_j \rangle$ bedingt konvergent sind. Aufgrund der (Sesqui-)Linearität des Skalarproduktes und der **Schwarz-Ungleichung** gilt $\left|\sum\limits_{i=0}^{m} \langle v_i, w \rangle\right| = \left|\left\langle \left(\sum\limits_{i=0}^{m} v_i\right), w \right\rangle\right| \leq \left\|\sum\limits_{i=0}^{m} v_i\right\| \cdot \|w\|$ für alle $m \in \mathbb{N}$, d.h., eine bedingt konvergente Vektorreihe ist insbesondere komponentenweise bedingt konvergent.

8.11 Folge der Summanden

Für eine komponentenweise bedingt konvergente Reihe $\sum_{i \in \mathbb{N}} v_i$ konvergiert die Folge der Summanden gegen Null: $\lim\limits_{i \to \infty} \|v_i\| = 0$.

Beweis: Angenommen, es gibt ein $\epsilon > 0$, so dass die Menge $E = \{n \in \mathbb{N} : \|v_i\| > \epsilon\}$ unendlich ist, und die Folge $\left(\frac{v_i}{\|v_i\|}\right)_{i \in E} \subset S$ auf der kompakten Menge S einen Häufungspunkt v_∞ besitzt mit einem $n_0 \in \mathbb{N}$, so dass $\left\|\frac{v_i}{\|v_i\|} - v_\infty\right\| < \frac{\epsilon}{2}$ für alle $i \geq n_0$. Dann gilt $|\langle v_i, v_\infty \rangle| = \|v_i\| \cdot \left|\left\langle \frac{v_i}{\|v_i\|}, v_\infty \right\rangle\right| = \|v_i\| \cdot \left|\left\langle \frac{v_i}{\|v_i\|}, v_\infty - \frac{v_i}{\|v_i\|} \right\rangle + \left\langle \frac{v_i}{\|v_i\|}, \frac{v_i}{\|v_i\|} \right\rangle\right| \geq \epsilon \left(1 - \frac{\epsilon}{2}\right) = \frac{\epsilon}{2}$ für alle $i \in E$, so dass die Reihe $\sum_{i \in \mathbb{N}} \langle v_i, v_\infty \rangle$ nicht in eine konvergente Reihe umgeordnet werden kann im Widerspruch zur Voraussetzung der komponentenweise bedingten Konvergenz.

8.12 Divergenzpunkte und absolute Konvergenz

Ein Punkt $x \in S = \{x \in \mathbb{R}^n : \|x\| = 1\}$ heißt **Divergenzpunkt** der Reihe $\sum_{i\in\mathbb{N}} v_i$, wenn die Menge $\mathbb{N}_U := \left\{i \in \mathbb{N} : \frac{v_i}{\|v_i\|} \in U\right\}$ für jede Umgebung $U \subset S$ von x unendlich ist und die Teilfolge $\sum_{i\in\mathbb{N}_U} \|v_i\| = \infty$ divergiert. Die Menge $D \subset S$ aller Divergenzrichtungen einer bedingt konvergenten Reihe ist nach Definition abgeschlossen in S und als Teilmenge der kompakten und insbesondere abgschlossenen **Einheitsspäre** S auch abgeschlossen in $\mathbb{R}^n$. Im Fall $D = \emptyset$ gibt es eine endliche Überdeckung $\mathcal{U}$ offener Mengen $U \subset S$, für die jeweils $\sum_{i\in\mathbb{N}_U} \|v_i\| < \infty$ und folglich $\sum_{i\in\mathbb{N}} \|v_i\| \leq \sum_{U\in\mathcal{U}} \sum_{i\in\mathbb{N}_U} \|v_i\| < \infty$, d.h., **absolute Konvergenz** mit $\Gamma = \mathbb{R}^n$, $\Gamma^\perp = \emptyset$ und $\Sigma = \{v\}$ für $v = \sum_{i\in\mathbb{N}} v_i$.

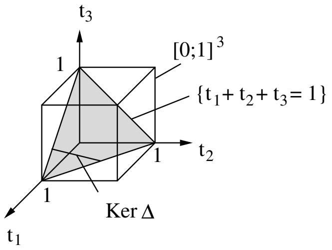
8.13 Konvexe Hülle der Divergenzpunkte

Im Fall $D \neq \emptyset$ enthält die **konvexe Hülle**

$$\operatorname{co}(D) = \left\{\sum_{k=0}^{m} t_k x_k : \sum_{k=0}^{m} t_k = 1,\ t_k \in [0;1],\ x_k \in D,\ 0 \leq k \leq m,\ m \in \mathbb{N}\right\}$$

den Koordinatenursprung, denn ansonsten existiert nach dem **Satz von Hahn-Banach** 4.2 ein $w \in \mathbb{R}^n$ und ein $\epsilon > 0$ mit $\langle x, w\rangle > \epsilon \forall x \in \operatorname{co}(D)$ sowie wegen der Stetigkeit des linearen Funktionals $\langle \ldots, w\rangle : \mathbb{R}^n \to \mathbb{R}$ auch eine Umgebung $U \in \mathcal{U}(x)$ mit $\langle x, w\rangle > \frac{\epsilon}{2} \forall x \in U$, so dass $\sum_{i\in\mathbb{N}} \left\langle v_{p(i)}, w\right\rangle$ für keine Permutation p konvergieren kann im Gegensatz zur Annahme der komponentenweisen bedingten Konvergenz von $\sum_{i\in\mathbb{N}} v_i$.

8.14 Konvexe Hülle der minimalen Divergenzpunkte



Die minimale Teilmenge $D_0 \subset D$ mit $0 \in \operatorname{co}(D_0)$ ist affin unabhängig und besteht daher aus höchstens $n+1$ Punkten, denn im Fall $D_0 = \{x_o, ..., x_{n+1}\}$ ist der **Kern** $\operatorname{Ker}\Delta = \left\{t = (t_1; ...; t_{n+1}) \in \mathbb{R}^{n+1} : \Delta(t) = 0\right\}$ der linearen Abbildung $\Delta : \mathbb{R}^{n+1} \to \mathbb{R}^n$ mit $\Delta(t) = \sum_{k=0}^{n+1} t_k x_k \wedge t_0 = 1 - \sum_{k=1}^{n+1} t_k$ für $x_0, ..., x_{n+1} \in D \subset \mathbb{R}^n$ ein mindestens eindimensionaler und höchstns $(n+1)$-dimensionaler Unterraum des $\mathbb{R}^{n+1}$ in der Form $\operatorname{Ker}\Delta = \left\{t \in \mathbb{R}^{n+1} : t = a + \sum_{r=1}^{n+1} s_r b_r,\ s_1, ..., s_{n+1} \in \mathbb{R}\right\}$ mit $a \in \mathbb{R}^{n+1}$ und linear unabhängigen $b_1, ..., b_{n+1} \in \mathbb{R}^{n+1}$ sowie $b_1 \neq 0$, der wegen 3.4 den Definitionsbereich $[0;1]^{n+2}$ der konvexen Hülle und für jedes $0 \leq j \leq n+1$ mit $\langle e_j, b_r\rangle \neq 0$ seinen **Rand** $\left\{(t_0, ..., t_{n+1}) \in [0;1]^{n+2} : \exists 0 \leq j \leq n+1 : t_j = 0\right\}$ schneidet. Dabei kann man sich das Urbild $\Delta^{-1}\left[\operatorname{co}(D_0)\right]$ der konvexen Hülle als mindestens eindimensionale und höchstns $(n+1)$-dimensionale Hyperfläche vorstellen, die durch den Schnitt der $(n+1)$-dimensionalen Hyperebene $\left\{t = (t_0; ...; t_{n+1}) \in \mathbb{R}^{n+2} : \sum_{k=0}^{n+1} t_k = 1\right\}$ mit dem Einheitswürfel $[0;1]^{n+2}$ entsteht. An jedem dieser Durchstoßpunkte ist mindestens ein $t_j = 0$ und $0 = \sum_{\substack{0\leq k\leq n+1 \\ k\neq j}} t_k x_k \wedge \sum_{\substack{0\leq k\leq n+1 \\ k\neq j}} t_k = 1$ erfüllt, d.h., $0 \in \operatorname{co}(D_0 \setminus \{x_j\})$. Die minimale konvexe Hülle enthält damit sogar eine **offene Umgebung** des Ursprungs: $0 \in U \subset \operatorname{co}(D_0)$ für ein $U \in \mathcal{U}(0)$, denn falls der Ursprung auf dem **Rand** $\left\{\sum_{k=1}^{m} t_k x_k \in \operatorname{co}(D_0) : \exists 1 \leq j \leq n+1 : t_j = 0\right\}$ liegt, könnte wieder der entsprechende Divergenzpunkt x_j entfernt werden, d.h. $0 \in \operatorname{co}(D_0 \setminus \{x_j\})$ im Widerspruch zum minimalen Charakter von D_0.

8.15 Lineare Hülle der minimalen Divergenzpunkte

Sei $X_0 = \left\{ \sum\limits_{k=1}^{m} t_k x_k : x_1, ..., x_m \in D_0 \right\}$ die **lineare Hülle** von $D_0 = \{x_1; ...; x_m\}$ mit der **Projektion** $pr_0 : \mathbb{R}^n \to X_0$, wobei $pr_0(v) = \sum\limits_{k=1}^{m} \left\langle v, \frac{x_k}{\|x_k\|} \right\rangle \frac{x_k}{\|x_k\|}$ und $X_1 = X_0^{\perp} = \{y \in \mathbb{R}^n : \langle x, y\rangle = 0 \forall x \in X_0\}$ das **orthogonale Komplement** mit der entsprechenden **Projektion** $pr_1 : \mathbb{R}^n \to X_1$, wobei $pr_1(v) = \sum\limits_{k=m+1}^{n} \left\langle v, \frac{x_k}{\|x_k\|} \right\rangle \frac{x_k}{\|x_k\|}$ für die **Basis** $\{x_{m+1}; ...; x_n\}$ von $D_0^{\perp}$. Dann gibt es eine Teilmenge $\Omega \subset \mathbb{N}$ mit $\sum_{i\in\Omega} \|\mathrm{pr}_1(v_i)\| < \infty$ und $\sum_{i\in\Omega\cap\mathbb{N}_U} \|v_i\| = \infty$ für jede Umgebung $U \subset S$ aller $x_k \in D_0$

Beweis: Für jedes $n \in \mathbb{N}$ und $v \in B_n := B_{2^{-n}}(x) \subset S$ gilt $\|\mathrm{pr}_1(v)\| = \|\mathrm{pr}_1(v - x) + \mathrm{pr}_1(x)\| = \|\mathrm{pr}_1(v - x) + 0\| \leq \|v - x\| < 2^{-n}$. Für jedes $x_k \in D_0 \subset D$ ist $\mathbb{N}_{B_n}$ unendlich und $\sum_{i\in\mathbb{N}_{Bn}} \|v_i\| = \infty$. Daher und nach 3.2 gibt es eine endliche Teilmenge $F_n \subset \mathbb{N}_{B_n}$ mit $n < \sum_{i\in F_n} \|v_i\| < n + 1$. Für $O_k = \bigcup_{n\in\mathbb{N}} F_n$ gilt dann $\sum_{i\in O_k} \|\mathrm{pr}_1(v_i)\| \leq \sum_{n\in\mathbb{N}} \sum_{i\in F_n} \|\mathrm{pr}_1(v_i)\| = \sum_{n\in\mathbb{N}} \sum_{i\in F_n} \left\|\mathrm{pr}_1\left(\frac{v_i}{\|v_i\|}\right)\right\| \cdot \|v_i\| \leq \sum_{n\in\mathbb{N}} \frac{n+1}{2^n} < \infty$ und für $\Omega = \bigcup\limits_{k=1}^{m} O_k$ folglich die erste Behauptung. Für eine beliebige Umgebung $U \subset S$ von $x_k \in D_0$ gibt es ein $n \in \mathbb{N}$ mit $B_{2^{-n}}(x_k) \subset U$ und damit $F_m \subset \mathbb{N}_{B_m} \subset \mathbb{N}_U$ für alle $m \geq n$, so dass $\sum_{i\in\Omega\cap\mathbb{N}_U} \|v_i\| \geq \sum_{i\in\bigcup_{m\geq n} F_m} \|v_i\| \geq \sup\limits_{m\geq n} m = \infty$, womit die zweite Behauptung gezeigt ist.

8.16 Näherung der linearen Hülle der minimalen Divergenzpunkte

Es gibt eine positive Konstante C, so dass für $x \in X_0$, $\epsilon > 0$ und jede endliche Teilmenge $F \subset \Omega$ eine weitere endliche Teilmenge $E \subset \Omega \setminus F$ existiert mit $\|x - \sum_{i\in E} v_i\| < \epsilon$ und $\|\sum_{i\in E'} v_i\| \leq C \cdot \max\{\|x\|, \epsilon\}$ für alle $E' \subset E$.

Beweis: Nach 8.5 gibt es ein $\delta < \frac{1}{4}$ mit $B_\delta(0) \subset \mathrm{co}(D_0)$ und $B_\delta(x_i) \cap B_\delta(x_j) = \emptyset \forall x_i, x_j \in D_0$. Seien $x \in X_0$, $\epsilon < \delta$ und die endliche Teilmenge $F \subset \Omega$ gegeben. Für $c = \frac{1}{\delta} \max\{\|x\|, \epsilon\}$ folgt $\frac{x}{c} \in B_\delta(0) \subset \mathrm{co}(D_0)$ und damit $x = \sum\limits_{k=1}^{m} c \cdot t_k x_k$ für $x_1, ..., x_m \in D_0$ und $0 \leq t_k \leq 1$ mit $\sum\limits_{k=1}^{m} t_k = 1$. Auf der sphärischen Kreisscheibe $S_k = B_\delta(x_k) \cap S$ gilt zunächst $x, y \in S_k \Rightarrow \langle x, y\rangle = \langle x, x\rangle + \langle x, y - x\rangle \geq 1 - \|x - y\| > 1 - 2\delta \geq \frac{1}{2}$, woraus für den zugehörigen Richtungskegel $\hat{S}_k = \{t \cdot x : x \in S_k, t > 0\}$ die Abschätzung $x, y \in \hat{S}_k \Rightarrow \|x + y\| \geq \left\langle x + y, \frac{x}{\|x\|} \right\rangle = \|x\| + \|y\| \left\langle \frac{y}{\|y\|}, \frac{x}{\|x\|} \right\rangle \geq \|x\| + \frac{1}{2}\|y\|$ folgt. Nach 8.6 sind die Mengen $\Omega_k := \left\{i \in \Omega : \frac{v_i}{\|v_i\|} \in S_k\right\} = \left\{i \in \Omega : v_i \in \hat{S}_k\right\}$ für $1 \leq k \leq m$ unendlich und $\sum_{i\in\Omega_k} \|v_i\| \geq \sum_{i\in\Omega_k\cap\mathbb{N}_U} \|v_i\| = \infty$. Da wegen $(v_i)_{i\in\Omega_k} \subset \hat{S}_k$ auch $\sum_{i\in\Omega_k} v_i \in \hat{S}_k$ und außerdem $\lim\limits_{i\to\infty} \|v_i\| = 0$ gilt, lässt sich für eine gegebene endliche Teilmenge $F \subset \Omega_k$ eine weitere endliche Teilmenge $E_k \subset \Omega_k \setminus F$ finden, so dass $s_k = \sum_{i\in E_k} v_i \in \hat{S}_k$ und $c \cdot t_k - \frac{\epsilon}{2m} < \|s_k\| \leq c \cdot t_k$. Damit folgt $\|s_k - c \cdot t_k x_k\| \leq \left\|\|s_k\| \cdot \frac{s_k}{\|s_k\|} - c \cdot t_k \cdot \frac{s_k}{\|s_k\|}\right\| + \left\|c \cdot t_k \cdot \frac{s_k}{\|s_k\|} - c \cdot t_k x_k\right\| = \|\|s_k\| - c \cdot t_k\| + c \cdot t_k \left\|\frac{s_k}{\|s_k\|} - x_k\right\| < \frac{\epsilon}{2m} + c \cdot t_k \cdot \frac{\epsilon}{2c}$ und für die endliche Teilmenge $E = \bigcup\limits_{k=1}^{m} E_k \subset \Omega \setminus F$ folgt $\|\sum_{i\in E} v_i - x\| = \left\|\sum_{i\in E} v_i - \sum\limits_{k=1}^{m} c \cdot t_k x_k\right\| \leq \sum\limits_{k=1}^{m} \left\|\sum_{i\in E_k} v_i - c \cdot t_k x_k\right\| < \sum\limits_{k=1}^{m} \left(\frac{\epsilon}{2m} + c \cdot t_k \cdot \frac{\epsilon}{2c}\right) = \frac{\epsilon}{2}\left(1 + \sum\limits_{k=1}^{m} t_k\right) = \epsilon.$

Für die zweite Behauptung sei $E' \subset E$ und $E'_k := E \cap E_k$. Dann ist $\sum_{i\in E'_k} v_i \in \hat{S}_k$ mit $\left\|\sum_{i\in E'_k} v_i\right\| \leq \left\|\sum_{i\in E_k} v_i\right\| \leq c \cdot t_k \leq c$, also $\sum_{i\in E'_k} v_i = t \cdot y$ mit $y \in S_k$ und $0 \leq t \leq c$, so dass $\|\sum_{i\in E'} v_i\| \leq c \cdot \delta \cdot C = C \cdot \max\{\|x\|, \epsilon\}$ mit $C := \frac{1}{\delta} \sup\left\{\left\|\sum\limits_{k=1}^{m} t_k y_k\right\| : 0 \leq t_k \leq 1, y_k \in B_\delta(x_k), x_k \in D_0\right\}$.

8.17 Die Richtungen absoluter Konvergenz

Die Richtungen absoluter Konvergenz sind orthogonal zur linearen Hülle der Divergenzpunkte: $\Gamma = \{y \in \mathbb{R}^n : \sum_{i\in\mathbb{N}} |\langle y, v_i\rangle| < \infty\} = \{y \in X_1 : \sum_{i\in\mathbb{N}} |\langle y, \mathrm{pr}_1(v_i)\rangle| < \infty\} = \Gamma_1$

Beweis: $\subset$: Angenommen, $\exists y \in \Gamma \setminus X_1$, dann existiert insbesondere ein $x_k \in D_0$ mit $\langle y, x_k\rangle \neq 0$ und die Menge $\mathbb{N}_U := \left\{i \in \mathbb{N} : \frac{v_i}{\|v_i\|} \in U\right\}$ für die offene Umgebung $U = \{x \in S : |\langle y, x\rangle| > |\langle y, x_k\rangle|\}$ von y ist unendlich und $\sum_{i\in\mathbb{N}_{\mathbb{U}}} \|v_i\| = \infty$. Damit folgt aber $|\langle y, v_i\rangle| > \|v_i\| \cdot |\langle y, x_k\rangle| \forall i \in \mathbb{N}_U$ und folglich $\sum_{i\in\mathbb{N}_{\mathbb{U}}} |\langle y, v_i\rangle| = \infty$ im Widerspruch zur Auswahl von y. $\supset$: Für $y \in X_1$ ist $\langle y, \mathrm{pr}_1(v_i)\rangle = \langle y, v_i\rangle - \langle y, \mathrm{pr}_0(v_i)\rangle = \langle y, v_i\rangle - 0 = \langle y, v_i\rangle$, woraus sich die Behauptung ergibt.

8.18 Satz von Lévy und Steinitz II

Die Menge Σ aller möglichen Summen $\sum_{i\in\mathbb{N}} v_{p(i)}$ zu den Permutationen $p : \mathbb{N} \to \mathbb{N}$ einer Folge $(v_i)_{i\in\mathbb{N}} \subset \mathbb{R}^n$ von Vektoren ist ein affiner Unterraum: $\Sigma = \Sigma + \Gamma^\perp$, wobei der Untervektorraum $\Gamma = \{y \in \mathbb{R}^n : \sum_{i\in\mathbb{N}} |\langle y, v_i\rangle| < \infty\}$ die Richtungen absoluter Konvergenz und $\Gamma^\perp = \{x \in \mathbb{R}^n : \langle y, x\rangle = 0 \forall y \in \Gamma\}$ das orthogonale Komplement zu Γ beschreiben. Dabei gilt $\Sigma \neq \emptyset$ genau dann, wenn $\sum_{i\in\mathbb{N}} v_i$ komponentenweise bedingt konvergent ist. Im Fall absoluter Konvergenz mit $\sum_{i\in\mathbb{N}} \|v_i\| < \infty$ ist $\Sigma = \{v\}$ mit $v = \sum_{i\in\mathbb{N}} v_i$ sowie $\Gamma = \mathbb{R}^n$und folglich $\Gamma^\perp = \emptyset$.

Beweis durch Induktion nach n: Aus der komponentenweise bedingten Konvergenz von $\sum_{i\in\mathbb{N}} v_i$ in $\mathbb{R}^n$ folgt nach 8.8 die komponentenweise bedingte Konvergenz von $\sum_{i\in\Omega} \mathrm{pr}_1(v_i)$ im Hilbertraum $X_1 \simeq \mathbb{R}^m$ mit der Dimension $m < n$, falls keine absolute Konvergenz vorliegt. (vgl. 8.3). Nach einem geeigneten Basiswechsel kann die Induktionsannahme auf $\sum_{i\in\mathbb{N}} \mathrm{pr}_1(v_i)$ angewandt werden, so dass die Menge Σ_1 aller möglichen Summen $\sum_{i\in\Omega} \mathrm{pr}_1\left(v_{p(i)}\right)$ die Struktur $\Sigma_1 = \Sigma_1 + \Gamma_1^\perp \cap X_1 = \Sigma_1 + \Gamma^\perp \cap X_1$ besitzt, denn nach 8.8 gilt $\Gamma_1 = \Gamma = \{y \in \mathbb{R}^n : \sum_{i\in\mathbb{N}} |\langle y, v_i\rangle| < \infty\}$. Man kann nun zeigen, dass für die Menge Σ aller möglichen Summen $\sum_{i\in\mathbb{N}} v_{p(i)}$ gilt $\Sigma = \Sigma_1 + \left(\Gamma^\perp \cap X_1\right) \oplus X_0 = \Sigma_1 + \Gamma^\perp$, wobei die Gleichung $\left(\Gamma^\perp \cap X_1\right) \oplus X_0 = \Gamma^\perp$ mit Hilfe der Zerlegung $x = \mathrm{pr}_0(x) + \mathrm{pr}_1(x)$ sowie der Beziehung $\Gamma \subset X_1$ eingesehen werden kann. Die Inklusion $\Sigma \subset \Sigma_1 + \left(\Gamma^\perp \cap X_1\right) \oplus X_0$ ergibt sich trivialerweise aus der Zerlegung $\mathbb{R}^n = X_0 \oplus X_1$. Für die Umkehrung ist zu zeigen, dass für jedes $x \in X_0$ und $y \in \Sigma_1$ eine Permutation p von $\mathbb{N}$ existiert mit $\sum_{i\in\mathbb{N}} v_{p(i)} = x + y$. Die gewünschte Permutation p wird zweckmäßigerweise durch eine Wohlordnung $\preccurlyeq$ auf $\mathbb{N}$ gemäß $p(i) = \max_{\leq} [0; i]_{\preccurlyeq}$ definiert, so dass $p(i) \leq p(j) \Leftrightarrow i \preccurlyeq j$ und $\sum_{i\in\mathbb{N}}^{\preccurlyeq} v_i = x + y$, d.h., für jedes $\epsilon > 0$ existiert ein $k \in \mathbb{N}$, so dass für alle $j \succcurlyeq k$ gilt $\left\|\sum_{i\succcurlyeq j} v_i - (x + y)\right\| < \epsilon$. Nach Induktionsannahme gibt es eine eine Permutation p von $\mathbb{N}$ mit $\sum_{i\in\mathbb{N}} \mathrm{pr}_1\left(v_{p(i)}\right) = y$ und damit wie eben beschrieben eine Wohlordnung $\preccurlyeq_1$ auf $\mathbb{N}$ und insbesondere auf $\Lambda = \mathbb{N} \setminus \Omega$ mit $i \preccurlyeq_1 j \Leftrightarrow p(i) \leq p(j)$. Damit erhält man bereits $\sum_{i\in\Lambda}^{\preccurlyeq_1} \mathrm{pr}_1(v_i) + \sum_{i\in\Omega} \mathrm{pr}_1(v_i) = y$. Auf Ω ist der Grenzwert der **absolut konvergenten** Teilreihe $\sum_{i\in\Omega} \mathrm{pr}_1(v_i) \in X_1$ **unabhängig von der Permutation bzw. Wohlordnung** und erlaubt daher weitere Anpassung der Wohlordnung $\preccurlyeq_1$ auf Ω, so dass die endgültige Wohlordnung $\preccurlyeq$ auf $\mathbb{N} = \Omega \cup \Lambda$ auf Λ mit $\preccurlyeq_1$ übereinstimmt und wie bisher $\sum_{i\in\mathbb{N}}^{\preccurlyeq} \mathrm{pr}_1(v_i) = y$ gilt sowie zusätzlich $\sum_{i\in\mathbb{N}}^{\preccurlyeq} \mathrm{pr}_0(v_i) = x$. Dazu konstruiert man induktiv die absteigenden wohlgeordneten Mengenfolgen $(\Lambda_k; \preccurlyeq)_{k\in\mathbb{N}}$ und $(\Omega_k; \leq)_{k\in\mathbb{N}}$ sowie eine aufsteigende Folge $(F_k)_{k\in\mathbb{N}}$ mit den folgenden Eigenschaften für $k \in \mathbb{N}$:

1. $\Lambda_{k+1} = \Lambda_k \setminus \{\min \Lambda_k\}$
2. $\Omega_{k+1} \subset \Omega_k \setminus \{\min \Omega_k\}$
3. $F_{k+1} = F_k \cup \{\min \Lambda_k\} \cup (\Omega_k \setminus \Omega_{k+1})$
4. $\left\|x - \sum_{i\in F_{k+1}} \mathrm{pr}_0(v_i)\right\| < 2^{-k}$
5. $\forall E \subset F_{k+1} \setminus F_k : \|\sum_{i\in E} v_i\| \leq C \cdot \left(2^{-k} + \|v_{\min \Lambda_k} + v_{\min \Omega_k}\|\right)$.

Nach 8.7 lässt sich eine endliche Menge $F_0 \subset \Omega$ finden mit $\|x - \sum_{i\in F_0} \mathrm{pr}_0(v_i)\| \leq \|x - \sum_{i\in F_0} v_i\| < 1$. Damit definiert man $\Omega_0 = \Omega \setminus F_0$ und $\Lambda_0 = \Lambda = \mathbb{N} \setminus \Omega$. Für breits konstruierte F_k, Ω_k, Λ_k sei $a_k := x - \mathrm{pr}_0\left(v_{\min \Lambda_k} + v_{\min \Omega_k} + \sum_{i\in F_k} v_i\right) \in X_0$ mit $\|a_k\| \leq \left\|x - \sum_{i\in F_k} \mathrm{pr}_0(v_i)\right\| + \|v_{\min \Lambda_k} + v_{\min \Omega_k}\| < 2^{-k} + \|v_{\min \Lambda_k} + v_{\min \Omega_k}\|$ wegen 4. Mit 8.7 lässt sich eine endliche Teilmenge $E_k \subset \Omega_k \setminus F_k \cup \{\min \Omega_k\}$ finden mit $\left\|a_k - \sum_{i\in E_k} v_i\right\| < 2^{-k-1}$ und für jede Teilmenge $E \subset E_k$ gilt $\|\sum_{i\in E} v_i\| < C \cdot \max\left\{2^{-k-1}; \|a_k\|\right\} \leq$

$C \cdot \left(2^{-k} + \left\|v_{\min \Lambda_k} + v_{\min \Omega_k}\right\|\right)$. Definiere nun $F_{k+1} := F_k \cup E_k \cup \{\min \Lambda_k; \min \Omega_k\}$, $\Omega_{k+1} := \Omega_k \setminus (E_k \cup \{\min \Omega_k\})$ und $\Lambda_{k+1} = \Lambda_k \setminus \{\min \Lambda_k\}$. Auf $\bigcup_{k \in \mathbb{N}} F_k = \mathbb{N}$ wird nun die angpasste Fortsetzung $\preccurlyeq$ der Wohlordnung $\preccurlyeq_1$ defniert mit $i \preccurlyeq j \Leftrightarrow \exists k \in \mathbb{N} : i \in F_k \wedge j \notin F_k$. Aufgrund der absoluten Konvergenz von $\sum_{i \in \Omega} \operatorname{pr}_1(v_i)$ undabhängig von der Ordnung gilt nach wie vor $\sum_{i \in \mathbb{N}}^{\preccurlyeq} \operatorname{pr}_1(v_i) = \sum_{i \in \Lambda}^{\preccurlyeq} \operatorname{pr}_1(v_i) + \sum_{i \in \Omega}^{\preccurlyeq} \operatorname{pr}_1(v_i) = y$. Wegen 4. und 5. sowie 8.2 gilt nun aber zusätzlich $\sum_{i \in \mathbb{N}}^{\preccurlyeq} \operatorname{pr}_0(v_i) = x$, womit $x+y \in \Sigma$ bewiesen ist. Die Behauptung ergibt sich nun aus $\Sigma + \Gamma^{\perp} = (X_0 \oplus \Sigma_1) + \left(X_0 \oplus \Gamma_1^{\perp}\right) = X_0 \oplus \left(\Sigma_1 + \Gamma_1^{\perp}\right) = X_0 \oplus \Sigma_1 = \Sigma$.

8.19 Beispiel I

Das erste Beispiel zeigt, dass die Mengen $X_0 \subset \Gamma^{\perp}$ bzw. $\Gamma \subset X_1$ nicht zusammenfallen müssen: Für Reihe $\sum_{i \in \mathbb{N}} v_i$ mit $v_i = (-1)^i \begin{pmatrix} i^{-0,5} \\ i^{-1} \end{pmatrix}$ ist $\sum_{i \in \mathbb{N}} |\langle y, v_i \rangle| = \infty$ u.a. für $y = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$, aber der einzige Divergenzpunkt ist $x = \lim\limits_{i \to \infty} \frac{v_i}{\|v_i\|} = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$. In diesem Fall ist $X_0 = \left\{\left(\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right) : x \in \mathbb{R}\right\}$, $X_1 = \left\{\left(\begin{smallmatrix} 0 \\ y \end{smallmatrix}\right) : y \in \mathbb{R}\right\}$, $\Gamma = \left\{\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)\right\}$ und $\Gamma^{\perp} = \mathbb{R}^2$. Die Menge der möglichen Summen bzw. Grenzwerte ist $\Sigma = \Sigma + \Gamma^{\perp} = \mathbb{R}^2$. Auch orthogonal zur linearen Hülle X_0 der Divergenzpunkte können also Divergenzrichtungen liegen, d.h., der Vektorraum Γ der Richtungen absoluter Konvergenz ist i.A. eine echte Teilmenge von X_1.

8.20 Beispiel II

Das zweite Beispiel zeigt, dass die Mengen $D_0 \subset D$ bzw. $\Gamma \subset X_1$ nicht zusammenfallen müssen: Für die Reihe $\sum_{i \in \mathbb{N}} v_i$ in $\mathbb{R}^3$ mit $v_i = \frac{(-i)^i}{i} \left(\begin{smallmatrix} 1 \\ 1 \\ 0 \end{smallmatrix}\right)$ sind die Divergenzrichtungen identisch mit dem Einheitskreis in der x-y-Ebene: $D = \left\{\left(\begin{smallmatrix} x \\ y \\ 0 \end{smallmatrix}\right) : x^2 + y^2 = 1\right\}$. Eine minimale Teilmenge, deren konvexe Hülle den Ursprung enthält, ist z.B. $D_0 = \left\{\left(\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right); \left(\begin{smallmatrix} -1 \\ 0 \\ 0 \end{smallmatrix}\right)\right\}$ mit $\operatorname{co}(D_0) = \left\{\left(\begin{smallmatrix} 1-2t \\ 0 \\ 0 \end{smallmatrix}\right) : 0 \leq t \leq 1\right\}$ und $X_0 = \left\{\left(\begin{smallmatrix} x \\ 0 \\ 0 \end{smallmatrix}\right) : x \in \mathbb{R}\right\}$. Das orthogonale Komplement ist $X_1 = \left\{\left(\begin{smallmatrix} 0 \\ y \\ z \end{smallmatrix}\right) : y, z \in \mathbb{R}\right\}$. Offensichtlich ist aber $\Gamma = \left\{\left(\begin{smallmatrix} 0 \\ 0 \\ z \end{smallmatrix}\right) : z \in \mathbb{R}\right\}$ und die Menge der möglichen Summen bzw. Grenzwerte ist $\Sigma = \Sigma + \Gamma^{\perp} = \left\{\left(\begin{smallmatrix} x \\ y \\ 0 \end{smallmatrix}\right) : x, y \in \mathbb{R}\right\}$. In diesem Fall gilt also $X_0 = \Gamma^{\perp}$.

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